

Statistics of blocks in k -divisible non-crossing partitions

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Abstract

We derive a formula for the expected number of blocks of a given size from a non-crossing partition chosen uniformly at random. Moreover, we refine this result subject to the restriction of having a number of blocks given. Furthermore, we generalize to k -divisible partitions. In particular, we find that, asymptotically, the expected number of blocks of size t of a k -divisible non-crossing partition of nk elements chosen uniformly at random is $\frac{kn+1}{(k+1)^{t+1}}$. Similar results are obtained for type B and type D k -divisible non-crossing partitions of Armstrong.

Introduction

In this paper we study some statistics of the block structure of non-crossing partitions. A first systematic study of non-crossing partitions was done by G. Kreweras [15]. More recently, much more attention has been paid to non-crossing partitions because, among other reasons, they play a central role in the combinatorial approach of Speicher to Voiculescu's free probability theory [20]. For an introduction to this combinatorial approach, see [16].

A non-crossing partition of $\{1, \dots, kn\}$ is called k -divisible if the size of each block is divisible by k . The poset of k -divisible non-crossing partitions was introduced by Edelman [10] and reduces to the poset of all non-crossing partitions for $k = 1$. We denote these posets by $NC^k(n)$ and $NC(n)$, respectively.

As can be seen in [1] and [2], k -divisible non-crossing partitions play an important role in the calculation of the free cumulants and moments of products of k free random variables. Moreover, in the approach given in [2] for studying asymptotic behavior of the size of the support, when $k \rightarrow \infty$, understanding the asymptotic behavior of the sizes of blocks was a crucial step.

In this direction, a recent paper by Ortmann [17] studies the asymptotic behavior of the sizes of the blocks of a uniformly chosen random partition. This lead him to a formula for

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the right-edge of the support of a measure in terms of the free cumulants, when these are positive. He noticed a very simple picture of this statistic as $n \rightarrow \infty$. Roughly speaking, in average, out of the $\frac{n+1}{2}$ blocks of this random partition, half of them are singletons, one fourth of the blocks are pairings, one eighth of the blocks have size 3, and so on.

Trying to get a better understanding of this asymptotic behavior, the question of the exact calculation of this statistic arose. In this paper, we answer this question and refine these results by considering the number of blocks given. Moreover, we generalize to k -divisible partitions, as follows.

Theorem 1. *The sum of the number of blocks of size tk over all the k -divisible non-crossing partitions of $\{1, 2, \dots, kn\}$ is given by*

$$\binom{n(k+1) - t - 1}{nk - 1}. \quad (1)$$

In particular, asymptotically, we have a similar phenomena as for the case $k = 1$; about a $\frac{k}{k+1}$ portion of all the blocks have size k , then a $\frac{k}{(k+1)^2}$ portion have size $2k$, then $\frac{k}{(k+1)^3}$ are of size $3k$, etc.

There is a natural way to identify a non-crossing partition of $1, 2, \dots, n$ with an intersection of reflecting hyperplanes of the Coxeter group A_{n-1} . More generally, for any Coxeter Group W , Bessis [7] and Brady and Watt [8] defined the poset $NC(W)$ of non-crossing partitions for a reflexion group, making $NC(A_{n-1})$ isomorphic to $NC(n)$. Furthermore, Armstrong [5] defined the poset $NC^k(W)$ of k -divisible non-crossing partitions for a reflexion group. Many enumerative results are now known for this noncrossing partitions for reflection groups (see e.g. [4, 12, 13, 14, 19]).

However, analogous results to Theorem 1 have not been studied. We address this problem for the types B and D .

Theorem 2. (1) *The sum of the number of non-zero pairs of blocks $\{-V, V\}$ of size tk over all the non-crossing partitions in $NC_B^k(n)$ is given by*

$$nk \binom{n(k+1) - t - 1}{nk - 1}.$$

(2) *The number of the non-crossing partitions in $NC(B_n)$ with a zero-block of size $2t$ is*

$$\binom{n(k+1) - t - 1}{nk - 1}.$$

While for k -divisible partitions type D we get the following.

Theorem 3. *The sum of the number of pairs of blocks $\{-V, V\}$ of size tk over all the non-crossing partitions in $NC_D^k(n)$ is given by*

$$(k(n-1) + 1) \binom{(k+1)(n-1) - t}{k(n-1) - 1} + k(n-1) \binom{(k+1)(n-1) - t - 1}{k(n-1) - 2}.$$

Theorems 2 and 3 show that, asymptotically, the behavior of sizes is the same for type B and type D non-crossing partitions as for the classical non-crossing partitions (or type A). Again, a $\frac{k}{k+1}$ portion of all the blocks have size k , then a $\frac{k}{k+1}$ portion of the remaining blocks are of size $2k$, etc.

Another consequence is that the expected number of blocks of a k -divisible non-crossing partition is given by $\frac{kn+1}{k+1}$. An equivalent formulation of this result was also observed by Armstrong [5, Theorem 3.9] for any Coxeter group. It is then a natural question if this simple formula can be derived in a bijective way. We end with a bijective proof of this fact, for type A and B k -divisible non-crossing partitions.

Let us finally mention that there exists a type B free probability. Free probability of type B was introduced by Biane, Goodman and Nica [9] and was later developed by Belinschi and Shlyakhtenko [6], Nica and Février [11] and Popa [18].

Apart from this introduction, the paper is organized as follows. In Section 1 we give basic definitions and collect known results on the enumeration of classical non-crossing partitions and non-crossing partitions of type B and D . The main new enumerative results concerning the numbers of blocks of a given size for a non-crossing partition of type A , B and D chosen at random are given in Sections 2, 3 and 4, respectively. Finally, in Section 5 we give a bijective proof of the fact that in average the number of blocks is given by $\frac{kn+1}{k+1}$ and some further consequences of this bijection.

1 Preliminaries

Classical non-crossing partitions

Definition 1.1. (1) We call $\pi = \{V_1, \dots, V_r\}$ a **partition** of the set $[n] := \{1, 2, \dots, n\}$ if and only if V_i ($1 \leq i \leq r$) are pairwise disjoint, non-void subsets of S , such that $V_1 \cup V_2 \cdots \cup V_r = \{1, 2, \dots, n\}$. We call V_1, V_2, \dots, V_r the **blocks** of π . The number of blocks of π is denoted by $|\pi|$.

(2) A partition $\pi = \{V_1, \dots, V_r\}$ is called **non-crossing** if for all $1 \leq a < b < c < d \leq n$ if $a, c \in V_i$ then there is no other subset V_j with $j \neq i$ containing b and d .

(3) We say that a partition π is **k -divisible** if the size of all the blocks is a multiple of k . If all the blocks are exactly of size k we say that π is **k -equal**.

We denote the set of non-crossing partitions of $[n]$ by $NC(n)$, the set of k -divisible non-crossing partitions of $[kn]$ by $NC^k(n)$ and the set of k -equal non-crossing partitions of $[kn]$ by $NC_k(n)$. $NC^k(n)$ is a poset (see Remark 1.2 below) and was first studied by Edelman [10], who calculated many of its enumerative invariants.

Remark 1.2. (1) $NC^k(n)$ can be equipped with the partial order \preceq of reverse refinement ($\pi \preceq \sigma$ if and only if every block of π is completely contained in a block of σ).

(2) For a given $\pi \in NC(n) \cong NC(\{1, 3, \dots, 2n - 1\})$ we define its Kreweras complement

$$Kr(\pi) := \max\{\sigma \in NC(2, 4, \dots, 2n) : \pi \cup \sigma \in NC(2n)\}.$$

The map $Kr : NC(n) \rightarrow NC(n)$ is an order reversing isomorphism. Furthermore, for all $\pi \in NC(n)$ we have that $|\pi| + |Kr(\pi)| = n + 1$, see [16] for details.

There is a graphical representation of a partition $\pi \in NC(n)$ which makes clear the property of being crossing or non-crossing, usually called the circular representation. We think of $[n]$ as labelling the vertices of a regular n -gon, clockwise. If we identify each block of π with the convex hull of its corresponding vertices, then we see that π is non-crossing precisely when its blocks are pairwise disjoint (that is, they don't cross).

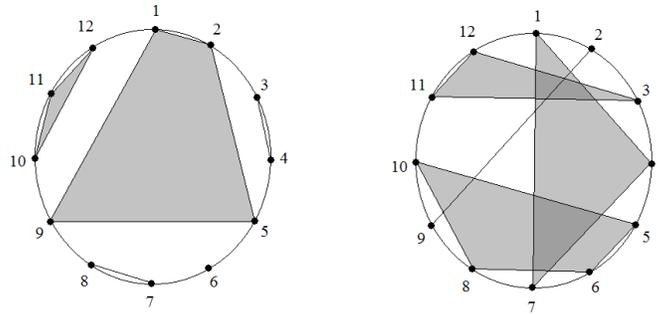


Figure 1: Crossing and Non-Crossing Partitions

Fig. 1 shows the non-crossing partition $\{\{1, 2, 5, 9\}, \{3, 4\}, \{6\}, \{7, 8\}, \{10, 11, 12\}\}$ of the set $[12]$, and the crossing partition $\{\{1, 4, 7\}, \{2, 9\}, \{3, 11, 12\}, \{5, 6, 8, 10\}\}$ of $[12]$ in their circular representation.

Remark 1.3. *The following characterization of non-crossing partitions is sometimes useful: for any $\pi \in NC(n)$, one can always find a block $V = \{r+1, \dots, r+s\}$ containing consecutive numbers. If one removes this block from π , the partition $\pi \setminus V \in NC(n-s)$ remains non-crossing.*

We recall the following result which gives a formula for the number of partitions with a given type [15].

Proposition 1.4. *Let r_1, r_2, \dots, r_n be nonnegative integers such that $r_1 + 2r_2 + \dots + nr_n = n$. Then the number of partitions of π in $NC(n)$ with r_1 blocks of size 1, r_2 blocks of size 2, \dots , r_n blocks of size n equals*

$$\frac{n!}{p_r(n-m+1)!}, \tag{2}$$

where $p_r = r_1!r_2! \cdots r_n!$ and $r_1 + r_2 + \dots + r_n = m$.

It is well known that the number of non-crossing partition is given by the Catalan numbers $\frac{1}{n+1} \binom{2n}{n}$. More generally, for k -divisible non-crossing partitions we have the following [10].

Proposition 1.5. *Let $NC^k(n)$ be the set of non-crossing partitions of $[nk]$ whose sizes of blocks are multiples of k . Then*

$$\#NC^k(n) = \frac{\binom{(k+1)n}{n}}{kn + 1}.$$

On the other hand, from Proposition 1.4, we can easily count k -equal partitions.

Corollary 1.6. *Let $NC_k(n)$ be the set of non-crossing partitions of nk whose blocks are all of size of k . Then*

$$\#NC_k(n) = \frac{\binom{kn}{n}}{(k-1)n + 1}.$$

The reader may have noticed from Proposition 1.5 and Corollary 1.6 that the number of $(k+1)$ -equal non-crossing partitions of $[n(k+1)]$ and the number of k -divisible non-crossing partitions of $[nk]$ coincide. We derive a bijective proof of this fact and study further consequences in Section 5.

Non-crossing partitions for Coxeter groups of classical types

For any Coxeter Group W , Bessis [7] and Brady and Watt [8] defined the poset of non-crossing partition for the reflexion group $NC(W)$.

For the three infinite families of finite irreducible Coxeter groups, A_{n-1} (the symmetric group), B_n (the hyperoctahedral group) and D_n (an index 2 subgroup of B_n) known as classical groups, there exists combinatorial realizations of $NC(W)$, which we will use as a definition of noncrossing partitions of W . While $NC(A_{n-1})$ is isomorphic to $NC(n)$, the combinatorial realizations of the lattices $NC(B_n)$ and $NC(D_n)$, that we explain in this section, were done by Reiner [19] and Reiner and Athanasiadis[4], respectively.

Furthermore, Armstrong [5] defined the poset $NC^k(W)$ of k -divisible non-crossing partitions for the reflexion group W , making $NC^k(A_{n-1})$ isomorphic to $NC^k(n)$. The construction of Reiner can be easily be generalized to $NC^k(B_n)$. However, a combinatorial realization of $NC^k(D_n)$ was not known, until the recent paper by Krattenthaler and Müller [13].

We use these combinatorial realizations to define non-crossing partitions for the Coxeter groups B_n and D_n .

Let us start with the definition of type B_n partitions.

Definition 1.7. *Let $[\pm n] := \{1, 2, \dots, n, -1, -2, \dots, -n\}$. A non-crossing partition of type B_n is a non-crossing partition π of $[\pm n]$ such that for each block V of π the block $-V = \{-x | x \in V\}$ is also a block of π , and there is at most one block, called the **zero-block**, which satisfies $V = -V$. We denote by $NC_B(n)$ the set of non-crossing partitions of type B_n .*

There is a circular representation for non-crossing partitions of type B_n obtained as follows. Arrange $2n$ vertices on a circle and label clockwise with the integers $1, 2, \dots, n, -1, -2, \dots, -n$. For each block V of π , draw the convex hull of the vertices whose labels are the integers in B . Then π is a non-crossing partition of type B_n if the convex hulls do not intersect. Notice that the circular representation of the non-crossing partitions of type B_n correspond to the circular representation partitions in $NC(2n)$ which are centrally symmetric.

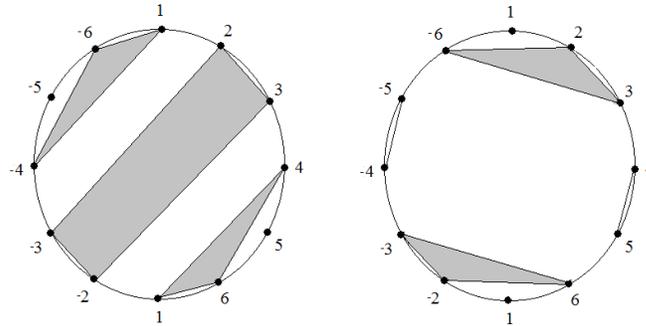


Figure 2: Type B partitions

The formula for the number of partitions of type B_n with the sizes of blocks given was obtained by Athanasiadis [3, Theorem 2.3].

Proposition 1.8. *Let s, r_1, r_2, \dots, r_n be nonnegative integers such that $s + r_1 + 2r_2 + \dots + nr_n = n$. Then the number of partitions of π in $NC_B(n)$ with r_i non-zero pairs of blocks of size i and with a zero-block of size $2s$, is given by*

$$\frac{n!}{p_r(n - m)!} \quad (3)$$

where $p_r = r_1!r_2! \cdots r_n!$ and $r_1 + r_2 + \dots + r_n = m$.

We define Type D partitions via their circular representation. The type D circular representation of a partition $\pi \in NC_B(n)$ is the drawing obtained as follows. Arrange $2n - 2$ vertices labeled with $1, 2, \dots, n - 1, -1, -2, \dots, -(n - 1)$ on a circle and put a vertex labeled with $\pm n$ at the center. For each block V of π , draw the convex hull of the vertices whose labels are in B .

Definition 1.9. *A partition $\pi \in NC_B(n)$ is a non-crossing partition of type D_n if the convex hulls of its type D_n circular representation do not intersect in their interiors and if there is a zero-block V then $\{n, -n\} \in V$. We denote by $NC_D(n)$ the set of non-crossing partitions of type D_n .*

The formula for the number of partitions of type D with the sizes of the blocks given was obtained by Athanasiadis and Reiner [4, Theorem 1.3].

Proposition 1.10. Let s, r_1, r_2, \dots, r_n be nonnegative integers such that $s + r_1 + 2r_2 + \dots + nr_n = n$. Then the number of partitions of π in $NC_D(n)$ with r_i non-zero pairs of blocks of size i and with a zero-block of size $2s$, is given by

$$\frac{(n-1)!}{p_r(n-m-1)!} \quad \text{if } s \geq 2 \quad (4)$$

$$2 \frac{(n-1)!}{p_r((n-1)-m)!} + r_1 \frac{(n-1)!}{p_r(n-m)!} \quad \text{if } s = 0 \quad (5)$$

where $p_r = r_1!r_2! \dots r_n!$ and $r_1 + r_2 + \dots + r_n = m$.

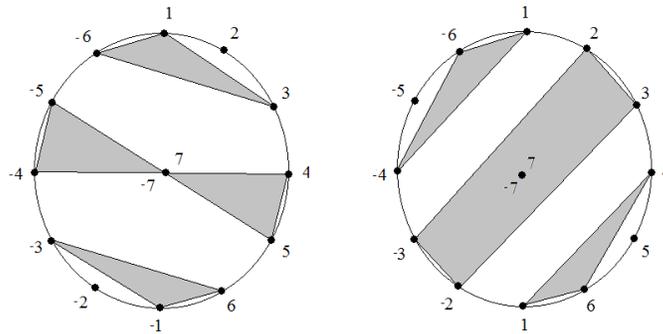


Figure 3: Type D partitions

Some basic combinatorial identities

For the convenience of the reader, we end this section by recalling some combinatorial identities that we use often in Sections 2, 3 and 4.

The following two summation lemmas will enable us to use Propositions 1.4, 1.8 and 1.10 to get the number of blocks of size t subject to the restriction of having a fixed number m of blocks.

Lemma 1.11. *The following identity holds*

$$\sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_1+2r_2+\dots+nr_n=n}} \frac{m!}{r_1! \dots r_n!} = \binom{n-1}{m-1}. \quad (6)$$

Proof. Writing $s_i = r_{i+1}$ and using that $r_1 = m - s_1 - s_2 - \dots - s_{n-1}$. We see that

$$\sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_1+2r_2+\dots+nr_n=n}} \frac{m!}{r_1! \dots r_n!} = \sum_{s_1+2s_2+\dots+(n-1)s_{n-1}=n-m} \binom{m}{s_2, \dots, s_{n-1}},$$

where the multinomial coefficient is defined by

$$\begin{aligned} \binom{m}{s_1, \dots, s_{n-m}} &= \frac{m!}{s_1! \cdots s_{n-m}! (m - s_1 - s_2 - \cdots - s_{n-m})!} \\ &= \binom{m}{s_1} \binom{m - s_1}{s_2} \cdots \binom{m - (s_1 + \cdots + s_{n-1})}{s_n}. \end{aligned}$$

The condition in the sum implies that $s_i = 0$, for $i > n - m$, which in turn implies that

$$\sum_{s_1+2s_2+\cdots+(n-1)s_{n-1}=n-m} \binom{m}{s_2, \dots, s_{n-1}} = \sum_{s_1+2s_2+\cdots+(n-m)s_{n-m}=n-m} \binom{m}{s_2, \dots, s_{n-m}}.$$

The desired identity now follows directly by comparing coefficients of z^{n-m} in both sides of the identity

$$(1 + z + z^2 + z^3 + \cdots)^m = (1 - z)^{-m}$$

□

Lemma 1.12. *The following identity holds*

$$\sum_{\substack{r_1+r_2+\cdots+r_n=m \\ r_1+2r_2+\cdots+nr_n=n}} \frac{(m-1)!r_t}{r_1! \cdots r_n!} = \binom{n-t-1}{m-2}. \quad (7)$$

Proof. We make the change of variable $\tilde{r}_t = r_t - 1$ and $\tilde{r}_i = r_i$ for $i \neq t$. Then

$$\begin{aligned} \sum_{\substack{r_1+r_2+\cdots+r_n=m \\ r_1+2r_2+\cdots+nr_n=n}} \frac{(m-1)!r_t}{r_1! \cdots r_n!} &= \sum_{\substack{\tilde{r}_1+\tilde{r}_2+\cdots+\tilde{r}_n=m-1 \\ \tilde{r}_1+2\tilde{r}_2+\cdots+(n-t)\tilde{r}_{n-t}=n-t}} \frac{(m-1)!}{\tilde{r}_1! \cdots \tilde{r}_n!} \\ &= \binom{n-t-1}{m-2}, \end{aligned}$$

where we used the Lemma 1.11 in the last equality.

□

Finally, we remind the so-called Chu-Vandermonde's identity which will enable us to remove the restriction of having a number of blocks given.

$$\sum_{m=0}^s \binom{y}{m} \binom{x}{s-m} = \binom{x+y}{s}. \quad (8)$$

2 Number of blocks in k -divisible non-crossing partitions

First we calculate the expected number of blocks of a given size t , subject to the restriction of having m blocks from which the main result will follow.

Proposition 2.1. *The sum of the number of blocks of size tk of all non-crossing partitions in $NC^k(n)$ with m blocks*

$$\binom{nk}{m-1} \binom{n-t-1}{m-2}. \quad (9)$$

Proof. First we treat the case $k = 1$. In order to count the number of blocks of size t of a given partition π with r_1 blocks with size 1, r_2 blocks of size 2, \dots , r_n blocks of size n , we need to multiply by r_t . So we want to calculate the following sum

$$\begin{aligned} \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_1+2r_2+\dots+nr_n=n}} \frac{n!r_t}{(n+1-m)!p_r} &= \binom{n}{m-1} \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_1+2r_2+\dots+nr_n=n}} \frac{(m-1)!r_t}{p_r} \\ &= \binom{n}{m-1} \binom{n-t-1}{m-2}. \end{aligned}$$

We used Lemma 1.12 in the last equality. This solves the case $k = 1$.

For the general case we follow the same strategy. In this case we need (r_1, \dots, r_n) such that $r_i = 0$ if k does not divide i . So the condition $r_1 + r_2 + \dots + r_n = m$ is really $r_k + r_{2k} + \dots + r_{nk} = m$ and the condition $r_1 + 2r_2 + \dots + nr_n = nk$ is really $kr_k + 2kr_{2k} + \dots + (nk)r_{nk} = nk$, or equivalently $r_k + 2r_{2k} + \dots + nr_{nk} = n$. Making the change of variable $r_{ik} = s_i$ we get.

$$\sum_{\substack{r_k+r_{2k}+\dots+r_{nk}=m \\ kr_k+2kr_{2k}+\dots+(nk)r_{nk}=nk}} \frac{(nk)!r_{tk}}{(nk+1-m)!r_k!r_{2k}!\dots r_{nk}!} = \sum_{\substack{s_1+s_2+\dots+s_n=m \\ s_1+2s_2+\dots+ns_n=n}} \frac{(nk)!s_t}{(nk+1-m)! \prod_{i=0}^n s_i!}. \quad (10)$$

Now, the last sum can be treated exactly as for the case $k = 1$, yielding the result. This reduction to the case $k = 1$ will be obviated for types B and D . \square

Now we can prove Theorem 1, which we state again for the convenience of the reader.

Theorem 1. *The sum of the number of blocks of size tk over all the k -divisible non-crossing partitions of $\{1, \dots, kn\}$ is given by*

$$\binom{n(k+1)-t-1}{nk-1}. \quad (11)$$

Proof. We use Proposition 2.1 and sum over all possible number of blocks. Letting $\tilde{m} = m-1$, we get

$$\sum_{m=1}^{nk} \binom{nk}{m-1} \binom{n-t-1}{m-2} = \sum_{\tilde{m}=0}^{nk-1} \binom{nk}{nk-\tilde{m}} \binom{n-t-1}{\tilde{m}-1}.$$

Now, using the Chu-Vandermonde's identity for $s = nk - 1$, $x = n(k + 1) - t - 1$ and $y = nk - 1$ we obtain the result. \square

Corollary 2.2. *The expected number of blocks of size tk of a non-crossing partition chosen uniformly at random in $NC^k(n)$ is given by*

$$\frac{(nk + 1) \binom{n(k+1)-t-1}{nk-1}}{\binom{(k+1)n}{n}}. \quad (12)$$

Moreover, similar to the case $k = 1$, asymptotically the picture is very simple, about a $\frac{k}{k+1}$ portion of all the blocks have size k , then $\frac{k}{k+1}$ of the remaining blocks are of size $2k$, and so on. This is easily seen using (12).

Corollary 2.3. *When $n \rightarrow \infty$ the expected number of blocks of size tk of a non-crossing partition chosen uniformly at random in $NC^k(n)$ is asymptotically $\frac{nk}{(k+1)^{t+1}}$.*

The following is a direct consequence of Theorem 1.

Corollary 2.4. *The sum of the number of blocks of all the k -divisible non-crossing partitions in $NC^k(n)$ is*

$$\binom{n(k+1) - 1}{nk}.$$

Proof. Summing over t , in (11), we easily get the result. \square

Finally, from Corollary 2.4 one can calculate the expected number of block of k -divisible non-crossing partition.

Corollary 2.5. *The expected number of blocks of a k -divisible partition of $[kn]$ chosen uniformly at random is given by $\frac{kn+1}{k+1}$.*

Remark 2.6. 1) *Corollary 2.5 was proved by Armstrong [5] for any Coxeter group.*

2) *When $k = 1$ there is a nice proof of Corollary 2.5. Recall from Remark 1.2 that the Kreweras complement $Kr : NC(n) \rightarrow NC(n)$ is a bijection such that $|Kr(\pi)| + |\pi| = n + 1$. Then summing over all $\pi \in NC(n)$ and dividing by 2 we obtain that the expected value is just $\frac{n+1}{2}$. This suggests that there should be a bijective proof of Corollary 2.5. This is done in Section 5.*

3 Number of blocks in k -divisible partitions of type B

In this section we give analogous results for k -divisible non-crossing partitions of type B defined in [5]

Definition 3.1. *A partition π in $NC_B(n)$ is **k -divisible** if the size of all blocks in π are multiples of k . We denote by $NC_B^k(n)$ the set of k -divisible non-crossing partitions of type B_{kn} .*

Remark 3.2. *The definition of a k -divisible partition of type B_n seems quite natural. However, there is small detail on what the size of the zero-block shall be. We believe that the size of the zero-block of a partition in $NC_B(n)$ should be defined as half of the size of this block seen as a partition in $NC(n)$. With this definition, a k -divisible partition must belong to $NC_B(kn)$ for some n integer, and for instance the partition $\{\{1, 2\}, \{-1, 1\}, \{-1, -2\}\} \in NC_B(3)$ would not be 2-divisible.*

There are nice properties that support this considerations, for instance the map $Abs : NC_B(n) \rightarrow NC(n)$ defined in [9] (and explained below) preserves block sizes; we will not continue with this discussion but rather refer the reader to Section 4.5 of [5].

The number of k -divisible non-crossing partitions of type B was given in [5].

Proposition 3.3. *Let $NC_B^k(n)$ be the set of non-crossing partitions of k -divisible non-crossing partitions of type B_{kn} . Then*

$$\#NC_B^k(n) = \binom{(k+1)n}{n}.$$

Of course, our counting results depend on the definition of the size of the zero-block. To avoid this problem we count the number of non-zero blocks and the number of zero-blocks separately. Also, to avoid confusion we talk about the size of a pair instead of talking about the size of a block.

Definition 3.4. *Let $\pi \in NC_B(n)$ and V a block of π . The size of the unordered pair $\{V, -V\}$ is $\frac{1}{2}|V \cup (-V)|$, where for a subset $A \subset \mathbb{Z}$, $|A|$ denotes its cardinality.*

We proceed as for the case of classical non-crossing partitions. We start by calculating the expected number of non-zero blocks of a given size t , subject to the restriction of having m non-zero blocks. Before this, we need also to fix the size of the zero-blocks as we state in the following lemma.

Lemma 3.5. *The number of pairs of non-zero blocks of size t of all non-crossing partitions in $NC(B_n)$ with m pairs of non-zero blocks and a zero-block of size $2s$ is given by*

$$n \binom{n-1}{m-1} \binom{n-t-s-1}{m-2}. \tag{13}$$

Proof. We use (3) and Lemma 1.12 in the same way as for the classical non-crossing partitions.

$$\begin{aligned} \sum_{\substack{r_1+r_2+\dots+r_n=m \\ s+r_1+2r_2+\dots+nr_n=n}} \frac{n!r_t}{(n-m)!p_r} &= n \binom{n-1}{m-1} \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_1+2r_2+\dots+nr_n=n-s}} \frac{(m-1)!r_t}{p_r} \\ &= n \binom{n-1}{m-1} \binom{n-t-s-1}{m-2}. \end{aligned}$$

□

Proposition 3.6. *The sum of non-zero pairs of blocks $\{-V, V\}$ of size t over all the non-crossing partitions in $NC_B(n)$ is given by*

$$n \binom{2n-t-1}{n-1}. \quad (14)$$

Proof. First, we use Lemma 3.5 and sum over all possible sizes s of the zero-block in (13). Thus, the number of non-zero blocks of size t of all the non-crossing partitions in $NC_B(n)$ with m pairs of non-zero blocks is given by

$$n \binom{n-1}{m-1} \binom{n-t}{m-1}. \quad (15)$$

Next, we sum over m in 15, using Chu-Vandermonde's identity for $x = n - 1$, $y = n - t$ and $s = n - 1$. \square

Now, we count the number of zero-blocks of size $2t$ over *all* the partitions.

Proposition 3.7. *The number of non-crossing partitions in $NC_B(n)$ with a zero-block of size $2t$ is given by*

$$\binom{2n-t-1}{n-1}. \quad (16)$$

Proof. For counting the number of zero-blocks of a certain size we first we calculate the number of partitions with a zero-block of size $2t$ and m non-zero blocks using 3 and Lemma 1.12, yielding

$$\binom{n}{m} \binom{n-t-1}{m-1}.$$

Summing over m by means of Chu-Vandermonde's identity we get the desired result. \square

It is straightforward to pass to k -divisible partitions, and obtain the Theorem 2, which we state again for the convenience of the reader.

Theorem 2. 1) *The sum of the non-zero pairs of blocks $\{-V, V\}$ of size tk of over all non-crossing partitions in $NC_B^k(n)$ is given*

$$nk \binom{n(k+1)-t-1}{nk-1}. \quad (17)$$

2) *The number of the non-crossing partitions in $NC_B^k(n)$ with a zero-block of size $2t$ then*

$$\binom{n(k+1)-t-1}{nk-1}. \quad (18)$$

Corollary 3.8. *The expected number of unordered pairs $\{-V, V\}$ of size tk of a non-crossing partition chosen uniformly at random in $NC_B^k(n)$ is given by*

$$\frac{(nk+1) \binom{n(k+1)-t-1}{nk-1}}{\binom{(k+1)n}{n}}. \quad (19)$$

The fact that formulas (12) and (19) coincide may be explained by the remarkable observation by Biane, Goodman and Nica [9] that there is an $(n + 1) - to - 1$ map from $NC_B(n)$ to $NC(n)$. More precisely, for any subset $X \subset \mathbb{Z}$, let $Abs(X) := \{|x| : x \in X\}$ and for a partition $\pi \in NC(n)$, denote by $Abs(\pi)$ the set $\{Abs(V) : V \in \pi\}$.

Proposition 3.9. *Let n be a positive integer. Then the map $Abs : NC_B(n) \rightarrow NC(n)$ is a $(n + 1) - to - 1$ map from $NC_B(n)$ onto $NC(n)$.*

Finally, we consider the asymptotics when $n \rightarrow \infty$. Since when n is large the number of zero-blocks is negligible, independent of the choice of how to count them, we obtain the same behavior as for the classical partitions.

Corollary 3.10. *When $n \rightarrow \infty$ the expected number of pairs of blocks of size t of a non-crossing partition chosen uniformly at random in $NC^k(B_n)$ is asymptotically $\frac{nk}{(k+1)^{t+1}}$.*

4 Number of blocks in k -divisible partitions of type D

Krattenthaler and Müller [13] solved the problem of finding a combinatorial realization of k -divisible non-crossing partitions of type D .

Let π be a partitions of $\{1, 2, \dots, kn, -1, -2, \dots, -kn\}$. We represent π in an annulus with the integers $1, 2, \dots, k(n-1), -1, \dots, -(kn-k)$ on the outer circle in clockwise order and the integers $k(n-1) + 1, \dots, kn, -(k(n-1) + 1), \dots, -kn$ in the inner circle in counterclockwise order. A block $V = \{e_1, e_2, \dots, e_t\}$ with e_1, e_2, \dots, e_s are arranged on the outer circle in the clockwise order and $e_{s+1}, e_{s+2}, \dots, e_t$ arranged on the inner circle in counterclockwise order for some i . Then we draw curves, which lie inside of the annulus, connecting e_i and e_{i+1} for all $i = 1, 2, \dots, t$, where $e_{t+1} = e_1$. If we can draw the curves for all the blocks of π such that they do not intersect, then we call π non-crossing partition on the $(2k(n-1), 2k)$ annulus. We say that a block on the outer circle is visible if there is no block between this block and the inner circle.

Following Kim [14] the combinatorial realization of k -divisible non-crossing partitions of type D is defined as follows.

Definition 4.1. *A k -divisible non-crossing partition π of type D_n is a non-crossing partition π on the $(2k(n-1), 2k)$ annulus satisfying the following conditions:*

1. *If $V \in \pi$, then $-V \in \pi$.*
2. *For each block $V \in \pi$ with t elements, if e_1, e_2, \dots, e_t are the elements of V such that e_1, e_2, \dots, e_s are arranged on the outer circle in clockwise order and $e_{s+1}, e_{s+2}, \dots, e_t$ are arranged on the inner circle in counterclockwise order, then $|e_{i+1}| \equiv |e_i| + 1 \pmod k$ for all $i = 1, 2, \dots, t$, where $e_{t+1} = e_1$.*
3. *If there is a zero-block V of π , then V contains all the integers on the inner circle and at least two integers on the outer circle.*

4. If π has no block with elements in both the inner circle and the outer circle, then for any inner block V_i and any visible block V_j and the partition $\tilde{\pi}$ with $V_i \cup V_j$ instead of V_i and V_j satisfies Condition 2.

We denote by $NC_D^k(n)$ the set of k -divisible non-crossing partitions of type D_n

Remark 4.2. As pointed out in [14], Condition 4 was overlooked in [13]. Also, Condition 4 is stated in [14] in a different but equivalent way.

The number of k -divisible non-crossing partitions of type D was given in [5].

Proposition 4.3. The number of k -divisible non-crossing partitions of type D_n is given by

$$\binom{(k+1)(n-1)}{n} + \binom{(k+1)(n-1)+1}{n}.$$

We consider two cases which we call type $D1$ and type $D2$ depending on whether there is a zero-block or not.

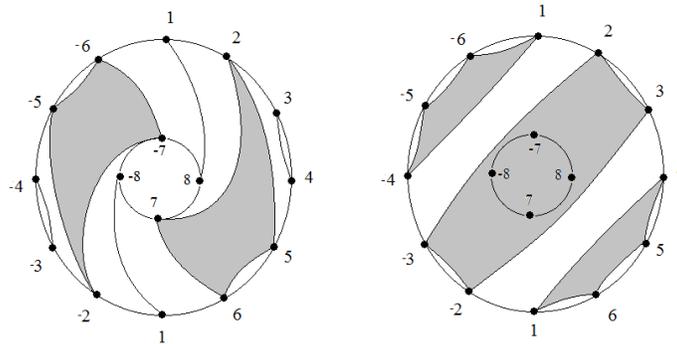


Figure 4: 2-divisible partitions of type $D1$ and type $D2$

Type D1

Definition 4.4. A k -divisible non-crossing partition of type D_n is of type $D1_n$ if it has a zero-block. We denote by $NC_{D1}^k(n)$ the set of k -divisible non-crossing partitions of type $D1_n$.

The number of k -divisible non-crossing partitions of type $D1$ was calculated in [13].

Proposition 4.5. Let n be a positive integer, $s \geq 2$ and $r_1, r_2, \dots, r_n \in \mathbb{N} \cup \{0\}$ be such that $s + r_1 + 2r_2 + \dots + nr_n = n$. Then the number of partitions of π in $NC_{D1}^k(n)$ with r_i pairs $\{-V, V\}$ of non-zero blocks size i a zero-block of size $2s$ is given by

$$\frac{(k(n-1))!}{(k(n-1)-m)!p_r}. \quad (20)$$

Lemma 4.6. *The number of pairs of non-zero blocks of size tk of all non-crossing partitions in $NC_{D_1}^k(n)$ with $2m$ non-zero blocks and a zero-block of size $2s$ is*

$$(k(n-1)) \binom{k(n-1)-1}{m-1} \binom{n-t-s-1}{m-2}. \quad (21)$$

Proof. In the same way as for the non-crossing partitions of type B , we use (20) and Lemma 1.12.

$$\begin{aligned} \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_1+2r_2+\dots+nr_n=n-s}} \frac{(k(n-1))!r_t}{(n-m)!p_r} &= (k(n-1)) \binom{k(n-1)-1}{m-1} \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_1+2r_2+\dots+nr_n=n-s}} \frac{(m-1)!r_t}{p_r} \\ &= n \binom{k(n-1)-1}{m-1} \binom{n-t-s-1}{m-2}. \end{aligned}$$

□

Proposition 4.7. *The number of non-zero blocks of size tk of all the non-crossing partitions in $NC_{D_1}^k(n)$ then*

$$k(n-1) \binom{(k+1)(n-1)-t-2}{k(n-1)-1}. \quad (22)$$

Proof. We use Lemma 4.6 and sum over all possible sizes s of the zero-block in (21). The only difference with type B is that now $s \geq 2$. Thus, the number of non-zero blocks of size tk of all the non-crossing partitions in $NC_{D_1}^k(n)$ with m non-zero block is given by

$$k(n-1) \binom{k(n-1)-1}{m-1} \binom{n-t-2}{m-1}.$$

Summing over m , using Chu-Vandermonde's identity for $x = k(n-1) - 1$, $y = n - t - 2$ and $s = k(n-1) - 1$, we get the result. □

In a similar way as for type B , we calculate the number of zero-blocks of a given size over all the non-crossing partitions of type D .

Proposition 4.8. *The number of non-crossing partitions in $NC_D^k(n)$ with a zero-block of size $2tk$ is given by*

$$\binom{(k+1)(n-1)-t}{k(n-1)-1}. \quad (23)$$

Proof. This is analogous to the case of $NC_B^k(n)$. □

Type D2

Definition 4.9. A k -divisible non-crossing partition of type D_n is of type $D2_n$ if it has no zero-block. We denote by $NC_{D2}^k(n)$ the set of k -divisible non-crossing partitions of type $D2_n$.

The number of k -divisible non-crossing partitions of type $D2$ was calculated in [13].

Proposition 4.10. Let n be a positive integer, $r_1, r_2, \dots, r_n \in \mathbb{N} \cup \{0\}$ be such that $r_1 + 2r_2 + \dots + nr_n = n$. Then the number of partitions of π in $NC^k(D2)$ with r_i non-zero pairs of size ik block is given by

$$2 \frac{(k(n-1))!}{(k(n-1)-m)!p_r} + r_1 \frac{(k(n-1))!}{(k(n-1)-m+1)!p_r}. \quad (24)$$

where $p_r = r_1!r_2! \cdots r_n!$ and $r_1 + r_2 + \dots + r_n = m$.

Proposition 4.11. The number of blocks of size tk of all non-crossing partitions in $NC_{D2}^k(n)$ is given by

$$k(n-1) \left(\binom{(k+1)(n-1)-t-2}{k(n-1)-2} + 2 \binom{(k+1)(n-1)-t-1}{k(n-1)-2} \right) \quad (25)$$

for $t > 1$ and by

$$k(n-1) \left(\binom{(k+1)(n-1)-3}{k(n-1)-2} + 2 \binom{(k+1)(n-1)-2}{k(n-1)-2} \right) + \binom{(k+1)(n-1)-1}{k(n-1)-1} \quad (26)$$

for $t = 1$.

Proof. Using (24) we count the number of blocks of size tk over all non-crossing partitions in $NC_D^k(n)$ with m non-zero pairs and then sum over m . This is

$$2 \sum_{\substack{r_1+\dots+r_n=m \\ r_1+\dots+nr_n=n}} \frac{r_t(k(n-1))!}{(k(n-1)-m)!p_r} + \sum_{\substack{r_1+\dots+r_n=m \\ r_1+\dots+nr_n=n}} \frac{r_t r_1(k(n-1))!}{(k(n-1)-m+1)!p_r}.$$

The first part of the sum can be treated in the same way as the classical case using Lemma 1.12.

$$\begin{aligned} 2 \sum_{\substack{r_1+\dots+r_n=m \\ r_1+\dots+nr_n=n}} \frac{r_t(k(n-1))!}{p_r(k(n-1)-m)!} &= 2(k(n-1)) \binom{k(n-1)-1}{m-1} \sum_{\substack{r_1+\dots+r_n=m \\ r_1+\dots+nr_n=n}} \frac{r_t(m-1)!}{p_r} \\ &= 2(k(n-1)) \binom{k(n-1)-1}{m-1} \binom{n-t-1}{m-2}. \end{aligned}$$

Summing over m , by means of the Chu-Vandermonde's formula, we get

$$2k(n-1) \binom{(k+1)(n-1)-t-1}{k(n-1)-2}. \quad (27)$$

For the second part of the sum we divide in two cases, $t = 1$ and $t > 1$.

If $t > 1$, then making the change of variable $\tilde{r}_1 = r_1 - 1$, and $\tilde{r}_i = r_i$, for $i \neq 1$, we get

$$\begin{aligned} \sum_{\substack{r_1+\dots+r_n=m \\ r_1+\dots+nr_n=n}} \frac{r_1 r_1 (k(n-1))!}{(k(n-1) - m + 1)! p_r} &= \sum_{\substack{\tilde{r}_1+\dots+\tilde{r}_n=m-1 \\ r_1+\dots+nr_n=n-1}} \frac{\tilde{r}_1 (k(n-1))!}{(k(n-1) - m + 1)! \prod_{i=1}^n \tilde{r}_i!} \\ &= (k(n-1)) \binom{k(n-1) - 1}{m-2} \binom{n-t-2}{m-3}. \end{aligned}$$

Summing over m we get

$$(k(n-1)) \binom{(k+1)(n-1) - t - 2}{k(n-1) - 2}. \quad (28)$$

Thus, combining (27) and (28) we get the result for $t > 1$.

Now, for $t = 1$, the sum

$$\sum_{\substack{r_1+\dots+r_n=m \\ r_1+\dots+nr_n=n}} \frac{r_1 r_1 (k(n-1))!}{(k(n-1) - m + 1)! p_r}$$

can be split as

$$\sum_{\substack{r_1+\dots+r_n=m \\ r_1+\dots+nr_n=n}} \frac{r_1 (r_1 - 1) (k(n-1))!}{(k(n-1) - m + 1)! p_r} + \sum_{\substack{r_1+\dots+r_n=m \\ r_1+\dots+nr_n=n}} \frac{r_1 (k(n-1))!}{(k(n-1) - m + 1)! p_r}.$$

Each of the terms can be treated as before, yielding

$$k(n-1) \binom{k(n-1) - 1}{m-2} \binom{n-3}{m-3} + \binom{k(n-1)}{m-1} \binom{n-2}{m-2}.$$

Again, summing over m we get

$$k(n-1) \binom{(k+1)(n-1) - 3}{k(n-1) - 2} + \binom{(k+1)(n-1) - 1}{k(n-1) - 2}. \quad (29)$$

Combining (27) and (29) we get the result for $t = 1$. □

Now putting together Propositions 4.8, 4.7 and 4.11 we get, after some small simplifications, Theorem 3, that we state again for the convenience of the reader.

Theorem 3. *The sum of the number of pairs of blocks $\{V, -V\}$ of size tk over all the partitions in $NC_D^k(n)$ is*

$$(k(n-1) + 1) \binom{(k+1)(n-1) - t}{k(n-1) - 1} + k(n-1) \binom{(k+1)(n-1) - t - 1}{k(n-1) - 2}.$$

Notice that even though the formulas of Proposition 4.11 look very different, for $t = 1$ and for $t > 1$, Theorem 3 gives the same formula for all t .

Finally, using the Theorem 3 and Proposition 4.3 we get the asymptotics when $n \rightarrow \infty$. Surprisingly, we get the same behavior for the cases of $NC^k(n)$ and $NC_B^k(n)$.

Corollary 4.12. *When $n \rightarrow \infty$ the expected number of pairs of size tk of a non-crossing partition chosen uniformly at random in $NC^k(D_n)$ is asymptotically $\frac{nk}{(k+1)^{t+1}}$.*

5 The bijection

In this section we give a bijective proof of the fact that $NC^k(n) = NC_{k+1}(n)$. From this bijection we derive Corollary 2.5.

Lemma 5.1. *For each n and each k let $f : NC_{k+1}(n) \rightarrow NC^k(n)$ be the map induced by the identification of the pairs $\{k+1, k+2\}, \{2(k+1), 2(k+1)+1\}, \dots, \{n(k+1), 1\}$. Then f is a bijection.*

Proof. First, we see that the image of this map is in $NC^k(n)$. So, let π be a $(k+1)$ -equal partition.

(i) Every block has one element on each congruence *mod* $k+1$. Indeed, because of the characterization of non-crossing partitions on Remark 1.3, there is at least one interval, which has of course this property. Removing this interval does not affect the congruence in the elements of other blocks. So by induction on n every block has one element of each congruence *mod* $k+1$.

(ii) Note that for each two elements identified we reduce 1 point. So suppose that m blocks (of size k) are identified in this bijection to form a big block V . Then the number of vertices in this big block equals $m(k+1) - \#(\text{identified vertices})/2$. Now, by (i), there are exactly two elements in each block to be identified with another element, that is $2m$. So

$$\begin{aligned} |V| &= m(k+1) - \#(\text{identified vertices})/2 \\ &= m(k+1) - (2m)/2 = mk. \end{aligned}$$

this proves that $f(\pi) \in NC^k(n)$.

Now, it is not hard to see that by splitting the points $1, k+2, \dots, nk+1$ of $\pi \in NC^k(n)$ we get a unique inverse $f^{-1}(\pi) \in NC_{k+1}(n)$. More specifically, f^{-1} is defined as follows. Let $\pi = \{V_i, \dots, V_t\}$ be a k -divisible partition and $V_i = \{e_{1,1}, \dots, e_{1,k}, e_{2,1}, \dots, e_{s_i,1}, \dots, e_{s_i,k}\}$, a block with $k(s_i)$ elements arranged clockwise and $e_{j,l} = ka_j + l$ for some $a_j \in \mathbb{N}$ (which is possible by a similar argument as (i) above). Then $f^{-1}(\pi)$ has, for each V_i , s_i blocks $V_i^1, V_i^2, \dots, V_i^{s_i}$ with $V_i^j = \{\hat{e}_{j,1}, \dots, \hat{e}_{j,k}, \hat{e}_{j,k+1}\}$ consisting of the elements $\hat{e}_{j,l} = (k+1)a_j + l$ for $l = 1, \dots, k$ and $\hat{e}_{j,k+1}$ comes from the splitting of $e_{j+1,k}$, that is, $\hat{e}_{j,k+1} = (k+1)a_j + l - 1$. \square

Now we can prove Corollary 2.5.

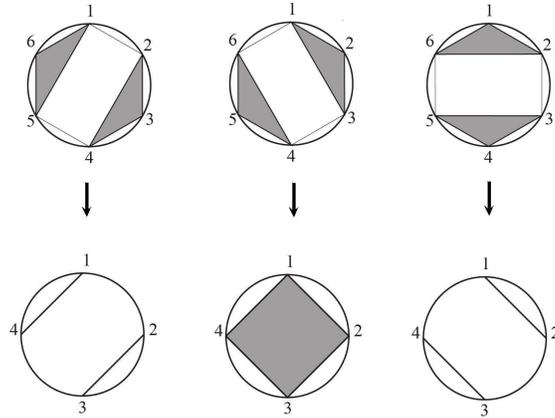


Figure 5: Bijection f between 3-equal and 2-divisible non-crossing partitions

Proof of Corollary 2.5. For each n and each k and each $0 < i \leq k + 1$ let $f_i : NC_{k+1}(n) \rightarrow NC^k(n)$ the map induced by the identification of the pairs $\{k + 1 + i, k + 1 + i + 1\}, \dots, \{2(k + 1) + i, 2(k + 1) + i + 1\}, \dots, \{n(k + 1) + i, n(k + 1) + i + 1\}$ (we consider elements $\text{mod } nk$). Then by the proof of the previous lemma, each f_i is a bijection. So, let π be a fixed $(k + 1)$ -equal partition. Considering all the bijections f_i on this fixed partition, we see that every point j is identified twice (one with f_{j-1} and one with f_j). Note that each block obtained by the identification corresponds to a block in the Kreweras complement. So, for each partition π in $NC_{k+1}(n)$, the collection $(f_i(\pi))_{i=1}^k$ consists of $k + 1$ partitions in $NC^k(n)$ whose number of blocks add $kn + 1$. \square

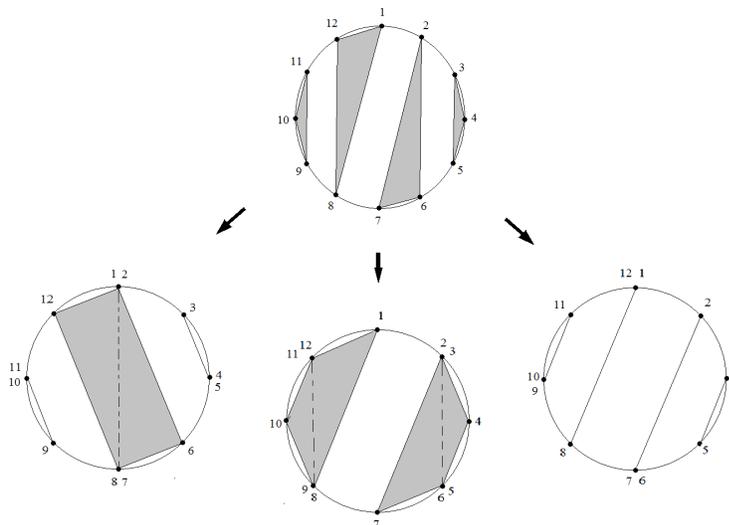


Figure 6: Bijections f_1 , f_2 and f_3 applied to $\pi = \{\{1, 8, 9\}, \{2, 6, 7\}, \{3, 4, 5\}, \{9, 10, 11\}\}$

Remark 5.2. Note that for a k -divisible partition on $[2kn]$ points, the property of being centrally symmetric is preserved under the bijections f_i (see e. g. Fig.6), and then the arguments given here also work for the partitions of type B . We expect that a similar argument works for type D .

In the following example we want to illustrate how the bijection given by Lemma 5.1 allows us to count k -divisible partitions with some restrictions by counting the preimage under f .

Example 5.3. Let $NC_{1 \rightarrow 2}^k(n)$ be the set of k -divisible non-crossing partitions of $[kn]$ such that 1 and 2 are in the same block. It is clear that $\pi \in NC_{1 \rightarrow 2}^k(n)$ if and only if $f^{-1}(\pi)$ satisfies that 1 and 2 are in the same block.

Now, counting the $(k + 1)$ -equal non-crossing partitions of $[(k + 1)n]$ such that 1 and 2 are in the same blocks is the same as counting non-crossing partitions of $[(k + 1)n - 1]$ with $n - 1$ blocks of size $k + 1$ and 1 block of size k containing the element 1, since 1 and 2 can be identified. From Proposition 1.4, the size of this set is easily seen to be

$$\frac{k}{(k + 1)n - 1} \binom{(k + 1)n - 1}{n - 1} = \frac{k}{n - 1} \binom{(k + 1)n - 2}{n - 2}$$

where the first factor of the LHS is the probability that the block of size k contains the element 1.

Let us finally mention that the bijections f_i are closely related to the Kreweras complement of a $(k + 1)$ -equal non-crossing partitions, which was considered in [2]. Indeed $Kr(\pi)$ can be divided in a canonical way into $k + 1$ partitions of $[n]$, π_1, \dots, π_{k+1} , such that $|\pi_i| = |f_i(\pi)|$. Fig. 6 shows the bijections f_1, f_2 and f_3 for $k = 3, n = 4$ and $\pi = \{\{1, 8, 9\}, \{2, 6, 7\}, \{3, 4, 5\}, \{9, 10, 11\}\}$, while Fig. 7 shows the same partition as Fig. 6 with its Kreweras complement divided into the partitions π_1, π_2 and π_3 .

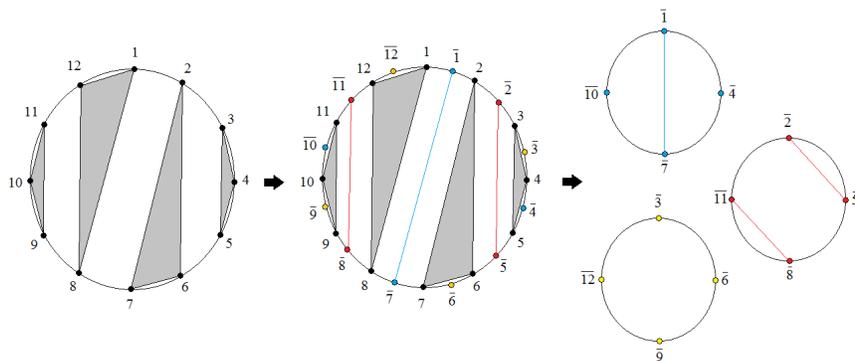


Figure 7: A 3-equal and its Kreweras complement divided mod 3.

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