Large 2-coloured matchings in 3-coloured complete hypergraphs

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Abstract

We prove the generalized Ramsey-type result on large 2-coloured matchings in a 3-coloured complete 3-uniform hypergraph, supporting a conjecture by A. Gyárfás.

1 Introduction and statement of result

In [3], the authors consider generalisations of Ramsey-type problems where the goal is not to find a monochromatic subgraph, but a subgraph that uses "few" colours. In particular, the following theorem is proven:

Theorem 1 ([3, Theorem 13]). *For $k \geqslant 1$, in every 3-colouring of a complete graph with $f(k) = \left\lfloor \frac{7k-1}{3} \right\rfloor$ vertices there is a 2-coloured matching of size k. This is sharp for every $k \geqslant 2$, i.e. there is a 3-colouring of $K_{f(k)-1}$ that does not contain a 2-coloured matching of size k.*

The example that shows the sharpness of the estimate is close to the colouring obtained by first colouring the vertices with the available colours in proportion close to 1 : 2 : 4 and then colouring the edges by the lowest index colour among its endpoints. The analogous question and construction make sense in the case of complete hypergraphs instead of K_n. At the 1. Emléktábla workshop held at Gyöngyöstarján in July 2010, the first nontrivial case of this question (with 3-uniform hypergraphs and 3 colours) was considered. The best known construction in this case is based on the proportion 1 : 3 : 9, and leads to the following conjecture by A. Gyárfás:

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Conjecture 2. *For any t-colouring of the complete r-uniform hypergraph on*

$$n \geqslant kr + \left\lfloor \frac{(k-1)(t-s)}{1+r+\cdots+r^{s-1}} \right\rfloor$$

vertices, there exists a s-coloured matching of size k.

While it is known that the conjecture fails for e.g. $t = 6$ and $s = 2$, several particular cases are open. We consider here only $t = 3$, $r = 3$ and $s = 2$, in which case the conjecture has the form

Theorem 3. *For any 3-colouring of the complete 3-uniform hypergraph on*

$$n \geqslant 3k + \left\lfloor \frac{k-1}{4} \right\rfloor$$

vertices, there exists a 2-coloured matching of size k.

The case $k = 4$ (the first case that is not a trivial consequence of the existing results for the monochromatic problem, see e.g. [1]) was confirmed at the workshop by a team consisting of N. Bushaw, A. Gyárfas, D. Gerbner, L. Merchant, D. Piguet, A. Riet, D. Vu and the author:

Theorem 4 ([2])**.** *For any 3-colouring of the complete 3-uniform hypergraph on 12 vertices there exists a perfect matching that uses at most 2 colours.*

In this paper, we prove Theorem 3 in the following equivalent form:

Theorem 5. *For any 3-colouring of the complete 3-uniform hypergraph on n vertices, there exists a 2-coloured matching of size*

$$m(n) = \left\lfloor \frac{4(n+1)}{13} \right\rfloor. \tag{1}$$

It is easy to check that these indeed are formulations of the same result as

$$n = 3k + \left\lfloor \frac{k-1}{4} \right\rfloor = \left\lfloor \frac{13k-1}{4} \right\rfloor$$

is the smallest integer for which $\left\lfloor \frac{4(n+1)}{13} \right\rfloor \geqslant k$ holds.

2 Proof

In the proof, the set of vertices of the hypergraph will be denoted by V, the colouring will be a function $c : \binom{V}{3} \to \{1, 2, 3\}$, and α, β, γ will be an arbitrary permutation of the colours 1, 2, 3. The colours are shifted cyclically, e.g. if $\alpha = 3$, then $\alpha + 1$ denotes the

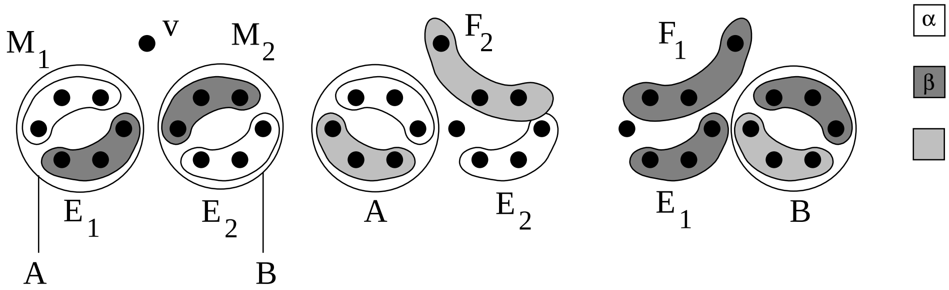

Figure 1: If the substitution of a single vertex v can change the colour of a dominant triple in two spreads of different colour, we have a universal set of size 13. Example: case when $c(E_1) = \beta$ and $c(E_2) = \alpha$.

colour 1 and if $\alpha = 1$, then $\alpha - 1$ denotes the colour 3. A matching on n vertices is *near perfect*, if it has size $\lfloor n/3 \rfloor$.

We call a sextuple A of points α*-dominated* for a colour $\alpha = 1, 2, 3$, if for all splittings $A = B_1 \cup B_2$ into two disjoint triples at least one of $c(B_1) = \alpha$ or $c(B_2) = \alpha$ holds. If A is not α-dominated for any α, we call it *universal*. Similarly, we call a set X of 13 points universal if it admits a near perfect matching in any pair of colours.

The proof proceeds by taking a maximal set of disjoint universal sets $A_1, \dots, A_l$, $X_1, \dots, X_m$ with $|A_i| = 6$ and $|X_j| = 13$. If we can now construct a 2-colour matching on $W = V \setminus (A_1 \cup \dots \cup X_m)$ of the size $m(|W|)$, then we can extend it by the appropriately coloured near perfect matchings in the universal sets A_i and X_j and keep the size of the matching at least $m(|V|)$. Indeed, in the case of an A_i decreasing n by 6 decreases $m(n)$ by at most $\lceil 4 \cdot 6/13 \rceil = 2$ and in the case of an X_j decreasing n by 13 decreases $m(n)$ by 4. Thus by switching to W we may assume that there are no universal sets of size 6 or 13, and the resulting structural properties of the colouring will give us the necessary large 2-colour matching.

If a vertex sextuple is α-dominated, and there are splittings of it into hyperedges of colours α and $\alpha + 1$ as well as into those of colours α and $\alpha + 2$, we call this sextuple a *spread* in colour α, and the splittings are its *demonstration splittings*. Depending of whether the hyperedges of colour α in the demonstration splittings overlap in 1 or 2 vertices, we assign the spread (with a fixed demonstration splitting implied) a level of 1 or 2 respectively.

Lemma 6. *Assume that there are two disjoint spreads A in colour α and B in colour β such that $\alpha \neq \beta$, and let v be an arbitrary vertex in the complement $V \setminus (A \cup B)$ of their union. Then the following property holds for $X = A$ or for $X = B$ (or both): if we substitute v for any vertex of the dominantly coloured triple in either demonstration splitting of the spread X, the colour of that triple stays the same, the dominating colour of X.*

Proof. Indirectly assume that $A = M_1 \cup E_1$ and $B = M_2 \cup E_2$ are splittings which v "spoils". That is, M_1 has the dominant colour α of A and $M_1 \cup \{v\}$ contains a triple F_1

of colour different from $c(M_1) = \alpha$, and analogously $c(M_2) = \beta$ and $M_2 \cup \{v\}$ contains a triple F_2 of colour different from β (see example on Figure 1). Then $A \cup B \cup \{v\}$ is a universal set of size 13. Indeed, it has a matching of size 4 that contains only colours α and β since both A and B admit perfect matchings in these colours. It also has a matching of size 4 that avoids the colour α: the spread B has such a matching of size 2, the triple E_1 has a colour different from α, and the remainder $M_1 \cup \{v\}$ contains F_1, also a triple of a colour different from α. The same argument with A and B reversed produces a near perfect matching that avoids the colour β, proving our claim and arriving at the contradiction that proves the lemma. □

This coupling property implies a very rigid structure of the colouring:

Proposition 7. *If there is a pair of disjoint spreads in two different colours, then there is a nearly perfect matching avoiding one colour.*

Proof. Without loss of generality we may assume that the two colours are 1 and 2. Let the two spreads be $A^{(1)}$ (colour 1) and $A^{(2)}$ (colour 2), and out of all disjoint pairs of spreads of colours 1 and 2 this one contains the most level 2 spreads. Then each of them is either level 2 or it is level 1 and there are no level 2 spreads of their colour that would be disjoint from the other spread.

In both $A^{(1)}$ and $A^{(2)}$, fix two demonstration splittings

$$A^{(i)} = M_+^{(i)} \cup P^{(i)} = M_-^{(i)} \cup N^{(i)}$$

such that $c(M_+^{(i)}) = c(M_-^{(i)}) = i$, $c(P^{(i)}) = i + 1$ and $c(N^{(i)}) = i - 1$. Depending on the level of $A^{(i)}$, we can label the vertices of $A^{(i)} = \{v_1^{(i)}, \ldots, v_6^{(i)}\}$ to satisfy the following equalities:

- in case of level 1:

$$M_+^{(i)} = \{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}\} \qquad P^{(i)} = \{v_4^{(i)}, v_5^{(i)}, v_6^{(i)}\}$$
$$M_-^{(i)} = \{v_1^{(i)}, v_4^{(i)}, v_5^{(i)}\} \qquad N^{(i)} = \{v_2^{(i)}, v_3^{(i)}, v_6^{(i)}\}$$

  We will call the vertex $v_1^{(i)}$ the *dominating* vertex and the rest of the vertices the *core* vertices.

- in case of level 2:

$$M_+^{(i)} = \{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}\} \qquad P^{(i)} = \{v_4^{(i)}, v_5^{(i)}, v_6^{(i)}\}$$
$$M_-^{(i)} = \{v_1^{(i)}, v_2^{(i)}, v_4^{(i)}\} \qquad N^{(i)} = \{v_3^{(i)}, v_5^{(i)}, v_6^{(i)}\}$$

  We will call the vertices $v_1^{(i)}$ and $v_2^{(i)}$ the *dominating* vertices and the rest of the vertices the *core* vertices.

In both cases, $D^{(i)}$ will denote the set of the dominating vertices and $C^{(i)}$ will denote the set of the core vertices. A pair of vertices will be called *critical*, if they are contained in either $M_+^{(i)}$ or $M_-^{(i)}$.

We choose sets $\hat{A}^{(1)}$ and $\hat{A}^{(2)}$ to be a maximal disjoint pair of sets satisfying the following properties:

- $D(i) \subseteq \hat{A}^{(i)} \subseteq V \setminus \left(C^{(i)} \cup A^{(3-i)}\right)$ for $i = 1, 2$.
- For any subset D of $\hat{A}^{(i)}$ of size $|D| = |D^{(i)}|$, the triples of the sextuple $C^{(i)} \cup D = \left(A^{(i)} \setminus D^{(i)}\right) \cup D$ complementary to $P^{(i)}$ and $N^{(i)}$ have colour i.
- For any subset D of $\hat{A}^{(i)}$ of size $|D| = |D^{(i)}|$, any pair of vertices $(u, v) \in V$ that is covered by the complement of either $P^{(i)}$ or $N^{(i)}$ in the sextuple $C^{(i)} \cup D = \left(A^{(i)} \setminus D^{(i)}\right) \cup D$, and any vertex $w \in \hat{A}^{(i)} \setminus D$, the triple $\{u, v, w\}$ has colour i.

That is, $\hat{A}^{(i)}$ is a maximal set of vertices (outside of $\hat{A}^{(3-i)}$) extending the set of dominating vertices of $A^{(i)}$ such that we can switch the dominating vertices of $A^{(i)}$ with any two vertices of $\hat{A}^{(i)}$ and still be unable to change the colour of the dominant triples of the modified sextuple by a single vertex change within the set $\hat{A}^{(i)}$. Such sets exist (for example, $D^{(i)}$ satisfies the requirements for $\hat{A}^{(i)}$), and their total size is bounded by $|V|$, so we can choose a maximal pair.

We claim that the sets $\hat{A}^{(1)} \cup \hat{A}^{(2)}$ cover $V \setminus (C^{(1)} \cup C^{(2)})$. Indeed, assume that there is a vertex $w \in V \setminus \left(C^{(1)} \cup C^{(2)} \cup \hat{A}^{(1)} \cup \hat{A}^{(2)}\right)$ such that it cannot be added to either $\hat{A}^{(1)}$ or $\hat{A}^{(2)}$ without violating their defining properties. This means that for $i = 1, 2$ we can switch the vertices in $D^{(i)}$ to some other vertices in $\hat{A}^{(i)}$ in such a way that for the resulting spread $\tilde{A}^{(i)}$ there is a pair of vertices $(u^{(i)}, v^{(i)})$ of a dominating triple such that

$$c(u^{(i)}, v^{(i)}, w) \neq i.$$

This contradicts Lemma 6 for the spreads $\tilde{A}^{(1)}$ and $\tilde{A}^{(2)}$ and the vertex w.

Additionally, these sets are already easy to colour with 2 colours:

Lemma 8. *The vertex set $\hat{A}^{(i)}$ is a clique of colour i.*

Proof. We suppress for brevity the indices $^{(i)}$. If A is level 2, then for any $\{x, y, z\} \subseteq \hat{A}$ we have by definition of $\hat{A}$ the property that z forms triples of colour i with all the critical vertex pairs of $C \cup \{x, y\}$, in particular, with $\{x, y\}$, and we are done.

If A is level 1, recall first that we also assume that there are no spreads of colour i and level 2 in $\hat{A} \cup C$. Indirectly assume furthermore that there is a triple $X = \{x_1, x_2, x_3\} \subseteq \hat{A}$ such that its colour is not i. For symmetry reasons it is enough to check the case when $c(X) = i + 1$. Then $P \cup X$ is covered by two disjoint triples of colour $i + 1$ and must be therefore $i + 1$-dominated - otherwise it would form a universal sextuple contrary to our assumptions. But $P \setminus N = \{v_4, v_5\}$ is a critical pair of vertices and hence $\{x_1, v_4, v_5\}$ has colour i; therefore its complement $Y = \{x_2, x_3, v_6\}$ has colour $i + 1$. This implies that the sextuple $X \cup N = Y \cup \{x_1, v_2, v_3\}$ can be split into colours $c(X) = i + 1$ and

$c(N) = i-1$ as well as into colours $c(Y) = i+1$ and $c(\{x_1, v_2, v_3\}) = i$ (the set $\{v_2, v_3, x_1\}$ is the complement of P in $C \cup \{x_1\}$ with $x_1 \in \hat{A}$), so this sextuple is $i+1$-dominated. Now use the fact that $\{x_2, v_3\}$ is covered by the complement of P in $(C \cup \{x_2\})$ and that $x_3 \in \hat{A} \setminus \{x_2\}$. By the last property of $\hat{A}$ this implies that $c(\{x_2, x_3, v_3\}) = i$, and consequently its complement in $X \cup N$ has colour $i+1$:

$$c(\{x_1, v_2, v_6\}) = i + 1.$$

By definition of $\hat{A}$, the sextuple $\{x_1\} \cup C$ is i-dominant as it cannot be dominant in any other colour. Hence the complement of $\{x_1, v_2, v_6\}$ in it has to have colour i:

$$c(\{v_3, v_4, v_5\}) = i.$$

Also, $\{v_4, v_5\}$ is a critical pair of vertices, so we have

$$c(\{x_1, v_4, v_5\}) = i$$

as well. But this means that $C \cup \{x_1\}$ is a level 2 spread of colour i as evidenced by splitting into $\{x_1, v_4, v_5\} \cup N$ (colours i and $i-1$ respectively) and into $\{v_3, v_4, v_5\} \cup \{x_1, v_2, v_6\}$ (colours i and $i+1$ respectively) - a contradiction with our initial assumption, hence $\hat{A}$ is indeed a clique of colour i as claimed. □

This also implies that $\hat{A}^{(1)} \cup M_+^{(1)}$ is a clique of colour 1 and $\hat{A}^{(2)} \cup M_-^{(2)}$ is a clique of colour 2 (we are adding a vertex or a critical pair of vertices to the appropriate $\hat{A}^{(i)}$). Notice that their complement is the union of the 2-coloured hyperedge $P^{(1)}$ and the 1-coloured hyperedge $N^{(2)}$.

Lemma 9. *If U and W are disjoint cliques of colours 1 and 2 respectively such that $|U| \geqslant 3$ and $|W| \geqslant 3$, then there exists an almost perfect matching in $U \cup W$ in colours 1 and 2.*

Proof. If $|U| + |W| \bmod 3 = |U| \bmod 3 + |W| \bmod 3$, that is, $|U| \bmod 3 + |W| \bmod 3 \leqslant 2$, then taking maximal disjoint sets of hyperedges in U and W separately gives an almost perfect matching in colours 1 and 2.

If this is not the case, then both $|U| \bmod 3$ and $|W| \bmod 3$ are at least 1 and at least one of them is equal to 2; without loss of generality, we may assume that $|U| \equiv 2 \bmod 3$. We claim that there is a hyperedge $E \subset U \cup W$ of colour 1 or 2 with the property that $|U \cap E| = 2$. Indeed, assume indirectly that all triples intersecting U in 2 vertices and W in 1 vertex have colour 3. Since $|U| \geqslant 3$ and $|U| \bmod 3 = 2$, we have $|U| \geqslant 5$. Consider any four distinct vertices $u_1, u_2, u_3, u_4 \in U$ and any two distinct vertices $w_1, w_2 \in W$. Then the set $X = \{u_1, u_2, u_3, u_4, w_1, w_2\}$ is covered by the triples $\{u_1, u_2, w_1\}$ and $\{u_3, u_4, w_3\}$, both of which have to have colour 3. Hence X can only be 3-dominated, consequently at least one of the members of the matching $\{u_1, w_1, w_2\} \cup \{u_2, u_3, u_4\}$ has colour 3. But the triple $\{u_2, u_3, u_4\}$ lies in the clique U and therefore has colour 1, so $c(\{u_1, w_1, w_2\}) = 3$. This implies that for any choice of a vertex $w_3 \in W \setminus \{w_1, w_2\}$ we have on one hand

$$c(\{u_1, w_1, w_2\}) = 3 \text{ and } c(\{u_2, u_3, w_3\}) = 3$$

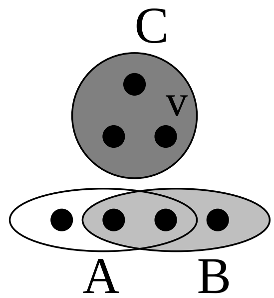

Figure 2: If there are no spreads, then only two colours may be used.

due to the latter triple intersecting U in 2 vertices, and on the other hand

$$c(\{u_1, u_2, u_3\}) = 1 \text{ and } c(\{w_1, w_2, w_3\}) = 2$$

due to U and V being cliques. Hence $\{u_1, u_2, u_3, w_1, w_2, w_3\}$ would be a universal sextuple, a contradiction that proves our claim.

Given a hyperedge $E \subset U \cup W$ of colour 1 or 2 with the property that $|U \cap E| = 2$, we can just add it to the union of any maximal matching of $U \setminus E$ and any maximal matching of $W \setminus E$ to get a nearly perfect matching of $U \cup W$ in colours 1 and 2. □

Applying Lemma 9 to the cliques $\hat{A}^{(1)} \cup M_+^{(1)}$ and $\hat{A}^{(2)} \cup M_-^{(2)}$ and adding the triples $P^{(1)}$ and $N^{(2)}$ yields a near perfect matching in colours 1 and 2 on V. This finishes the proof of Proposition 7. □

Once we can exclude two disjoint spreads of different colours, we have two possibilities: either there are no spreads at all, or there is a spread of, say, colour 1, and any spread in its complement is also of colour 1. We will also assume that $|V| \geqslant 9$ as otherwise the 2-colour condition is trivially fulfilled by any near perfect matching.

Case 1: there are no spreads. If there are no spreads, then no sextuple can contain triples of all three colours: one of them would be dominating, and any two instances of the other two colours could be chosen to be P and N of a spread. We will first look for a pair of triples of different colours that share two vertices, $c(A) \neq c(B)$, $|A \cap B| = 2$. If there are no such pairs, then all triples have the same colour and any nearly perfect matching is monochromatic, we are done. If, on the other hand, such triples A and B exist, we may assume without loss of generality that $c(A) = 1$ and $c(B) = 2$. Could there be triples of colour 3 (see Figure 2)? Any such triple C would have to be disjoint from $A \cup B$, because otherwise their union $A \cup B \cup C$ (together with any other vertex if it has only 5 elements) would form a sextuple of vertices that contains all the three colours. Then for any vertex $v \in C$ the triple $T = (A \cap B) \cup \{v\}$ is covered by both $A \cup C$ (covering only triples of colour 1 and 3) and $B \cup C$ (covering only triples of colour 2 and 3) and therefore can only be of colour 3. But then $A \cup B \cup \{v\}$ together with any other vertex form a sextuple that contains triples of all three colours, A, B and T - a contradiction.

Therefore in this case only two colours may be used at all, so any near perfect matching automatically satisfies our desired condition.

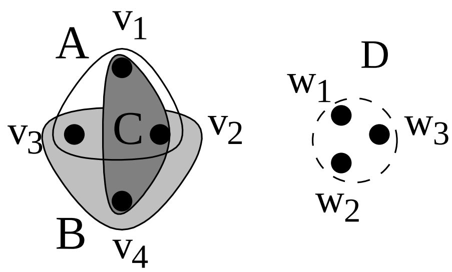

Figure 3: Case $|C \cap B| = 2$.

Case 2: there exists a spread (of colour 1, say). We first investigate what happens if there are no spreads of other colour at all. This results in a highly ordered structure:

Proposition 10. *If a colouring is such that all spreads are of colour* 1*, then either*

- *there exists a near perfect matching avoiding colour* 2 *or colour* 3*, or*
- *there are no triples of colour* 1 *at all.*

Proof. First note that the condition on the spreads means that whenever a sextuple contains triples of all three colours, it is 1-dominated. In particular, if a triple is covered by a disjoint union of a 2-coloured and a 3-coloured triple, it cannot have colour 1 - the union in question can only be 2- or 3-dominated. We show that this statement can also be used for non-disjoint pairs of triples of colours 2 and 3.

Lemma 11. *Assume that all spreads in the colouring are of colour* 1*. Then either*

- *there are no triples of colour* 1 *covered by the union of a triple of colour* 2 *and a triple of colour* 3*, or*
- *there exists a near perfect matching avoiding colour* 2 *as well as one avoiding colour* 3*.*

Proof. Assume $A = \{v_1, v_2, v_3\}$ is a colour 1 triple that is covered by triples B and C of colours 2 and 3 respectively. At least one of these has to cover 2 vertices of A, so after a renumbering of colours, triples and vertices we may assume that $B = \{v_2, v_3, v_4\}$ and $v_1 \in C$. We now have three cases for the situation of C with respect to A and B:

1. $C \cap B = \emptyset$. Then B and C are disjoint triples of colour 2 and 3 respectively which cover A, a triple of colour 1 - a contradiction.

2. $C \cap B = \{v_2, v_4\}$ (or analogously $\{v_3, v_4\}$); that is, C is covered by $A \cup B$ (see Figure 3). The union $A \cup B = A \cup B \cup C$ has 4 elements and contains all three colours, so adding any pair of vertices x, y makes it a 1-dominated sextuple. In this sextuple, the triples $\{x, y, v_1\}$ and $\{x, y, v_3\}$ have non-1-coloured complements, so they have to have colour 1 themselves. Now assume there is a triple $D = \{w_1, w_2, w_3\}$ of colour 2 disjoint from $A \cup B$ (the case of $c(D) = 3$ is similar). Then $D \cup C$ is a disjoint

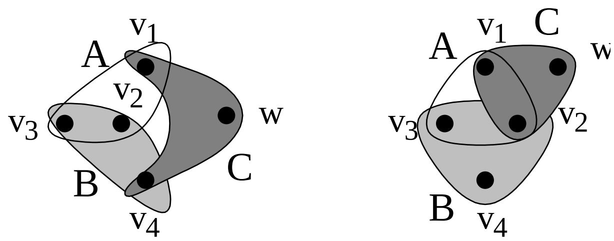

Figure 4: Case $|C \cap B| = 1$.

union of a 2-coloured triple and a 3-coloured one, and it covers the 1-coloured triple $\{w_1, w_2, v_1\}$ - a contradiction. Hence all triples disjoint from $A \cup B$ have colour 1. Consequently we can choose a near perfect matching either in colour 1 only, or at will in colours 1 and 2, or in colours 1 and 3 - if the total number of vertices is congruent to 1 or 2 modulo 3, we take a near perfect matching in the complement of $A \cup B$ and add A, otherwise we take a near perfect matching in the complement of $A \cup B$, add the triple B or C depending on which colour out of 2 and 3 is wanted and match up the remaining two vertices with either v_1 or v_3 (whichever is left out).

3. $|C \cap B| = 1$; let w denote the single vertex in $C \setminus (A \cup B)$ (see Figure 4). By the same argument as before, for any vertex x not in $A \cup B \cup C$ we have that $A \cup B \cup C \cup \{x\}$ is 1-dominated, so the complements of the non-1-coloured triples B and C must have colour 1:

$$c(\{x\} \cup ((A \cup B \cup C) \setminus B)) = 1,$$
$$c(\{x\} \cup ((A \cup B \cup C) \setminus C)) = 1.$$

This makes it impossible to have triples of colour other than 1 disjoint from $A \cup B \cup C$, as they would cover a 1-coloured triple together with either B or C (whichever has the colour other from that of the selected triple). Now taking a maximal matching outside $A \cup B \cup C$, we can extend it to a near perfect matching avoiding the colour 2 or the colour 3 as follows. If there are no vertices left outside the matching, add A to get a 1-coloured matching. If there is 1 vertex left, join it to $(A \cup B \cup C) \setminus B$ and add B to avoid the colour 3; do the same with B and C switched to avoid the colour 2. Finally, if there are 2 vertices left, join them respectively to the disjoint vertex pairs $(A \cup B \cup C) \setminus B$ and $(A \cup B \cup C) \setminus C$ to obtain a matching of colour 1.

□

Thus we may restrict our attention to the case when the union of a triple of colour 2 and a triple of colour 3 cannot cover a triple of colour 1, even if they are not disjoint.

We now try to find a vertex such that all triples containing it are of colour 1; we will call such a vertex 1-*forcing*. If there are no triples of colours 1 and 2 or 1 and 3 such that they intersect in two vertices, then either there are no triples of colour 1 - in which case there is a near perfect matching in colours 2 and 3 - or there are no triples of colour different from 1 - in which case there is a near perfect matching in colour 1. Hence we

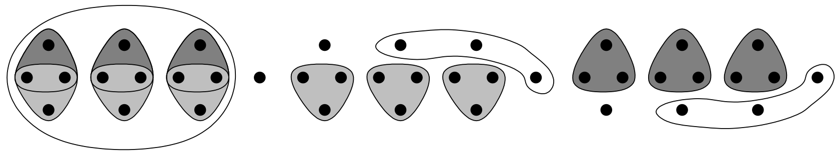

Figure 5: A 1-forcing vertex with 3 disjoint "neighbouring" triples of colours 2 and 3 implies the existence of a universal 13-vertex set.

might assume that there is a pair of triples of the form $A = \{v_1, v_2, v_3\}$, $B = \{v_2, v_3, v_4\}$ with $c(A) = 1$ and $c(B) = 2$, say. By our previous lemma there are no triples of colour 3 that contain v_1. If there are no such triples disjoint from $A \cup B$ either, then any near perfect matching containing A or B will be 1 and 2-coloured. Assume therefore that there is a triple C of colour $c(C) = 3$ in the complement of $A \cup B$. No triple covered by $B \cup C$ can have colour 1, in particular the triple $D = \{v_2, v_3, x\}$ with some $x \in C$ has to have colour 2 or 3. If $c(D) = 3$, then we can repeat the argument with A and D instead of A and B to get that no colour 2 triples contain v_1 either, so v_1 is 1-forcing. If $c(D) = 2$, then $A \cup C$ contains triples of all three colours and thus is 1-dominated, in particular the triple $E = \{v_1\} \cup (C \setminus \{x\}) = (A \cup C) \setminus D$ has to have colour 1 due to its complement having colour 2. In this case, the application of the same argument to E and C gives us the same result of no triples of colour 2 containing v_1 and v_1 is 1-forcing again.

Putting such a 1-forcing vertex aside and repeating the procedure, we end up with a set of 1-forcing vertices and a remainder set where either there are no triples of colour 1 or there is a near perfect matching in colours 1 and 2 or 3. In the latter case, we can just complete the matching with 1-forcing vertices at will, so we assume now that the remainder, denoted henceforth by R, only has triples of colours 2 and 3.

If $R = V$, then we get the second conclusion of our proposition, so we may assume that there is at least one 1-forcing vertex. If, moreover, R had three disjoint pairs of triples of colours 2 and 3 that intersect in 2 vertices, we could add a 1-forcing vertex and get a universal 13-vertex set (see Figure 5) - a contradiction. If there are no three disjoint pairs like that, then after picking at most two of them the rest (denoted by R') has to be a clique, of colour 2, say. We can then take a 1-forcing vertex and add to it those vertices of the triples of colour 3 among the chosen pairs of colour 2 and 3 that are not covered by the corresponding triples of colour 2, and add another vertex from R' if we still don't have three vertices. Choose a near perfect matching from the rest of R containing the selected triples of colour 2 and then cover the rest with 1-forcing vertices if there are any left. This yields a near perfect matching in colours 1 and 2, and finishes the proof of the proposition. □

Since in both cases of Proposition 10 we get a near perfect matching in 2 colours, we only need to consider the case where there exist spreads of a different colour. By symmetry, assume that U is a spread of colour 1 and W is a spread of colour 2. By Proposition 7, we can apply Proposition 10 to $V \setminus U$, so we either get a near perfect matching avoiding colour 2 or 3 or no triples of colour 1 at all. In the first case, we can add one of the demonstration splittings of U to get a near perfect matching of V avoiding either colour 2 or colour 3. The same argument of applying Proposition 10 to $V \setminus W$ yields either a near perfect matching on V in 2 colours or no triples of colour 2 at all. We may hence assume that we got the second result in both attempts, and the colouring is such that all triples of colour 1 intersect U while all triples of colour 2 intersect W.

We see that in such a setup, the vertex set $V \setminus (U \cup W)$ is a clique in colour 3. Additionally, there is at least one triple of colour 3 in U (and in W, but it may not be disjoint from those in U), so if we take a maximal matching in colour 3 that contains a maximal matching of $V \setminus (U \cup W)$, we end up with at most $2+|U|+|W|-|U \cap W|-3 \leqslant 10$ vertices not covered by this matching and consequently only containing triples of colours 1 and 2. We distinguish between three possibilities for the number m of vertices left out:

- $m \leqslant 8$ and $m \neq 6$. By the theorem of Alon, Frankl and Lovász ([1]) there is an almost perfect monochromatic matching in this 2-coloured subgraph: 3 vertices needed for 1 triple, 7 for two triples. Adding it to the initial colour 3 matching, we obtain a near perfect matching of V in 2 colours.

- $m = 6$ or $m = 9$. In this case $|V|$ is a multiple of 3, so either it is at most 12, in which case we apply Theorem 4, or $|V|$ is at least 15, hence the prediction (1) gives a size at least one less than that of a perfect matching. In this latter case, the result of [1] is sufficient (a size 1 matching for $m = 6$ and a size 2 matching for $m = 9$).

- $m = 10$. Here all of our estimates have to be sharp, that is, $|U \cap W| = 1$ and we must have 2 vertices from $V \setminus (U \cup W)$ and 8 vertices from $U \cup W$ not covered by the matching in colour 3. If choosing a different maximal matching in colour 3 leads to a different case, we are done, so we may assume that no matter which 2 vertices a and b of $V \setminus (U \cup W)$ are left out from the initial matching, there do not exist 2 disjoint triples of colour 3 in $U \cup W \cup \{a, b\}$. But any vertex in $U \cup W \setminus (U \cap W)$ lies in the complement of a triple of colour 3 - the elements of $U \setminus W$ miss the colour 3 triple in W and vice versa. Therefore any vertex in $U \cup W \setminus (U \cap W)$ together with any two vertices in $V \setminus (U \cup W)$, and any two vertices in $U \setminus W$ (or $W \setminus U$) together with any vertex in $V \setminus (U \cup W)$, give a hyperedge of colour 1 or 2.

  If now $|V| \leqslant 31$, we can cover all of $V \setminus (U \cap W)$ (at most 30 vertices) by at most 10 such hyperedges (adding a suitable splitting of W or applying Theorem 4 if $|V| = 13$). If, on the other hand, $|V| \geqslant 32$, then the formula (1) predicts a matching at least 1 less than a near perfect one. Such a matching can be found with direct application of [1] to the 10-vertex remainder as before.

In all three cases we arrive at a matching in 2 colours of size at least that predicted by (1), finishing the proof of Theorem 5.

References

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