Polynomials with real zeros and compatible sequences *

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Submitted: Jan 31, 2012; Accepted: Aug 31, 2012; Published: Sep. 13, 2012
Mathematics Subject Classification: 26C10

Abstract

In this paper, we study polynomials with only real zeros based on the method of compatible zeros. We obtain a necessary and sufficient condition for the compatible property of two polynomials whose leading coefficients have opposite sign. As applications, we partially answer a question proposed by M. Chudnovsky and P. Seymour in the recent publication [M. Chudnovsky, P. Seymour, The roots of the independence polynomial of a clawfree graph, J. Combin. Theory Ser. B 97 (2007) 350–357]. We also establish the connection between the interlacing property and the compatible property of two polynomials and give a simple proof of some known results.

Keywords: Polynomials with only real zeros; Compatible sequences; Common interleaver

1 Introduction

Polynomials with only real zeros arise often in combinatorics and other branches of mathematics. We refer the reader to [1, 2, 3, 4, 8, 10, 11, 12, 13, 14, 15, 16] for many results on this subject. There are various methods for proving that polynomials have only real zeros. One basic method is to prove by induction that polynomials have interlaced

*This work was supported in part by the National Science Foundation of China (No. 11126234, 11201260), by the Key Project of Chinese Ministry of Education (No. 212098), by the Specialized Research Fund for the Doctoral Program of Higher Education of China (No. 20113705120002), by the National Science Foundation of Shandong Province of China (No. ZR2011AL018), and by the STPF of University in Shandong Province of China (No. J10LA03).

real zeros. So far there have been quite a few papers obtained by this method [1, 2, 3, 5, 8, 9, 10, 11, 13, 16]. Recently using the method of compatible zeros, Chudnovsky and Seymour [4] gave that all zeros of the independence polynomial of a clawfree graph are real, which shows that this method is also a powerful tool to attack problems of the reality of zeros of polynomials. This is the motivation for us to study polynomials with only real zeros using the method of compatible zeros in this paper.

Following Wagner [15], a real polynomial is said to be *standard* if either it is identically zero or its leading coefficient is positive. Denote by RZ the set of real polynomials with only real zeros. Given a polynomial $f(x)$ in RZ of degree n, we arrange all zeros (counting multiplicities) of $f(x)$ in a nonincreasing sequence $R(f) = (r_1, r_2, \ldots, r_n)_\geqslant$. Let $\{f_i(x)\}_{i=1}^k$ be a sequence of polynomials with real coefficients. We say that it is *compatible* if for all $c_1, \ldots, c_k \geqslant 0$, the polynomial $\sum_{i=1}^k c_i f_i(x) \in \mathrm{RZ}$. We say that it is *pairwise compatible* if for all $i, j \in \{1, \ldots, k\}$, polynomials $f_i(x)$ and $f_j(x)$ are compatible.

Given two nonincreasing sequences $(r_1, r_2, \ldots, r_n)_\geqslant$ and $(s_1, s_2, \ldots, s_m)_\geqslant$ of real numbers, we say that these two sequences are *compatible* if $\mid m - n \mid \leqslant 1$ and $\max\{r_i, s_i\} \leqslant \min\{r_{i-1}, s_{i-1}\}$ for all $1 \leqslant i \leqslant \min\{m, n\}$. For a polynomial $f(x)$, denote by $n_f(x)$ the number of real zeros of $f(x)$ that lie in the interval $[x, \infty)$ (counted with their multiplicities). Chudnovsky and Seymour [4] gave the following result which shows the equivalence of the compatible property of two polynomials and their sequences of zeros when leading coefficients of these two polynomials have the same sign.

Proposition 1.1 ([4])**.** *Let f and g be real polynomials with positive leading coefficient. Then f and g are compatible if and only if $|n_f(x) - n_g(x)| \leqslant 1$ for all $x \in \mathbb{R}$.*

Suppose that two polynomials $f \in \mathrm{RZ}$ and $g \in \mathrm{RZ}$. Let $R(f) = (r_1, r_2, \ldots, r_n)_\geqslant$ and $R(g) = (s_1, s_2, \ldots, s_m)_\geqslant$. Following [15], we say that g *alternates left of* f (g *alternates* f for short) ($g \preccurlyeq_{alt} f$) if $\deg f = \deg g = n$ and

$$s_n \leqslant r_n \leqslant s_{n-1} \leqslant \cdots \leqslant s_2 \leqslant r_2 \leqslant s_1 \leqslant r_1. \tag{1.1}$$

We say that g *interlaces* f ($g \preccurlyeq_{int} f$) if $\deg f = \deg g + 1 = n$ and

$$r_n \leqslant s_{n-1} \leqslant \cdots \leqslant s_2 \leqslant r_2 \leqslant s_1 \leqslant r_1. \tag{1.2}$$

Let $g \preccurlyeq f$ denote "either g alternates f or g interlaces f". We say that g *interleaves* f if $g \preccurlyeq f$. If $g \preccurlyeq_{int} f$, then we also say that their sequences of zeros $R(g) \preccurlyeq_{int} R(f)$. Similarly, we can define $R(g) \preccurlyeq_{alt} R(f)$ and $R(g) \preccurlyeq R(f)$. For notational convenience, let $a \preccurlyeq bx + c$ for any real constants a, b, c and $f \preccurlyeq 0, 0 \preccurlyeq f$ for any $f \in \mathrm{RZ}$.

Let $\{f_i(x)\}_{i=0}^n$ be a sequence of polynomials with all zeros real. Following [4], a *common interleaver* for $\{f_i(x)\}_{i=0}^n$ is a sequence of real numbers that interleaves the sequence of zeros of $f_i(x)$ for all $1 \leqslant i \leqslant n$. In order to prove the reality of zeros of the independence polynomial of a clawfree graph, Chudnovsky and Seymour [4] gave the following result by means of the Helly property of linear interval.

Proposition 1.2 ([4, Theorem 3.6])**.** *Let $\{f_i(x)\}_{i=1}^n$ be polynomials with positive leading coefficients and $f_i(x) \in \mathrm{RZ}$ for all $1 \leqslant i \leqslant n$. Then the following four statements are equivalent.*

(i) $f_1, \ldots, f_n$ *are pairwise compatible.*

(ii) f_i, f_j *have a common interleaver for all* $1 \leqslant i < j \leqslant n$.

(iii) $f_1, \ldots, f_n$ *have a common interleaver.*

(iv) $f_1, \ldots, f_n$ *are compatible.*

Without the assumption that leading coefficients of polynomials are all positive, Chudnovsky and Seymour [4] showed that Proposition 1.2 (i) and (iv) are no longer equivalent. And they also proposed the following:

"…it is possible that statements Proposition 1.2 (iii) and (iv) are equivalent under an appropriate modification of the definition of a common interleaver."

This leads to the following two problems.

Problem 1.3. *Suppose that leading coefficients of* f, g *have opposite sign and* $f, g \in \mathrm{RZ}$. *Under what conditions* f *and* g *are compatible?*

Problem 1.4. *Suppose that* $\{f_i(x)\}_{i=1}^n$ *be real polynomials and* $f_i(x) \in \mathrm{RZ}$ *for all* $1 \leqslant i \leqslant n$. *Under what conditions Proposition 1.2 (iii) can imply (iv)?*

The object of this paper is to study polynomials with only real zeros by the method of compatible zeros. In Section 2, we show a necessary and sufficient condition for the compatible property of two polynomials whose leading coefficients have opposite sign. As applications, we can give a solution to Problems 1.3 and 1.4 respectively. We also obtain a direct proof of the equivalence of Proposition 1.2 (i) and (iv). In Section 3, we establish the connection between the interlacing property and the compatible property of two polynomials and give a simple proof of some known results.

2 Main results

Let sgn denote the sign function defined on $\mathbb{R}$ by

$$\operatorname{sgn}(x) = \begin{cases} +1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Let $f(x)$ be a real function. Denote $\operatorname{sgn} f(+\infty) = +1$ (resp. -1) if $\operatorname{sgn} f(x) = +1$ (resp. -1) for sufficiently large x. The meaning of $\operatorname{sgn} f(-\infty)$ is similar.

Let $f(x) \in \mathrm{RZ}$ and $R(f) = (r_1, r_2, \ldots, r_n)_\geqslant$. Denote by $\bar{f} = \frac{f(x)}{(x-r_1)}$. Then $R(\bar{f}) = (r_2, r_3, \ldots, r_n)_\geqslant$. The following theorem gives a solution to Problem 1.3.

Theorem 2.1. *Suppose that leading coefficients of* f *and* g *have opposite sign. Denote by* $R(f) = (r_1, r_2, \ldots, r_n)_\geqslant$ *and* $R(g) = (s_1, s_2, \ldots, s_m)_\geqslant$. *Then* f *and* g *are compatible if and only if either*

(i) $r_1 \geqslant s_1$, *then* $|n_{\bar{f}}(x) - n_g(x)| \leqslant 1$ *for all* $x \in \mathbb{R}$, *or*

(ii) $r_1 < s_1$, *then* $|n_f(x) - n_{\bar{g}}(x)| \leqslant 1$ *for all* $x \in \mathbb{R}$.

Proof. Without loss of generality, we may assume that f and g have no common zeros, which implies that f and g have only simple zeros. We may also assume that the leading coefficient of f and g is 1 and -1 respectively. We prove this statement only for the case (i) since the case (ii) is similar. We arrange points in $R(f)$ and $R(g)$ in a nonincreasing sequence $(a_1, a_2, \ldots, a_{m+n})_{\geqslant}$. So these $m+n$ points determine $m+n+1$ intervals in the real line, and going from right to left, we label these intervals by $I_1, I_2, \ldots, I_{m+n+1}$, where $I_1 = (a_1, +\infty), I_{m+n+1} = (-\infty, a_{m+n}]$, and $I_i = (a_i, a_{i-1}]$ for $2 \leqslant i \leqslant m+n$. Clearly, if polynomials f and g have the same sign in an interval, then they will have opposite sign in the adjacent one. Since leading coefficients of f and g have opposite sign, they will have opposite sign in all odd indexed intervals and have the same sign in all even indexed ones.

($\Rightarrow$) Let $F_\theta = \theta f + (1-\theta)g$ for all $0 \leqslant \theta \leqslant 1$. So $F_\theta \in \mathrm{RZ}$ by the condition that f and g are compatible. We will show that all odd indexed intervals with limited length have one endpoint a zero of f and the other a zero of g. Otherwise, we may assume that there is an odd indexed interval I_{2i+1} with limited length whose endpoints are two zeros of f, since the case for zeros of g is similar. Now we have $\operatorname{sgn} F_\theta(a_{2i+1} + \varepsilon_1) F_\theta(a_{2i} - \varepsilon_2) = 1$ for sufficiently small $\varepsilon_1, \varepsilon_2 \geqslant 0$. Then F_θ has even number of zeros in $(a_{2i+1} + \varepsilon_1, a_{2i} - \varepsilon_2)$. Note that zeros of a polynomial is the continuous function of coefficients of the polynomial (see [6] for instance). Then both f and g have the same number of zeros as F_θ in the interval $(a_{2i+1} + \varepsilon_1, a_{2i} - \varepsilon_2)$. Since f and g have no real zeros in this interval, so does F_θ. According to previously mentioned, we also have that F_θ has exactly one zero in each odd indexed interval with limited length whose endpoints are a zero of f and g respectively. Hence, when $n = m$, we obtain

$$n = \deg(F_\theta) < \sharp\{\text{odd indexed intervals}\} - 1 = \left\lfloor \frac{n+m+2}{2} \right\rfloor - 1 = n.$$

When $n > m$, we obtain

$$n = \deg(F_\theta) \leqslant \sharp\{\text{odd indexed intervals}\} - 1 = \left\lfloor \frac{n+m+2}{2} \right\rfloor - 1 < n.$$

Either case yields a contradiction. So all odd indexed intervals with limited length have one endpoint a zero of f and the other a zero of g. Hence we have $|n_{\bar{f}}(x) - n_g(x)| \leqslant 1$ for all $x \in \mathbb{R}$.

($\Leftarrow$) Set $p_i \in I_{2i}$ for all $1 \leqslant i \leqslant \lfloor \frac{n+m+1}{2} \rfloor$. Denote by

$$p(x) = \prod_{i=1}^{\lfloor \frac{n+m+1}{2} \rfloor} (x - p_i).$$

It follows that $g \preccurlyeq p \preccurlyeq f$. By the condition, we get $n = m$, $n = m+1$ or $n = m+2$. So $\lfloor \frac{n+m+1}{2} \rfloor = n$ or $n-1$. Let $F = \alpha f + \beta g$ for all $\alpha, \beta \geqslant 0$. We may assume that F is standard (otherwise replaced F by $-F$). Note that $\operatorname{sgn} F(p_i) = (-1)^i$ and $\operatorname{sgn} F(+\infty) = 1$. By Weierstrass Intermediate Value Theorem, $F(x)$ has one zero in each of the following n intervals $(p_n, p_{n-1}), (p_{n-1}, p_{n-2}), \ldots, (p_2, p_1), (p_1, +\infty)$, where $p_n = -\infty$ provided $\lfloor \frac{n+m+1}{2} \rfloor = n - 1$. Thus $F(x) \in \mathrm{RZ}$ for all $\alpha, \beta \geqslant 0$. So we have f and g are compatible. $\square$

Corollary 2.2. *Suppose that leading coefficients of f and g have opposite sign. If f and g are compatible, then $|\deg(f) - \deg(g)| \leqslant 2$.*

The following theorem is a generalization of the sufficiency of Theorem 2.1. Although it can be proved by the same technique used in the proof of Theorem 2.1, we include its proof for completeness.

Theorem 2.3. *Let $\{f_i(x)\}_{i=1}^n$ be a sequence of real polynomials satisfying the following conditions.*

(a) *$f_i(x) \in \mathrm{RZ}$ and $R(f_i) = (r_1^{(i)}, r_2^{(i)}, \ldots, r_{m_i}^{(i)})_{\geqslant}$ for all $1 \leqslant i \leqslant n$.*

(b) *$f_1, \ldots, f_s$ have positive leading coefficients and $f_{s+1}, \ldots, f_n$ have negative leading coefficients.*

(c) *$r_1^{(i)} \geqslant r_1^{(j)}$ for all $1 \leqslant i \leqslant s$ and $s+1 \leqslant j \leqslant n$.*

(d) *$|n_{\bar{f}_i}(x) - n_{f_j}(x)| \leqslant 1$ for all $1 \leqslant i \leqslant s, s+1 \leqslant j \leqslant n$ and $x \in \mathbb{R}$.*

Then $f_1, f_2, \ldots, f_n$ are compatible.

Proof. Let $F(x) = \sum_{t=1}^n \alpha_t f_t$ for all $\alpha_1, \alpha_2, \ldots, \alpha_n \geqslant 0$. Then it suffices to prove that $F \in \mathrm{RZ}$. We may assume that F is standard (otherwise replaced F by $-F$). For all $1 \leqslant i \leqslant s$ and $s+1 \leqslant j \leqslant n$, we set $p_k \in \mathbb{R}$ by conditions (c) and (d), such that $r_{k+1}^{(i)} \leqslant p_k \leqslant r_k^{(i)}$ and $r_k^{(j)} \leqslant p_k \leqslant r_{k-1}^{(j)}$, where $r_0^{(j)} = +\infty$. Let

$$p(x) = \prod_{k=1}^{\lfloor \frac{m_1+m_n+1}{2} \rfloor} (x - p_k).$$

Then we have $R(f_j) \preccurlyeq R(p) \preccurlyeq R(f_i)$ for all $1 \leqslant i \leqslant s$ and $s+1 \leqslant j \leqslant n$. By the condition (d), we get $m_1 = m_n$, $m_1 = m_n + 1$ or $m_1 = m_n + 2$. So $\lfloor \frac{m_1+m_n+1}{2} \rfloor = m_1$ or $m_1 - 1$. Note that $\operatorname{sgn} F(p_i) = (-1)^i$ and $\operatorname{sgn} F(+\infty) = 1$. By Weierstrass Intermediate Value Theorem, we can obtain that $F(x)$ has one zero in each of the following n intervals $(p_{m_1}, p_{m_1-1}), (p_{m_1-1}, p_{m_1-2}), \ldots, (p_2, p_1), (p_1, +\infty)$, where $p_{m_1} = -\infty$ provided $\lfloor \frac{m_1+m_n+1}{2} \rfloor = m_1 - 1$. Thus $F(x) \in \mathrm{RZ}$ for all $\alpha_1, \alpha_2, \ldots, \alpha_n \geqslant 0$. Then $f_1, f_2, \ldots, f_n$ are compatible. $\square$

From the proof of Theorem 2.3, we say that a *modification common interleaver* for polynomials $f_1, f_2, \ldots, f_n$ is a sequence $R(p)$ of real numbers satisfying $R(f_j) \preccurlyeq R(p) \preccurlyeq R(f_i)$, where f_i has positive leading coefficients and f_j has negative leading coefficients for all $1 \leqslant i \leqslant s$ and $s+1 \leqslant j \leqslant n$.

Remark 2.4. Under the definition of a modification common interleaver, Theorem 2.3 gives a solution to Problem 1.4.

Using Proposition 1.1, we are able to prove the following result.

Proposition 2.5. *Let f_1, f_2, g be pairwise compatible polynomials with positive leading coefficients. Then $f = \alpha_1 f_1 + \alpha_2 f_2$ and g are compatible for all $\alpha_1, \alpha_2 \geqslant 0$.*

Proof. Without loss of generality, we may assume that f and g have no common zeros. For $i = 1, 2$, denote by $R(f_i) = (r_1^{(i)}, r_2^{(i)}, \ldots, r_{n_i}^{(i)})_{\geqslant}$, $R(f) = (r_1, r_2, \ldots, r_n)_{\geqslant}$ and $R(g) = (s_1, s_2, \ldots, s_m)_{\geqslant}$. We arrange points in $R(f)$ and $R(g)$ in a nonincreasing sequence $(a_1, a_2, \ldots, a_{m+n})_{\geqslant}$. Following Proposition 1.1, it suffices to prove $\mid n_f(x) - n_g(x) \mid \leqslant 1$ for all $x \in \mathbb{R}$. For convenience, set $a_{m+n+1} = -\infty$. Note that

$$\mid n_f(x) - n_g(x) \mid = \mid n_f(a_i) - n_g(a_i) \mid$$

for all $x \in (a_{i+1}, a_i]$ where $1 \leqslant i \leqslant n + m$, and $\mid n_f(x) - n_g(x) \mid = 0$ for all $x \in (a_1, +\infty)$. So we only need to show that $\mid n_f(a_i) - n_g(a_i) \mid \leqslant 1$ for all $1 \leqslant i \leqslant n + m$. We do this in two steps.

First, we claim that a_1, a_2 are a zero of f and g respectively.

Otherwise assume for a contradiction that $a_1, a_2 \in R(f)$ or $a_1, a_2 \in R(g)$. If $a_1, a_2 \in R(f)$, then we assume that the right endpoint of the interval, where a_2 lies in, is a zero of f_1. If $a_1, a_2 \in R(g)$. Let i be the minimum index such that $a_i \in R(f)$. Then, without loss of generality, we may also assume that the left endpoint of the interval, where a_i lies in, is a zero of f_1. So we have $\mid n_{f_1}(a_2) - n_g(a_2) \mid = 2$, which is contrary to Proposition 1.1.

Second, we claim that the number of consecutive numbers a_i, which is zeros of f or g, is not exceed 2 for $i \geqslant 2$.

Otherwise, let i be the minimum index such that $a_i, a_{i+1}, a_{i+2} \in R(f)$ or $a_i, a_{i+1}, a_{i+2} \in R(g)$. If $a_i, a_{i+1}, a_{i+2} \in R(f)$, then we have $n_f(a_{i-1}) - n_g(a_{i-1}) = k$, where $k \in \{-1, 0, 1\}$. Assume that the right endpoint of the interval, where a_{i+2} lies in, is a zero of f_1. If $a_i, a_{i+1}, a_{i+2} \in R(g)$, then we have $n_g(a_{i-1}) - n_f(a_{i-1}) = k$, where $k \in \{-1, 0, 1\}$. Let t be the minimum index such that $t \geqslant i + 2$ and $a_t \in R(f)$. Without loss of generality, we may also assume that the left endpoint of the interval, where a_t lies in, is a zero of f_1. So we have $\mid n_{f_1}(a_{i+2}) - n_g(a_{i+2}) \mid \geqslant k + 3$, where $k \in \{-1, 0, 1\}$, which is also contrary to Proposition 1.1.

Thus we have $\mid n_f(a_i) - n_g(a_i) \mid \leqslant 1$ for all $1 \leqslant i \leqslant m + n$. Hence sequences $R(f)$ and $R(g)$ are compatible. This completes our proof. □

Proposition 2.5 now allows us to give a direct proof of the following theorem from [4], using a very straightforward induction.

Theorem 2.6 ([4])**.** *Let $f_1(x), f_2(x), \ldots, f_n(x)$ be pairwise compatible polynomials with positive leading coefficients. Then $f_1(x), f_2(x), \ldots, f_n(x)$ are compatible.*

3 Applications

In this section we establish the connection between the interlacing property and the compatible property of real zeros of two polynomials. And we give a simple proof of several known facts.

Proposition 3.1. *Suppose that f, g have leading coefficients of the same sign and $g \preccurlyeq f$. Then sequences $R(f) \cup \{a\}$ and $R(g) \cup \{b\}$ are compatible for all $a \leqslant b$.*

Proof. Let $R(f) = (r_1, \ldots, r_n)_{\geqslant}$ and $R(g) = (s_1, \ldots, s_m)_{\geqslant}$. We arrange points in $R(f)$ and $R(g)$ in a nonincreasing sequence $(a_1, a_2, \ldots, a_{m+n})_{\geqslant}$. Denote by $R(F) = R(f) \cup \{a\}$ and $R(G) = R(g) \cup \{b\}$. Then it suffices to prove that $|n_F(x) - n_G(x)| \leqslant 1$, where $x \in \{a_1, \ldots, a_{m+n}, a, b\}$ by the proof of Proposition 2.5.

Note that $n_f(x) - n_g(x) = 0$ or 1 since $g \preccurlyeq f$. For convenience, set $a_0 = +\infty$ and $a_{m+n+1} = -\infty$. Assume that $a \in (a_{i+1}, a_i]$ and $b \in (a_{j+1}, a_j]$ for $1 \leqslant j < i \leqslant m+n$. Then we have $n_F(a_k) - n_G(a_k) = 0$ or 1 for $1 \leqslant k < j$; $n_F(a_k) - n_G(a_k) = -1$ or 0 for $j \leqslant k < i$; $n_F(a_k) - n_G(a_k) = 0$ or 1 for $i \leqslant k \leqslant m+n$ and $n_F(a) - n_G(a) = 0$ or 1; $n_F(b) - n_G(b) = -1$ or 0. This completes the proof of the proposition. $\square$

Remark 3.2. Suppose that f, g have leading coefficients of the same sign and $g \preccurlyeq f$. Then sequences $R(f)$ and $R(g) \cup \{a\}$ are compatible for all $a \in \mathbb{R}$.

3.1 Fisk's results

Fisk [7] wrote a book to extend the study of zeros of polynomials. In this subsection, we can obtain a short and simple proof of several results of Fisk. Let $f \in \mathrm{RZ}$ and $R(f) = (r_1, r_2, \ldots, r_n)_{\geqslant}$. Then we define $R(f)(i) = (r_1, \ldots, r_i)_{\geqslant}$ and $R(f)/R(f)(i) = (r_{i+1}, \ldots, r_n)_{\geqslant}$ for all $1 \leqslant i \leqslant n$.

Theorem 3.3 ([7])**.** *Let f_1, f_2, g_1, g_2 be real polynomials whose leading coefficients have the same sign. Suppose that $f_1, g_1, f_2, g_2 \in \mathrm{RZ}$ and $g_1 \preccurlyeq f_1$. Then the following results hold.*

(i) *If $g_2 \preccurlyeq f_2$, then $\alpha f_1 g_2 + \beta f_2 g_1 \in \mathrm{RZ}$ for all $\alpha, \beta \geqslant 0$.*

(ii) *If $f_2 \preccurlyeq g_2$, then $\alpha f_1 g_2 - \beta f_2 g_1 \in \mathrm{RZ}$ for all $\alpha, \beta \geqslant 0$.*

Proof. Let $R(f_i) = (r_1^{(i)}, \ldots, r_{n_i}^{(i)})_{\geqslant}$ and $R(g_i) = (s_1^{(i)}, \ldots, s_{m_i}^{(i)})_{\geqslant}$ for $i = 1, 2$. Without loss of generality, we may assume that f_1, f_2, g_1, g_2 have no common zeros and $r_1^{(1)} > s_1^{(2)}$. Denote by $R(F) = R(f_1) \cup R(g_2)$ and $R(G) = R(f_2) \cup R(g_1)$. By Proposition 1.1, to prove the case (i), it suffices to prove sequences $R(F)$ and $R(G)$ are compatible under the condition $R(g_i) \preccurlyeq R(f_i)$ for $i = 1, 2$. By Theorem 2.1, to prove the case (ii), it suffices to prove sequences $R(\bar{F})$ and $R(G)$ are compatible under the condition $R(\bar{f}_1) \preccurlyeq R(g_1)$ and $R(f_2) \preccurlyeq R(g_2)$. So we prove it only for the case (i) since the case (ii) is similar.

Now we show that the case (i) holds by induction on $n_2 + m_2$. For $n_2 + m_2 \leqslant 1$, we have $\deg f_2 \leqslant 1, \deg g_2 = 0$ since $g_2 \preccurlyeq f_2$. Then sequences $R(f_1)$ and $R(G)$ are compatible from Remark 3.2. Next we may assume that sequences $R(F)$ and $R(G)$ are compatible for $n_2 + m_2 \leqslant n$. For $n_2 + m_2 = n + 1$, we distinguish two cases.

Case 1. When $g_2 \preccurlyeq_{alt} f_2$. If $r^{(2)}_{n_2} \in (s^{(1)}_i, r^{(1)}_i)$ or $(r^{(1)}_i, s^{(1)}_{i-1})$, then we get that sequences $R(f_1)(i) \cup R(g_2)/\{s^{(2)}_{m_2}\}$ and $R(g_1)(i) \cup R(f_2)/\{r^{(2)}_{n_2}\}$ are compatible by inductive hypothesis since $R(g_1)(i) \preccurlyeq_{alt} R(f_1)(i)$ and $R(g_2)/\{s^{(2)}_{m_2}\} \preccurlyeq_{alt} R(f_2)/\{r^{(2)}_{n_2}\}$. If $r^{(2)}_{n_2}, s^{(2)}_{m_2-1} \in (s^{(1)}_i, r^{(1)}_i)$, then we exchange points $s^{(2)}_{m_2-1}$ and $r^{(1)}_i$, i.e., $s^{(2)}_{m_2-1}$ is the i-th largest zero of $f^{(1)}$. Note that $R(g_1)/R(g_1)(i) \preccurlyeq R(f_1)/R(f_1)(i)$. By Proposition 3.1, we have $R(g_1)/R(g_1)(i) \cup \{r^{(2)}_{n_2}\}$ and $R(f_1)/R(f_1)(i) \cup \{s^{(2)}_{m_2}\}$ are compatible. Then $R(F)$ and $R(G)$ are compatible.

Case 2. When $g_2 \preccurlyeq_{int} f_2$. If $r^{(2)}_{n_2} \in (s^{(1)}_i, r^{(1)}_i)$ or $(r^{(1)}_i, s^{(1)}_{i-1})$, then we get that sequences $R(f_1)(i) \cup R(g_2)$ and $R(g_1)(i) \cup R(f_2)/\{r^{(2)}_{n_2}\}$ are compatible by inductive hypothesis since $R(g_1)(i) \preccurlyeq_{alt} R(f_1)(i)$ and $R(g_2) \preccurlyeq_{alt} R(f_2)/\{r^{(2)}_{n_2}\}$. If $r^{(2)}_{n_2}, s^{(2)}_{m_2-1} \in (s^{(1)}_i, r^{(1)}_i)$, then we also exchange points $s^{(2)}_{m_2-1}$ and $r^{(1)}_i$. Note that $R(g_1)/R(g_1)(i) \preccurlyeq R(f_1)/R(f_1)(i)$. By Remark 3.2, we have $R(g_1)/R(g_1)(i) \cup \{r^{(2)}_{n_2}\}$ and $R(f_1)/R(f_1)(i)$ are compatible. Then $R(F), R(G)$ are compatible and the proof is complete. $\square$

Let $f_1, f_2, \ldots, f_n$ be real polynomials with positive leading coefficients and $f_i \in \mathrm{RZ}$ for all $1 \leqslant i \leqslant n$. Following Fisk [7], we say that polynomials $f_1, f_2, \ldots, f_n$ are *mutually interlacing* if $f_i \preccurlyeq_{alt} f_j$ for all $1 \leqslant i < j \leqslant n$. Then we give a simple proof of the following result.

Corollary 3.4 ([7])**.** *If $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ are two sequences of mutually interlacing polynomials whose leading coefficients have the same sign, then*

$$\sum_{i=1}^{n} f_i g_{n+1-i} = f_1 g_n + f_2 g_{n-1} + \cdots + f_n g_1 \in \mathrm{RZ}.$$

Equivalently, $\sum_{i=1}^{n} f_i(x) g_i(-x) \in \mathrm{RZ}$.

Proof. Without loss of generality, we may assume that f, g are monic. Note that for all $1 \leqslant i < j \leqslant n$, we have $f_i \preccurlyeq_{alt} f_j$ and $g_{n-j} \preccurlyeq_{alt} g_{n-i}$ by the condition. From Theorem 3.3, polynomials $f_i g_{n-i}, f_j g_{n-j}$ are compatible for all $1 \leqslant i < j \leqslant n$. Then polynomials $f_1 g_n, f_2 g_{n-1}, \ldots, f_n g_1$ are pairwise compatible. By Theorem 2.6, we can obtain $f_1 g_n, f_2 g_{n-1}, \ldots, f_n g_1$ are compatible. Hence $f_1 g_n + f_2 g_{n-1} + \cdots + f_n g_1 \in \mathrm{RZ}$. This completes the proof. $\square$

Let $R(f) = (r_1, r_2, \ldots, r_n)_{\geqslant}$. It is obviously that $\frac{f(x)}{(x-r_1)}, \frac{f(x)}{(x-r_2)}, \ldots, \frac{f(x)}{(x-r_n)}$ are mutually interlacing. A special case of Corollary 3.4 is the following.

Corollary 3.5 ([7])**.** *Suppose that $R(f) = (r_1, r_2, \ldots, r_n)_{\geqslant}$ and $R(g) = (s_1, s_2, \ldots, s_n)_{\geqslant}$. Then the polynomial*

$$h(x) = \sum_{i=1}^{n} \left(\frac{f}{x - r_i} \right) \left(\frac{g}{x - s_{n+1-i}} \right) \in \mathrm{RZ}.$$

3.2 Wang-Yeh's results

Wang and Yeh [16] established the following result which has been proved to be an extremely useful tool. In fact, it provides a unified approach to unimodality and log-concavity of many well-known sequences in combinatorics. See [16] for detail. A simple proof of this theorem has been given using the method of interlacing zeros by Liu and Wang [10]. Here we give another simple proof using the method of compatible zeros.

Theorem 3.6 ([16])**.** *Let f and g be real polynomials whose leading coefficients have the same sign. Suppose that $f, g \in \mathrm{RZ}$ and $g \preccurlyeq f$. If $ad \leqslant bc$, then the polynomial*

$$F(x) = (ax + b)f(x) + (cx + d)g(x) \in \mathrm{RZ}.$$

Proof. Without loss of generality, we may assume that f, g are monic and have no zeros in common, which implies that they have only simple zeros.

If $ac = 0$, then the statement follows from Theorem 3.3. So let $ac \neq 0$. We distinguish two cases according to the sign of ac.

Case 1. If $ac > 0$, then $-b/a \leqslant -d/c$ by $ad \leqslant bc$. So sequences $R(f) \cup \{-b/a\}$ and $R(g) \cup \{-d/c\}$ are compatible by Proposition 3.1. Then polynomials $(ax + b)f(x)$ and $(cx + d)g(x)$ are compatible. Thus we have $F(x) \in \mathrm{RZ}$.

Case 2. If $ac < 0$, then $-b/a \geqslant -d/c$ by $ad \leqslant bc$. Note that leading coefficients of $(ax + b)f(x)$ and $(cx + d)g(x)$ have opposite sign. By Theorem 2.1, we need to prove that polynomials $\overline{(ax + b)f(x)}$ and $(cx+d)g(x)$ are compatible. Let $R(f) = (r_1, \ldots, r_n)_{\geqslant}$. When $-b/a > r_1$, we exchange these two points, i.e., $-b/a$ is the largest zero of f. By the condition $g \preccurlyeq f$, we have $\bar{f} \preccurlyeq g$. Now the result follows from Case 1.

Thus the proof of the theorem is complete. □

Denote by PF those polynomials in RZ whose coefficients are nonnegative. So all zeros of a polynomial in PF are nonpositive. It is obviously that if $f, g \in \mathrm{PF}$ and $g \preccurlyeq f$, then $f \preccurlyeq xg$. Thus the following corollary is an immediate consequence of Theorem 3.6.

Corollary 3.7 ([16])**.** *Suppose that $f, g \in \mathrm{PF}$ and $g \preccurlyeq_{int} f$. If $ad \geqslant bc$, then the polynomial*

$$G(x) = (ax + b)f(x) + x(cx + d)g(x) \in \mathrm{RZ}.$$

4 Concluding remarks

As shown in Theorem 2.1, we get a necessary and sufficient condition for the compatible property of two polynomials whose leading coefficients have opposite sign. But there is no similar analogue for general k. We only get a sufficient condition in Theorem 2.3, which may be necessary. It is a challenge to give a necessary and sufficient condition for the compatible property of a sequence of k polynomials when leading coefficients have different sign, where $k \geqslant 3$. On the other hand, it is worthwhile to look at those sequences whose associated polynomial has only real zeros, such as some sequences related to graph theory, using the method of compatible zeros, and to discuss deeper relations between the compatible property and the interlacing property of real zeros of polynomials.

Acknowledgments

The author thanks Prof. Y. Wang, who first asked her about the compatible property of real zeros of polynomials and gave helpful suggestions in the preparation of this paper. She thanks Prof. F.M. Dong and B.-X. Zhu for their kind assistance.

The author thanks the anonymous referee for careful reading and valuable suggestions.

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