

Some quotients of chain products are symmetric chain orders

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Abstract

Canfield and Mason have conjectured that for all subgroups G of the automorphism group of the Boolean lattice B_n (which can be regarded as the symmetric group S_n) the quotient order B_n/G is a symmetric chain order. We provide a straightforward proof of a generalization of a result of K. K. Jordan: namely, B_n/G is an SCO whenever G is generated by powers of disjoint cycles. In addition, the Boolean lattice B_n can be replaced by any product of finite chains. The symmetric chain decompositions of Greene and Kleitman provide the basis for partitions of these quotients.

Keywords: symmetric chain decomposition, Boolean lattice, quotients

1 Introduction

There are several familiar notions of symmetry for the family of finite ranked partially ordered sets. This family can be defined in more general ways (see [9]), but for our purposes, all of our finite partially ordered sets P have a minimum element 0_P and for all $x \in P$, all saturated chains $C \subseteq P$ with minimum element 0_P and maximum x have the same length $r_P(x) := |C| - 1$. Such P are called *ranked* posets, $r = r_P$ is the *rank function* and $r(P)$, the maximum over all $r(x)$, $x \in P$, is the rank of P . Note that a ranked ordered

set satisfies the Jordan-Dedekind chain condition: for all $x \leq y$ in P , all saturated chains in the interval $[x, y]$ have the same length.

In a ranked order P the chain $x_1 < x_2 < \dots < x_k$ is a *symmetric chain* if it is saturated and if $r(x_1) + r(x_k) = r(P)$. A *symmetric chain decomposition* or *SCD* of P is a partition of P into symmetric chains. If P has an SCD, call P a *symmetric chain order*, or an *SCO*. Here, we are concerned with ordered sets based on the *Boolean lattice*, denoted B_n , which is the power set of $[n] = \{1, 2, \dots, n\}$ ordered by containment. Clearly B_n is a ranked poset, with \emptyset being the minimum element, and $r(A) = |A|$ for all $A \subseteq [n]$. In fact, it is an SCO [2].

We are interested in ordered sets defined by actions of the automorphism group of B_n . It is well-known that this group is faithfully induced by the symmetric group S_n of all permutations on the underlying set $[n]$, so we will refer to S_n as the automorphism group of B_n . Given any subgroup G of S_n , the *quotient* B_n/G has as its elements the orbits in B_n under G

$$[A] = \{B \mid B = \sigma(A), \text{ for some } \sigma \in G\},$$

$A \in B_n$, ordered by

$$[A] \leq [B] \iff X \subseteq Y \text{ for some } X \in [A] \text{ and } Y \in [B].$$

In studying Venn diagrams, Griggs, Killian and Savage [12] explicitly constructed an SCD of the quotient B_n/G for n prime and given that G is generated by a single n -cycle. They asked if this *necklace* poset is an SCO for arbitrary n . Canfield and Mason [3] made a much more general conjecture: for all subgroups G of S_n , B_n/G is a symmetric chain order.

Jordan [14] gave a positive answer to the question of Griggs, Killian and Savage, basing the SCD of the quotient on the explicit construction of an SCD in B_n by Greene and Kleitman [10]. The construction in [14] requires an intermediate equivalence relation and some careful analysis. Here we provide a more direct proof of a generalization of Jordan's theorem by "pruning" the Greene-Kleitman SCD. More generally, we show that B_n/G is an SCO provided that G is generated by powers of disjoint cycles (see Theorem 1). We also provide a different proof that B_n/G is an SCO when G is a 2-element subgroup generated by a reflection, based on an SCD of $B_{\lfloor n/2 \rfloor}$.

The ordered sets B_n/G do share several forms of symmetry or regularity with the Boolean lattice. An SCO P is necessarily rank-symmetric, rank-unimodal, and strongly Sperner (see, for instance, [14] for definitions). A result of Stanley [17] shows that B_n/G has these three properties for all subgroups G of S_n . However, these three conditions are not sufficient to yield symmetric chain decompositions.

On the other hand, Griggs [11] showed that a ranked ordered set with the LYM property, rank-symmetry and rank-unimodality is an SCO. It is not the case that all SCOs have the LYM property (see [11] for examples), in fact, not all quotients B_n/G have the LYM property. However, if n is prime and G is generated by an n -cycle, then B_n/G does satisfy LYM, giving the Griggs, Killian and Savage result. Pouzet and Rosenberg [16] obtain Stanley's results and "local" families of symmetric chains for more general structures than the quotients B_n/G , but their results do not show that B_n/G is an SCO.

2 The Main Results

There are two results for quotients of B_n by groups generated by powers of disjoint cycles and for a particular 2-element group. The third theorem concerns quotients of powers of a finite chain, with a corollary for products of chains of differing length.

Theorem 1. *Let G be a subgroup of S_n generated by powers of disjoint cycles. Then the partially ordered set B_n/G is a symmetric chain order.*

The proof of Theorem 1 follows from this sequence of results. The new proof of Lemma 2, which is a modest generalization of Jordan's result, is given in Section 3.

Lemma 2. *Let σ be an n -cycle in S_n and let H be a subgroup of the group generated by σ . Then B_n/H is a symmetric chain order.*

The following fact is well-known and can be proved by an argument much like the original proof in [2] that the divisor lattice of an integer is an SCO. (In [9], this is credited to Alekseev [1].)

Lemma 3. *Let P and Q be partially ordered sets. If P and Q are symmetric chain orders then so is $P \times Q$.*

In the following lemma, use this notation. Suppose that σ_j ($j = 1, 2, \dots, t$) are disjoint cycles in S_n and that $\rho_j = \sigma_j^{r_j}$, for integers r_1, r_2, \dots, r_t . Let X_j be the subset of $[n]$ of elements moved by ρ_j ($j = 1, 2, \dots, t$), and let X_0 be all elements of $[n]$ fixed by all the ρ_j 's. Let $B(X)$ denote the Boolean lattice of all subsets of a set X .

Lemma 4. *Let H_j be the subgroup of S_n generated by ρ_j ($j = 1, 2, \dots, t$) and let G be the subgroup generated by $\{\rho_1, \rho_2, \dots, \rho_t\}$. Then*

$$B_n/G \cong B(X_0) \times B(X_1)/H_1 \times \cdots \times B(X_t)/H_t.$$

Proof. For any $A \subseteq [n]$, let $[A]$ denote its equivalence class in B_n/G , and let $A_j = A \cap X_j$, $j = 0, 1, \dots, t$. Define a map Φ on B_n/G by $\Phi([A]) = (A_0, [A_1], \dots, [A_t])$. From the definition of the ordering of the quotient, $[A] \leq [B]$ in B_n/G if and only if there is some $\tau \in G$ such that $A \subseteq \tau(B)$. Then $\tau = \rho_1^{i_1} \rho_2^{i_2} \cdots \rho_t^{i_t}$ for nonnegative integers i_1, i_2, \dots, i_t . The claimed isomorphism follows from this fact:

$$A \subseteq \tau(B) \text{ if and only if } A_0 \subseteq B_0 \text{ and } A_j \subseteq \sigma^{i_j}(B_j) \text{ for } j = 1, 2, \dots, t.$$

□

The following is actually a corollary of Theorem 1. Indeed, a proof based on an approach like that used in the proof of Theorem 1 – a greedy pruning of a Greene-Kleitman SCD – can be shown to provide a basis for the proof offered in Section 4. However, the proof in Section 4 provides some insight into the Greene-Kleitman SCD and may be of use for other choices for the group of permutations, such as the dihedral group, which is an appealing next case for study.

Theorem 5. *Let G be a 2-element subgroup with non-unit element a product of disjoint transpositions. Then the partially ordered set B_n/G is a symmetric chain order.*

The last result concerns quotients defined by automorphism groups of products of chains. Given any partially ordered set P and subgroup G of its automorphism group $\text{Aut}(P)$, the quotient P/G has elements the orbits $[x]$ on P defined by G with $[x] \leq [y]$ in P/G if there are $x' \in [x]$ and $y' \in [y]$ such that $x' \leq y'$ in P . For any finite chain C , positive integer m and $\alpha \in \text{Aut}(C^m)$, there is some $\phi \in S_m$ such that

$$\alpha(c_1, c_2, \dots, c_m) = (c_{\phi^{-1}(1)}, c_{\phi^{-1}(2)}, \dots, c_{\phi^{-1}(m)}), \text{ for all } (c_1, c_2, \dots, c_m) \in C^m.$$

This follows easily from the action of the automorphisms on the covers of the minimum element of the chain product. (It is also a consequence of a result of Chang, Jónsson and Tarski [4], on the strict refinement property for product decompositions of partially ordered sets.) In particular, automorphism groups of powers of chains behave as those of the Boolean lattice and we can regard $\text{Aut}(C^m)$ as the symmetric group S_m acting on the coordinates of C^m .

Theorem 6. *Let C be a chain and let K be a subgroup of S_m generated by powers of disjoint cycles. Then C^m/K is an SCO.*

The proof, presented in Section 5, is a consequence of the proof of Lemma 2 and some observations on the Greene-Kleitman SCD. V. Dhand [5] has a new, very interesting result that is more general than the essential part of Theorem 6: if P is any SCO then so is P^n/\mathbb{Z}_n . His arguments depend upon algebraic tools. P. Hersh and A. Schilling give a new proof of Jordan's result via an explicit combinatorial construction of an SCD in B_n/\mathbb{Z}_n based on representation of the special linear group [13].

We note that Theorem 6 can be stated more generally for chain products. Let $P = \prod_{i=1}^n C_i^{m_i}$ where $C_j \not\cong C_k$ for $j \neq k$. It is easy to see that each automorphism of P factors into an n -tuple from $\prod_{i=1}^n \text{Aut}(C_i^{m_i})$ and that each $\text{Aut}(C_i^{m_i}) \cong S_{m_i}$. (For a much more general result based [4], see [6].) Thus, if K is a subgroup of $\text{Aut}(P)$ which also factors into a product of subgroups of S_{m_i} of the form covered by Theorem 6 then, by Lemmas 3 and 4, P/K is an SCO. In particular, we have this consequence.

Corollary 7. *Let P be a product of chains and let K be a subgroup of $\text{Aut}(P)$ that is generated by powers of disjoint cycles. Then P/K is an SCO.*

We use Corollary 7 to deal with some cases where K does not factor so nicely in [8].

3 The Proof of Lemma 2

We use the natural order $1 < 2 < \dots < n$ on $[n]$ and may assume that the n -cycle σ is $(1\ 2\ \dots\ n)$. This is valid because any n -cycle ρ is a conjugate of $(1\ 2\ \dots\ n)$ and for any subgroup K of S_n and any $\pi \in S_n$, $B_n/K \cong B_n/\pi^{-1}K\pi$ via $[A] \mapsto [\pi(A)]$.

We first describe the procedure for obtaining an SCD of B_n/H based on the Greene-Kleitman SCD of B_n then verify that the procedure yields the claimed SCD.

Let C_1, C_2, \dots, C_t , where $t = \binom{n}{\lfloor n/2 \rfloor}$, be the symmetric chains in the Greene-Kleitman decomposition, ordered by decreasing length. For all $A \in B_n$, $[A]$ is the equivalence class containing A in B_n/H where H is the subgroup of S_n generated by $\rho = \sigma^s$.

Claim 1: There is a family $\mathcal{C} = \{C'_{i_1}, C'_{i_2}, \dots, C'_{i_m}\}$, with (i_1, i_2, \dots, i_m) a subsequence of $(1, 2, \dots, t)$, that satisfies these conditions:

- (3.1) for all $1 \leq j \leq m$, $C'_{i_j} \subseteq C_{i_j}$ and is a symmetric chain in B_n ;
- (3.2) for all $1 \leq r < s \leq m$ and for all $A \in C'_{i_r}, B \in C'_{i_s}, A \notin [B]$; and,
- (3.3) for all $[X]$ there is some $Y \in [X]$ such that $Y \in C'_{i_j}$ for some j .

For $j = 1, 2, \dots, m$, let $\widehat{C}_j = \{[A] \mid A \in C'_{i_j}\}$. Then the chains $\widehat{C}_1, \widehat{C}_2, \dots, \widehat{C}_m$ cover B_n/H (by (3.3)), the sets are disjoint (by (3.2)), and form symmetric chains (by (3.1)). Thus, it is enough to verify **Claim 1** in order to prove Lemma 2.

Several properties of the Greene-Kleitman SCD of B_n are needed. For the most part, these are well-known – see, for instance, the descriptions in [9] and [14]. It is useful to regard members of B_n both as subsets of $[n]$ and as binary sequences of length n , defined with respect to the natural order. (Indeed, one needs to fix an order to speak of *the* Greene-Kleitman SCD.) The SCD is obtained by a bracketing or pairing procedure that has several equivalent descriptions. Here are two that are useful to us. Let $A \subseteq [n]$.

- (3.4) If $1 \notin A$ and $2 \in A$, pair 1 and 2; define $p_A(2) = 1$. Suppose that we have considered $1, 2, \dots, k-1$. If $k \in A$ and there is some $j < k, j \notin A$ such that j is unpaired, then let $p_A(k)$ be the maximum such j and say $p_A(k)$ and k are *paired*. Continue for all k in $[n]$.
- (3.5) For all $x \in A$ such that precisely half of the elements of the interval $[y, x]$ are members of A , for some $1 \leq y < x$, let $p_A(x)$ be the maximum such y .

Let $R(A)$ be the set of all x for which $p_A(x)$ is defined, let $L(A) = \{p_A(x) \mid x \in R(A)\}$, and let $P(A) = L(A) \cup R(A)$. Now set

$$f(A) = A \cup \{z\}, \quad z = \min([n] - (A \cup L(A))),$$

if $[n] - (A \cup P(A)) \neq \emptyset$; otherwise $f(A)$ is undefined. Then this rule inverts f :

$$f^{-1}(B) = B - \{z\}, \quad z = \max(B - R(B)).$$

Let $\mathcal{C}(A) = \{f^k(A) \mid k \in \mathbb{Z}\}$. As A runs over all of B_n , the distinct $\mathcal{C}(A)$'s provide the Greene-Kleitman SCD of B_n .

The following hold for all $A \in B_n$.

- (3.6) For all $x \in R(A)$, $[p_A(x), x] \subseteq P(A)$.

(3.7) $\mathcal{C}(A) = \{X \in B_n \mid R(X) = R(A)\}$ and $p_X(a) = p_A(a)$ for all $X \in \mathcal{C}(A)$ and for all $a \in R(A)$.

(3.8) $\min(\mathcal{C}(A)) = R(A)$, $\max(\mathcal{C}(A)) = [n] - L(A)$; in fact, $\mathcal{C}(A)$ is the chain

$$R(A) \subset R(A) \cup \{a_1\} \subset R(A) \cup \{a_1, a_2\} \subset \dots \subset R(A) \cup \{a_1, a_2, \dots, a_t\} = [n] - L(A),$$

where $[n] - (R(A) \cup L(A)) = \{a_1 < a_2 < \dots < a_t\}$.

The following two lemmas provide properties of this SCD that substantiate **Claim 1**. Given a symmetric chain C in B_n and $X \in C$ with $|X| \leq \lfloor n/2 \rfloor$, let X^* be the member of C with $|X^*| = n - |X|$.

Lemma 8. *For $i = 1, 2, \dots, t$ and for all $X \in C_i$ with $|X| \leq \lfloor n/2 \rfloor$, $(\sigma(X))^* = \sigma(X^*)$. Thus $(\sigma^j(X))^* = \sigma^j(X^*)$ for all integers j , so $(\rho(X))^* = \rho(X^*)$ for all $\rho \in H$.*

A special case of the preceding lemma is in [7]. Since this reference is a technical report and the result does not appear to be available in the literature, we prove this below. (We note that for n odd, the middle edges of the Greene-Kleitman SCD are the edges of the well-known lexicographic matching in the middle two levels graph. The invariance of this matching under rotation is a special case of a result of Kierstead and Trotter [15] on what they call i -lexical matchings.)

Here is the second property of the Greene-Kleitman SCD that we require.

Lemma 9. *Let $w \in \{1, 2, \dots, t\}$, and let $A \in C_w$ with $|A| \leq \lfloor n/2 \rfloor$. Suppose that there is some $B \in [A]$ such that $B \in C_j$ for some $j < w$. Then there is some $k < w$ and $D \in C_k$ such that $D \in [f^{-1}(A)]$, provided that $f^{-1}(A)$ is defined.*

To prove **Claim 1** from these facts, define \mathcal{C} inductively.

First, let $i_1 = 1$ and $C'_{i_1} = C_1$. Suppose that $C'_{i_1}, C'_{i_2}, \dots, C'_{i_k}$ are defined. If there exists $i \in \{i_k + 1, \dots, t\}$ such that for some $X \in C_i$,

$$[X] \cap \left(\bigcup_{j=1}^k C'_{i_j} \right) = \emptyset \tag{1}$$

let i_{k+1} be the least such i and let

$$C'_{i_{k+1}} = \left\{ Y \in C_{i_{k+1}} \mid [Y] \cap \left(\bigcup_{j=1}^k C'_{i_j} \right) = \emptyset \right\}. \tag{2}$$

If there is no such i then $m = k$, that is, C'_{i_m} is the last chain required by **Claim 1** and the procedure is complete.

If $Y \in C'_{i_{k+1}}$, with $|Y| \leq \lfloor n/2 \rfloor$ then $Y^* \in C'_{i_{k+1}}$, by the dual of Lemma 8. Also, if $Z \in C_{i_{k+1}}$ and $Y \subseteq Z \subseteq Y^*$ where $Y \in C'_{i_{k+1}}$ then $Z \in C'_{i_{k+1}}$ by Lemma 9 and Lemma 8. Thus, $C'_{i_{k+1}}$ is symmetric in B_n and (3.1) holds. Equation (2) verifies (3.2); (3.3) follows from (1) and (2).

Proof of Lemma 8. The proof is divided into cases depending upon which of $R(X) \subseteq X \subseteq X^*$ contain n . It is not possible that $n \in X - R(X)$, because $|X| \leq \lfloor n/2 \rfloor$ means that for some $y < n$ precisely half the elements of $[y, n]$ are in X , and, hence, $n \in R(X)$ by (3.5). Consequently, there are three cases. In each case, we show that

$$R(\sigma(X^*)) = R(\sigma(X)),$$

apply (3.7) to see that $\sigma(X^*)$ and $(\sigma(X))^*$ are both members of $\mathcal{C}(\sigma(X))$, and conclude that $\sigma(X^*) = (\sigma(X))^*$ since these sets both have cardinality $n - |\sigma(X)|$.

Case 1: $n \notin X^*$

Since $n \in [n] - L(X) = \max(\mathcal{C}(X))$ and $n \notin X^*$, $X \neq \min(\mathcal{C}(X)) = R(X)$. Thus, there exists $y = \min(X - R(X))$. If $y = 1$ then $p_{\sigma(X)}(2) = 1 = p_{\sigma(X^*)}(2)$. For each $z \in R(X)$, $\sigma(z) = z + 1 \in R(\sigma(X))$ and each $z + 1 \in R(\sigma(X))$ has $z \in R(X)$ apart from $z + 1 = 2$. Thus,

$$\begin{aligned} R(\sigma(X)) &= \sigma(R(X) \cup \{2\}) && \text{since } p_{\sigma(X)}(2) = 1, \\ &= \sigma(R(X^*)) \cup \{2\} && \text{by (3.7),} \\ &= R(\sigma(X^*)) && \text{since } p_{\sigma(X^*)}(2) = 1. \end{aligned}$$

If $y > 1$ we claim that $[1, y - 1] \subseteq P(X)$. Note that $y - 1 \in X$ as otherwise $y \in R(X)$ with $p_X(y) = y - 1$, contradicting the choice of y . By the minimality of y , $y - 1 \in R(X)$ and, by (3.7), $[p_X(y - 1), y - 1] \subseteq P(X)$. Continue in the same manner, with $p_X(y - 1) - 1$ in place of y , and thereby verify the claim that $[1, y - 1] \subseteq P(X)$. The argument is just about the same as when $y = 1$ except we use the fact that $[1, y - 1] \subseteq P(X)$ and $1 \notin X$, so, $p_{\sigma(X)}(y + 1) = 1$:

$$R(\sigma(X^*)) = \sigma(R(X^*)) \cup \{y + 1\} = \sigma(R(X)) \cup \{y + 1\} = R(\sigma(X)).$$

Case 2: $n \in R(X)$

Every element of $R(\sigma(X))$ is in $\sigma(R(X))$ and every element of $\sigma(R(X))$, except for 1, is in $R(\sigma(X))$. Thus,

$$R(\sigma(X)) = \sigma(R(X)) - \{1\} = \sigma(R(X^*)) - \{1\} = R(\sigma(X^*)).$$

Case 3: $n \in X^* - X$

Since $n \in X^* - X$, (3.8) shows that $X^* = \max(\mathcal{C}(X)) = [n] - L(X)$ and, thus, $X = \min(\mathcal{C}(X)) = R(X)$.

If $z + 1 \in R(\sigma(X))$ then $z \in X = R(X)$, so $R(\sigma(X)) \subseteq \sigma(R(X))$. Conversely, $1 \notin \sigma(R(X))$, and any $z + 1 \in \sigma(R(X))$ is obviously a member of $R(\sigma(X))$. Thus, $R(\sigma(X)) = \sigma(R(X))$. Similarly, since $n \notin R(X)$, it follows that $R(\sigma(X^*)) = \sigma(R(X^*))$. Hence, $R(\sigma(X^*)) = R(\sigma(X))$.

Since $(\sigma(X))^* = \sigma(X^*)$ for all X with $|X| \leq \lfloor n/2 \rfloor$, we can apply induction on j to conclude that $(\sigma^j(X))^* = \sigma^j(X^*)$:

$$(\sigma^j(X))^* = (\sigma(\sigma^{j-1}(X)))^* = \sigma((\sigma^{j-1}(X))^*) = \sigma(\sigma^{j-1}(X^*)) = \sigma^j(X^*). \quad \square$$

Proof of Lemma 9. As before, let C_1, C_2, \dots, C_t , where $t = \binom{n}{\lfloor n/2 \rfloor}$, be the symmetric chains in the Greene-Kleitman decomposition, ordered by decreasing length, and let $\sigma = (1\ 2\ \dots\ n)$. Let $A \in C_w$ with $|A| \leq \lfloor \frac{n}{2} \rfloor$. Suppose that there exists a $j < w$ such that $B \in C_j$ and $B \in [A]$. Hence there is an integer r such that $B = \sigma^r(A)$.

Assume that $f^{-1}(A)$ is defined. We show that there is a $k < w$ such that $D \in C_k$ and $D \in [f^{-1}(A)]$. Since $f^{-1}(A)$ is defined, $A - R(A) \neq \emptyset$. Let $y = \max(A - R(A))$. We may assume that $-(y - 1) \leq r \leq n - y$, $r \neq 0$. We consider two cases:

Case 1: $y + r \in R(B)$

Then $r > 0$ since otherwise y would also be paired in A , contrary to its choice. Each $z \in B$ with $y + r < z$ must be in $R(B)$ since $y < z - r$ so, by the choice of y , $z - r \in R(A)$. Now consider the binary sequence $\sigma^r(f^{-1}(A))$ contained in some chain C_k . Recall that $f^{-1}(A) = A - \{y\}$ and note that $\sigma^r(f^{-1}(A)) = B - \{y + r\}$. It follows from this that $\sigma^r(f^{-1}(A))$ must have one fewer pairs than B , since $y + r$ will be unpaired in $\sigma^r(f^{-1}(A))$ while $y + r$ is paired in B , and there are no other differences in the pairings. By (3.8), $|C_k| > |C_j|$, so $k < j < w$, as desired.

Case 2: $y + r \in B - R(B)$

If $y + r = \max(B - R(B))$ then we are done, since then we have $f^{-1}(B) = \sigma^r(f^{-1}(A))$. So suppose instead $z = \max(B - R(B))$ where $y + r < z$. If $r > 0$ then $z - r \in (A - R(A))$, contrary to the choice of y . Thus $r < 0$. If $z - r \leq n$ then $z - r$ would be an unpaired element of A , since $y < z - r$ remains unpaired in A . This would contradict the choice of y . Thus $n < z - r$.

We now prove that for some p , $\sigma^p(B)$ is the maximum element of its chain, and its chain is not a singleton. This will contradict the fact that $|A| \leq \lfloor \frac{n}{2} \rfloor$, since $|\sigma^p(B)| = |A|$.

Since $\sigma^{-r}(B) = A$, $-r > 0$, we obtain A from B by applying σ^{-r} times. Since $n < z - r$ there is some p such that $\sigma^p(z) = n$. Let $X = \sigma^p(B)$. Then, $\sigma^p(z) \in X - R(X)$. Because $n \in X - R(X)$, $X = [n] - L(X)$. By (3.8), X is the maximum element of its chain. The chain containing X is not a singleton, since $X - R(X) \neq \emptyset$. \square

4 The proof of Theorem 5

Let $\rho = (i_1 j_1)(i_2 j_2) \cdots (i_k j_k)$, where the transpositions are pairwise disjoint, let $X = \bigcup_{r=1}^k \{i_r, j_r\}$, and let $G = \{1, \rho\}$. Then

$$B_n/G \cong B(X)/G \times B([n] - X)$$

via the mapping $[A] \mapsto ([A \cap X], A - X)$ for all $A \subseteq [n]$. By Lemma 3, we may assume that n is even and that $n = 2k$. Using the remark about conjugation at the beginning of Section 3, we may assume that $\rho = (1\ 2k)(2\ 2k - 1) \cdots (k\ k + 1)$. As noted in the introduction, G can be generated by a power of a $2k$ -cycle, so Theorem 1 applies. (In fact, $\rho = \tau^{-1} \sigma \tau$ where $\sigma = (1\ 2 \cdots 2k)$ and $\tau = (k + 1\ 2k)(k + 2\ 2k - 1) \cdots$.) And, as we shall see, its proof method can be adapted to give the proof we offer here. However, the argument below might help with the most interesting open case, namely, showing that B_n/D_{2n} is an SCO for the dihedral group D_{2n} .

Regard each $A \in B_{2k}$ as a concatenated pair of binary strings of length k . That is, $A = \mathbf{b}_1\mathbf{b}_2^r$, where $\mathbf{b}_1, \mathbf{b}_2 \in \{0, 1\}^k$ and \mathbf{b}^r is the reverse of the binary k -sequence \mathbf{b} . Then the equivalence classes in B_n/G are the sets $\{\mathbf{b}_1\mathbf{b}_2^r, \mathbf{b}_2\mathbf{b}_1^r\}$; these sets have 2 elements except in the case that $\mathbf{b}_1 = \mathbf{b}_2$.

Let C_1, C_2, \dots, C_t , where $t = \binom{k}{\lfloor k/2 \rfloor}$, be any symmetric chain decomposition of B_k , ordered by decreasing length. We define a total ordering \preceq on $B_k = \{0, 1\}^k$ as follows:

$$\mathbf{b}_r \preceq \mathbf{b}_s \text{ if } \mathbf{b}_r \in C_i, \mathbf{b}_s \in C_j, i < j, \text{ or if } \mathbf{b}_r \subseteq \mathbf{b}_s \text{ in } C_i \text{ for some } i. \quad (3)$$

For $1 \leq i < j \leq t$, let $P_{ij} = C_i \times C_j$, with the coordinate-wise ordering induced by the containment order on B_k , for each $i = 1, 2, \dots, t$, let

$$P_{ii} = \{(\mathbf{b}_r, \mathbf{b}_s) \in C_i \times C_i \mid \mathbf{b}_r \subseteq \mathbf{b}_s\},$$

ordered coordinate-wise, and let

$$P = \bigcup_{1 \leq i \leq j \leq t} P_{ij},$$

again, ordered coordinate-wise. Thus, P is a subset of B_{2k} with the exactly the ordering inherited from the Boolean lattice.

In fact, with r_P and r_{B_k} as the rank functions in P and B_k , respectively, then for $1 \leq i \leq j \leq t$ and with $r_{B_k}(\min C_i) = r_i$, and $r_{B_k}(\max C_i) = k - r_i$,

$$r_P(\min P_{ij}) = r_i + r_j, \quad r_P(\max P_{ij}) = 2k - (r_i + r_j), \quad \text{and } l(P_{ij}) = 2k - 2(r_i + r_j).$$

We see that each P_{ij} is a symmetric subset of B_{2k} in which the covering relation is preserved, that is, $(\mathbf{b}_p, \mathbf{b}_q)$ is covered by $(\mathbf{b}_u, \mathbf{b}_v)$ in some P_{ij} if and only if $\mathbf{b}_p\mathbf{b}_q^r$ is covered by $\mathbf{b}_u\mathbf{b}_v^r$ in B_{2k} .

Consider the map ϕ of P to B_{2k}/G defined by $\phi((\mathbf{b}_1, \mathbf{b}_2)) = \{\mathbf{b}_1\mathbf{b}_2^r, \mathbf{b}_2\mathbf{b}_1^r\}$. Since $\mathbf{b}_1 \preceq \mathbf{b}_2$ for all $(\mathbf{b}_1, \mathbf{b}_2) \in P$, ϕ is injective. It is obviously a surjection. It is also order-preserving: if $(\mathbf{b}_p, \mathbf{b}_q) \leq (\mathbf{b}_u, \mathbf{b}_v)$ in P then $\mathbf{b}_p\mathbf{b}_q^r \leq \mathbf{b}_u\mathbf{b}_v^r$ in B_{2k} .

Since the rank of an equivalence class in B_{2k}/G is the rank of its members in B_{2k} , it follows that a symmetric chain in P is a symmetric chain in B_{2k}/G . Thus, it is enough to proof the following.

Claim 2: P has a symmetric chain decomposition.

Since P is partitioned by P_{ij} , $1 \leq i \leq j \leq t$, each of which preserve the covering relation in P , it is enough to prove that each P_{ij} has a partition into chains, each of which is symmetric in P .

For $1 \leq i < j \leq t$, $P_{ij} = C_i \times C_j$ is a cover-preserving subset of P , with minimum element at level $r_i + r_j$ and maximum element at level $2k - (r_i + r_l)$ in P , a partially ordered set of length $2k$. Then the ‘‘standard’’ symmetric partition of a product of two chains (the original partition in [2]) provides symmetric chains in P .

For $i = 1, 2, \dots, t$, $P_{ii} = \{(\mathbf{b}_r, \mathbf{b}_s) \in C_i \times C_i \mid \mathbf{b}_r \subseteq \mathbf{b}_s\}$, where C_i is the chain of binary strings $\mathbf{b}_{r_i} \subset \mathbf{b}_{r_i+1} \subset \dots \subset \mathbf{b}_{k-r_i}$, where $r_{B_k}(\mathbf{b}_s) = s$ in B_k , that is, is an s -element set,

for $s = r_i, r_i + 1, \dots, k - r_i$. Then P_{ii} is an interval in P with minimum element at level $2r_i$ and maximum element at level $2k - 2r_i$ in P . Also,

$$(\mathbf{b}_{r_i}, \mathbf{b}_{r_i}) < (\mathbf{b}_{r_i}, \mathbf{b}_{r_i+1}) < \dots < (\mathbf{b}_{r_i}, \mathbf{b}_{k-r_i}) < (\mathbf{b}_{r_i+1}, \mathbf{b}_{k-r_i}) < \dots < (\mathbf{b}_{k-r_i}, \mathbf{b}_{k-r_i})$$

is a symmetric chain in P and $P_{ii} - C$ is a cover-preserving subset of P , isomorphic to the product of two chains, with minimum element $(\mathbf{b}_{r_i+1}, \mathbf{b}_{r_i+1})$ at level $2r_i + 2$ and maximum element $(\mathbf{b}_{k-r_i-1}, \mathbf{b}_{k-r_i-1})$ at level $2k - 2r_i - 2$. By induction, we have a decomposition of P_{ii} by chains symmetric in P . This verifies **Claim 2** and completes the proof of Theorem 5.

5 The proof of Theorem 6

With Lemmas 3 and 4, it is enough to prove the result for K generated by a single m -cycle. We assume that C is the k -element chain $00\dots 0, 10\dots 0, 11\dots 0, 111\dots 1$ in the Boolean lattice B_{k-1} . Let $n = (k - 1)m$. Then C^m is the sublattice of B_n consisting of all binary sequences of length n of the form

$$\mathbf{b} = \mathbf{b}_1 \mathbf{b}_2 \dots \mathbf{b}_m, \text{ where each } \mathbf{b}_i \in C.$$

That is, the elements of C^m are exactly those n -sequences which are m $(k - 1)$ -sequences, each comprised of 1's followed by 0's.

Let C_1, C_2, \dots, C_t , where $t = \binom{n}{\lfloor n/2 \rfloor}$, be the symmetric chains in the Greene-Kleitman SCD of B_n , ordered by decreasing length, as in Section 3. We claim that for each j , $C_j \subseteq C^m$ or $C_j \cap C^m = \emptyset$.

Suppose that $\mathbf{b} = \mathbf{b}_1 \mathbf{b}_2 \dots \mathbf{b}_m \in C_j \cap C^m$. With the notation in (3.6) - (3.8), and applying these to $A = \mathbf{b}$, we can see that i^{th} entry b_i of \mathbf{b} is determined as follows:

$$b_i = \begin{cases} 0 & i \in L(\mathbf{b}) \cup \{a_r, a_{r+1}, \dots, a_t\} \\ 1 & i \in R(\mathbf{b}) \cup \{a_1, a_2, \dots, a_{r-1}\}. \end{cases}$$

for some r .

If \mathbf{b} is not the maximum element of C_j then its successor \mathbf{b}' is obtained by changing the 0 in position a_r to a 1. Either $a_r = 1$ or the entry in \mathbf{b} in position $a_r - 1$ is a 1, by (3.6). Thus, \mathbf{b}' consists of m $(k - 1)$ -sequences of 1's followed by 0's and, so, belongs to $C_j \cap C^m$. If \mathbf{b} is not the minimum element of C_j then its predecessor \mathbf{b}'' is obtained by changing the 1 in position a_{r-1} to a 0. Either $a_{r-1} = n$ or the entry in \mathbf{b} in position $a_{r-1} + 1$ is a 0, by (3.6). Again \mathbf{b}'' consists of m $(k - 1)$ -sequences of 1's followed by 0's and, so, belongs to $C_j \cap C^m$. Hence, if $C_j \cap C^m \neq \emptyset$ then $C_j \subseteq C^m$.

Let $K = \langle \phi^r \rangle$ where we may assume that $\phi = (12 \dots m)$. We need an SCD for C^m/K . We know that C^m is a sublattice of B_n , as noted above, and that $\phi^r = \sigma^{(k-1)r}|_{C^m}$ where $\sigma = (12 \dots n) \in S_n$. As in the proof of Lemma 2, **Claim 1** gives an SCD $\widehat{C}_j = C'_{i_j}$, $j = 1, 2, \dots, m$, of B_n/H where $H = \langle \sigma^{(k-1)r} \rangle$. Thus, the subfamily of those \widehat{C}_j , $j = 1, 2, \dots, m$, such that $\widehat{C}_j \subseteq C^m$ is an SCD for C^m/K .

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