Invariant Principal Order Ideals under Foata’s Transformation

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Abstract

Let Φ denote Foata’s second fundamental transformation on permutations. For a permutation σ in the symmetric group $S_n$, let $\tilde{\Lambda}_\sigma = \{\pi \in S_n: \pi \preceq_w \sigma\}$ be the principal order ideal generated by σ in the weak order $\preceq_w$. Björner and Wachs have shown that $\tilde{\Lambda}_\sigma$ is invariant under Φ if and only if σ is a 132-avoiding permutation. In this paper, we consider the invariance property of Φ on the principal order ideals $\Lambda_\sigma = \{\pi \in S_n: \pi \preceq \sigma\}$ with respect to the Bruhat order $\preceq$. We obtain a characterization of permutations σ such that $\Lambda_\sigma$ are invariant under Φ. We also consider the invariant principal order ideals with respect to the Bruhat order under Han’s bijection $H$. We find that $\Lambda_\sigma$ is invariant under the bijection $H$ if and only if it is invariant under the transformation Φ.

Keywords: Foata’s second fundamental transformation; Han’s bijection; Bruhat order; principal order ideal

1 Introduction

Let $S_n$ denote the symmetric group on $[n] = \{1, 2, \ldots, n\}$. Foata’s second fundamental transformation Φ on permutations in $S_n$ maps the major index of a permutation π to the inversion number of Φ(π), see Foata [5]. Björner and Wachs [2] have shown that Foata’s second fundamental transformation can also be used in the study of subsets $U$ of $S_n$ over which the inversion number and the major index are equidistributed. In particular, they

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showed that if the subset $U$ is an order ideal of permutations in $S_n$ with respect to the weak order, then $U$ is invariant under $\Phi$ if and only if the maximal elements of $U$ are 132-avoiding permutations.

In this paper, we investigate principal order ideals $\Lambda_\sigma = \{ \pi \in S_n : \pi \leq \sigma \}$ with respect to the Bruhat order $\leq$ that are invariant under Foata’s transformation. We obtain a characterization of permutations $\sigma$ for which $\Lambda_\sigma$ is invariant under $\Phi$.

We also consider the principal order ideals $\Lambda_\sigma$ that are invariant under Han’s bijection $H$ while restricted to permutations. Recall that Han’s bijection, denoted $H$, is a Foata-style bijection defined on words, which can be used to show that the Z-statistic introduced by Zeilberger and Bressoud [7] is Mahonian. We shall show that a principal order ideal $\Lambda_\sigma$ with respect to the Bruhat order is invariant under $H$ if and only if it is invariant under $\Phi$.

Let us give a brief review of Foata’s transformation $\Phi$ and Han’s bijection $H$ on permutations. To describe $\Phi$, we need to define a factorization for permutations on a finite set of positive integers. Let $A$ be a set of $n$ positive integers, and let $x$ be an integer not belonging to $A$. For any permutation $w = w_1 w_2 \cdots w_n$ on $A$, $x$ induces a factorization $w = \gamma_1 \gamma_2 \cdots \gamma_j$, where each subword $\gamma_i \ (1 \leq i \leq j)$ is determined uniquely as follows:

(i) If $w_n < x$, then the last element of $\gamma_i$ is smaller than $x$ and all the remaining elements of $\gamma_i$ are greater than $x$;

(ii) If $w_n > x$, then the last element of $\gamma_i$ is greater than $x$ and all the remaining elements of $\gamma_i$ are smaller than $x$.

For example, let $w = 387125$. Then the factorization induced by $x = 4$ is $w = 38 \cdot 7 \cdot 125$, while the factorization induced by $x = 6$ is $w = 3 \cdot 871 \cdot 2 \cdot 5$, where we use dots to separate the factors.

For a factorization $w = \gamma_1 \gamma_2 \cdots \gamma_j$ induced by $x$, let

$$\delta_x(w) = \gamma'_1 \gamma'_2 \cdots \gamma'_j,$$

where $\gamma'_i \ (1 \leq i \leq j)$ is obtained from $\gamma_i$ by moving the last element to the beginning of $\gamma_i$. For example, for the permutation $w = 387125$, based on the above factorization, we have $\delta_4(w) = 837512$.

To define $\Phi$, we still need the $k$-th ($1 \leq k \leq n$) Foata bijection $\phi_k : S_n \rightarrow S_n$. Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$. For $k = 1$, define $\phi_k(\sigma) = \sigma$. For $k > 1$, define

$$\phi_k(\sigma) = \delta_{\sigma_k}(\sigma_1 \sigma_2 \cdots \sigma_{k-1}) \cdot \sigma_k \sigma_{k+1} \cdots \sigma_n.$$

The transformation $\Phi$ is defined to be the composition $\phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1$, that is, for $\sigma \in S_n$,

$$\Phi(\sigma) = \phi_n(\cdots \phi_2((\phi_1(\sigma))) \cdots).$$
We next describe Han’s bijection $H$ on permutations, and we need two maps $C_x$ and $C^x$ ($x \in [n]$) from the set of permutations on $[n]\{x\}$ to $S_{n-1}$. Assume that $w = w_1w_2\cdots w_{n-1}$ is a permutation on $[n]\{x\}$. Let $\tau_i = w_i - x \pmod{n}$, namely,
\[
\tau_i = \begin{cases} 
    w_i - x + n, & \text{if } w_i < x; \\
    w_i - x, & \text{if } w_i > x,
\end{cases}
\]
and let
\[
\nu_i = \begin{cases} 
    w_i, & \text{if } w_i < x; \\
    w_i - 1, & \text{if } w_i > x.
\end{cases}
\]
The maps $C_x$ and $C^x$ are defined by
\[
C^x(w) = \tau_1\tau_2\cdots\tau_{n-1} \quad \text{and} \quad C_x(w) = \nu_1\nu_2\cdots\nu_{n-1}.
\]

It is easy to check that both $C^x$ and $C_x$ are one-to-one correspondences between the set of permutations on $[n]\{x\}$ and $S_{n-1}$. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$. The bijection $H$ can be defined recursively as follows
\[
H(\sigma) = C^{-1}_{\sigma_n}(H(C^{\sigma_n}(\sigma')))|_{\sigma_n},
\]
where we set $H(1) = 1$ and $\sigma' = \sigma_1\sigma_2\cdots\sigma_{n-1}$. Note that the bijection $H$ can also be described in terms of permutation codes, see Chen, Fan and Li [3].

We now recall the Bruhat order on permutations. To describe this partial order, we need to clarify the definition of the multiplication of permutations. First, we regard a permutation $\pi \in S_n$ as a bijection on $[n]$ by setting $\pi(i) = i_i$. The product $\pi\sigma$ of two permutations $\pi, \sigma \in S_n$ is defined as the composition of $\pi$ and $\sigma$ as functions, that is, $\pi\sigma(i) = \pi(\sigma(i))$ for $i \in [n]$. For $1 \leq i < j \leq n$, let $(i,j)$ denote the transposition of $S_n$ that interchanges the elements $i$ and $j$. Thus, the multiplication on the right of a permutation $\pi$ by a transposition $(i,j)$ has the same effect as interchanging the elements $\pi_i$ and $\pi_j$. Similarly, the multiplication on the left of a permutation $\pi$ by a transposition $(i,j)$ is equivalent to the exchange of the elements $i$ and $j$. For example, for $\pi = 236514$, we have $\pi(2,5) = 216534$ and $(2,5)\pi = 536214$.

The Bruhat order of $S_n$ is defined as follows. For two permutations $\pi, \sigma \in S_n$, we say that $\pi \leq \sigma$ if there exists a sequence of transpositions $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$ such that
\[
\sigma = \pi(i_1, j_1)(i_2, j_2)\cdots(i_k, j_k)
\]
and
\[
\text{inv}(\pi(i_1, j_1)\cdots(i_{t-1}, j_{t-1})) < \text{inv}(\pi(i_1, j_1)\cdots(i_t, j_t)), \quad \text{for } t = 1, 2, \ldots, k,
\]
where inv(\pi) is the inversion number of $\pi$, namely,
\[
\text{inv}(\pi) = |\{(\pi_i, \pi_j) : 1 \leq i < j \leq n \text{ and } \pi_i > \pi_j\}|.
\]
If replacing transpositions by adjacent transpositions in the above definition, then the Bruhat order reduces to the weak order \(\leq_w\). Denote by \(\Lambda_\sigma\) and \(\tilde{\Lambda}_\sigma\) the principal order ideals generated by \(\sigma\) in the Bruhat order and the weak order respectively.

The following theorem gives a characterization of the covering relation in the Bruhat order.

**Theorem 1** (Björner and Brenti [1], Lemma 2.1.4). Let \(\pi, \sigma \in S_n\). Then \(\pi\) is covered by \(\sigma\) in the Bruhat order if and only if \(\sigma = \pi(i,j)\) for some \(1 \leq i < j \leq n\) such that \(\pi_i < \pi_j\) and there does not exist \(k\) such that \(i < k < j\) and \(\pi_i < \pi_k < \pi_j\).

The following theorem is due to Ehresmann [4], see also Björner and Brenti [1, Theorem 2.1.5], which gives a criterion for the comparison of two permutations in the Bruhat order.

**Theorem 2** (Ehresmann [4]). Let \(\pi, \sigma \in S_n\). Then \(\pi \leq \sigma\) if and only if

\[
\pi[i,j] \leq \sigma[i,j]
\]

for any \(1 \leq i, j \leq n\).

This paper is organized as follows. In Section 2, we present a characterization of permutations \(\sigma\) such that the principal order ideals \(\Lambda_\sigma\) are invariant under \(\Phi\). Section 3 is devoted to the proof of the fact that \(\Lambda_\sigma\) is invariant under Han’s bijection \(H\) if and only if it is invariant under Foata’s transformation \(\Phi\).

**2 Invariant principal order ideals under Foata’s map**

The objective of this section is to give a characterization of invariant principal ideals \(\Lambda_\sigma\) under Foata’s transformation \(\Phi\). To this end, we need a property on the maximal elements of the following subset \(\Lambda_\sigma(k)\) of \(\Lambda_\sigma\) which is defined by

\[
\Lambda_\sigma(k) = \{\tau: \tau \in \Lambda_\sigma\text{ and } \tau_n = k\},
\]

where \(1 \leq k \leq n\). It should be noted that by Theorem 2, \(\Lambda_\sigma(k)\) is nonempty unless \(k \geq \sigma_n\). We need to consider a special element in \(\Lambda_\sigma(k)\), denoted \(M(\sigma,k)\). Define \(M(\sigma,k) = \sigma\) if \(k = \sigma_n\). The definition of \(M(\sigma,k)\) for \(k > \sigma_n\) can be described as follows.

Let \(i_1\) be the largest element in \(\sigma\) such that \(i_1\) is to the right of \(k\) and \(\sigma_n \leq i_1 < k\). If \(i_1 = \sigma_n\), then define \(M(\sigma,k) = (k,i_1)\sigma\). Otherwise, we continue to consider the permutation \((k,i_1)\sigma\). Let \(i_2\) be the largest element in \((k,i_1)\sigma\) such that \(i_2\) is to the right of \(k\) and \(\sigma_n \leq i_2 < k\). If \(i_2 = \sigma_n\), then define \(M(\sigma,k) = (k,i_2)(k,i_1)\sigma\). Otherwise, we consider the permutation \((k,i_2)(k,i_1)\sigma\), and let \(i_3\) be the largest element in \((k,i_2)(k,i_1)\sigma\)
such that $i_3$ is to the right of $k$ and $\sigma_n \leq i_3 < k$. Repeating this procedure, we end up with an element $i_s$ ($1 \leq s \leq n$) such that $i_s = \sigma_n$. Define

$$M(\sigma, k) = (k, i_s) \cdots (k, i_2)(k, i_1)\sigma.$$ 

For example, for $\sigma = 875169423$, $M(\sigma, 7)$ is constructed as follows. It is clear that $i_1 = 6$. So, we have $(7, i_1)\sigma = 865179423$. Now we see that $i_2 = 4$, and hence $(7, i_2)(7, i_1)\sigma = 865149723$. Since $i_3 = 3 = \sigma_n$, we find $M(\sigma, 7) = 865149327$.

The following theorem will be employed in the proof of Theorem 5.

**Theorem 3.** Suppose that $\sigma$ is a permutation in $S_n$ and $k$ is an integer such that $\sigma_n \leq k \leq n$. Then, $M(\sigma, k)$ is the unique maximal element of $\Lambda_{\sigma}(k)$, that is, $M(\sigma, k) \geq \tau$ for any $\tau \in \Lambda_{\sigma}(k)$.

**Proof.** It is clear that the theorem holds when $k = \sigma_n$. Now we consider the case $k > \sigma_n$. Assume that

$$M(\sigma, k) = (k, i_s) \cdots (k, i_2)(k, i_1)\sigma.$$ 

Let $\tau \in \Lambda_{\sigma}(k)$, and denote $\sigma^{(1)} = (k, i_1)\sigma$. To prove $\tau \leq M(\sigma, k)$, we first show that

$$\tau \leq \sigma^{(1)}. \quad (1)$$

Let $m$ and $t$ be the indices such that $\sigma_m = k$ and $\sigma_t = i_1$. By the choice of $i_1$, we see that $1 \leq m < t \leq n$. It is easy to verify the following relation

$$\sigma^{(1)}[i, j] = \begin{cases} \sigma[i, j] - 1, & \text{if } m \leq i < t \text{ and } i_1 < j \leq k; \\ \sigma[i, j], & \text{otherwise}. \end{cases} \quad (2)$$

Since $\tau[i, j] \leq \sigma[i, j]$ for $1 \leq i, j \leq n$, we see that (1) can be deduced from the following relation

$$\tau[i, j] < \sigma[i, j], \quad \text{for } m \leq i < t \text{ and } i_1 < j \leq k. \quad (3)$$

We now proceed to prove (3). We shall present detailed argument for the case $i = m$ and $j = i_1 + 1$ and the remaining cases can be dealt with in the same vein. Suppose to the contrary that $\tau[m, i_1 + 1] = \sigma[m, i_1 + 1]$. By the choice of $i_1$, we see that the elements $i_1 + 1, i_1 + 2, \ldots, k - 1$ in $\sigma$ are all to the left of $\sigma_m$. Thus, we get

$$\sigma[m, i_1 + 2] = \sigma[m, i_1 + 1] - 1.$$ 

It follows that

$$\tau[m, i_1 + 2] \geq \tau[m, i_1 + 1] - 1 = \sigma[m, i_1 + 1] - 1 = \sigma[m, i_1 + 2]. \quad (4)$$

On the other hand, since $\tau \leq \sigma$, we see that $\tau[m, i_1 + 2] \leq \sigma[m, i_1 + 2]$. Hence,

$$\tau[m, i_1 + 2] = \sigma[m, i_1 + 2].$$
In a similar fashion, we find that
\[ \tau[m, i_1 + 3] = \sigma[m, i_1 + 3], \quad \tau[m, i_1 + 4] = \sigma[m, i_1 + 4], \quad \ldots, \quad \tau[m, k] = \sigma[m, k]. \]

From the relation \( \tau[m, k] = \sigma[m, k] \), we assert that the number \( k \) belongs to the set \( \{ \tau_1, \tau_2, \ldots, \tau_m \} \). This can be seen as follows. Suppose to the contrary that \( k \not\in \{ \tau_1, \tau_2, \ldots, \tau_m \} \). Then, we have \( \tau[m, k + 1] = \tau[m, k] \). But, since \( \sigma_m = k \), we get \( \sigma[m, k + 1] = \sigma[m, k] - 1 \). Thus we obtain that \( \tau[m, k + 1] > \sigma[m, k + 1] \), a contradiction.

Obviously, the above assertion that \( k \in \{ \tau_1, \tau_2, \ldots, \tau_m \} \) is contrary to the assumption that \( \tau_n = k \). Thus, relation (3) holds for \( i = m \) and \( j = i_1 + 1 \). This completes the proof of the relation in (1).

Using the same argument, we can deduce that
\[ \tau \leq (k, i_2)\sigma^{(1)}. \]

Repeating this procedure, we finally get
\[ \tau \leq (k, i_s) \cdots (k, i_2)\sigma^{(1)} = M(\sigma, k). \]

This completes the proof.

Before stating our main theorem, we need a result of Björner and Wachs [2] on the weak order. Let \( \leq_w \) denote the weak order on permutations. For a given permutation \( \sigma \in S_n \), they constructed a permutation \( \gamma(\sigma) \in S_n \) such that \( \gamma(\sigma) \leq_w \sigma, \text{inv}(\sigma) \leq \text{maj}(\gamma(\sigma)) \), and the last element of \( \gamma(\sigma) \) is equal to \( \sigma_n \). Moreover, they showed that \( \text{inv}(\sigma) = \text{maj}(\gamma(\sigma)) \) if and only if \( \sigma \) is 132-avoiding. Recall that a permutation \( \sigma \) is 132-avoiding if there do not exist numbers \( 1 \leq a < b < c \leq n \) such that \( \sigma_a < \sigma_c < \sigma_b \). We shall not give the precise definition of \( \gamma(\sigma) \), since we only require the properties of \( \gamma(\sigma) \) as mentioned above.

The following lemma will be used in the proof of Theorem 5.

**Lemma 4.** Suppose that \( \sigma \in S_n \) is a permutation such that \( \Lambda_{\sigma} \) is invariant under Foata’s map \( \Phi \). Then \( \sigma \) is a 132-avoiding permutation.

**Proof.** Since \( \Lambda_{\sigma} \) is invariant under \( \Phi \), it is easy to see that \( \Phi(\gamma(\sigma)) \leq \sigma \). Thus we have
\[ \text{inv}(\Phi(\gamma(\sigma))) \leq \text{inv}(\sigma). \] (5)

Recall that \( \text{inv}(\Phi(\gamma(\sigma))) = \text{maj}(\gamma(\sigma)) \). Hence, by (5), we deduce that \( \text{maj}(\gamma(\sigma)) \leq \text{inv}(\sigma) \). On the other hand, as we have mentioned above, \( \gamma(\sigma) \) possesses the property that \( \text{maj}(\gamma(\sigma)) \geq \text{inv}(\sigma) \). So we get \( \text{maj}(\gamma(\sigma)) = \text{inv}(\sigma) \). In other words, \( \sigma \) is 132-avoiding. This completes the proof.

We are now ready to state the main result in this paper.

**Theorem 5.** Suppose that \( \sigma \) is a permutation in \( S_n \). Then the following assertions hold:

1. If \( \sigma_n = n \), then \( \Lambda_{\sigma} \) is invariant under \( \Phi \) if and only if \( \Lambda_{\sigma'} \) is invariant under \( \Phi \), where
   \[ \sigma' = \sigma_1 \sigma_2 \cdots \sigma_{n-1}. \]
If $\sigma_n = k < n$, then $\Lambda_\sigma$ is invariant under $\Phi$ if and only if

$$\sigma = n (n-1) \cdots (k+1)(k-1) \cdots 2 1 k,$$

where $\sigma = n (n-1) \cdots 2 1$ for $k = 1$.

**Proof.** We first prove (1). Assume that $\Lambda_\sigma$ is invariant under Foata's map $\Phi$. To prove that $\Lambda_\sigma'$ is invariant under $\Phi$, let $\pi'$ be a permutation in $S_{n-1}$ such that $\pi' \leq \sigma'$. It is easy to see that $\pi' n \leq \sigma' n = \sigma$. Thus, $\Phi(\pi'n) \leq \sigma$. Since $\Phi(\pi'n) = \Phi(\pi)n$, we have

$$\Phi(\pi)n \leq \sigma.$$

By the fact that $\sigma_n = n$, we find $\Phi(\pi') \leq \sigma'$. Hence $\Lambda_\sigma'$ is invariant under $\Phi$. Conversely, it can be easily checked that if $\Lambda_\sigma'$ is invariant under $\Phi$, then $\Lambda_\sigma$ is invariant under $\Phi$.

Next, we proceed to prove (2). In this case, $\sigma_n = k < n$. Assume that $\sigma$ is a permutation in (6). It follows from Theorem 2 that

$$\Lambda_\sigma = \{ \pi \in S_n : \pi_n \geq k \}.$$

Let $\pi$ be a permutation in $\Lambda_\sigma$. Since the last element of $\Phi(\pi)$ is equal to $\pi_n$, from (7) we see that $\Phi(\pi)$ belongs to $\Lambda_\sigma$. So we deduce that $\Phi(\Lambda_\sigma) \subseteq \Lambda_\sigma$. Since $\Phi$ is a bijection, we have $\Phi(\Lambda_\sigma) = \Lambda_\sigma$, that is, $\Lambda_\sigma$ is invariant under $\Phi$.

It remains to prove the reverse direction of (2), that is, if $\Lambda_\sigma$ is invariant under $\Phi$, then $\sigma$ is a permutation of form (6). We have the following two cases.

**Case 1:** $\sigma_n = k = n - 1$. We use induction on $n$. Assume that the assertion in (2) is true for permutations in $S_{n-1}$. Let $\sigma$ be a permutation in $S_n$ such that $\Lambda_\sigma$ is invariant under $\Phi$.

By Lemma 4, we see that $\sigma$ is 132-avoiding. It is readily checked that $\sigma_1 = n$. Let $\tau = (n-1,n)\sigma$. Evidently,

$$\tau = M(\sigma,n) \quad \text{and} \quad \Lambda_\tau = \{ \pi : \pi \leq \sigma, \pi_n = n \},$$

which implies that $\Lambda_\tau$ is invariant under $\Phi$. Since $\tau_n = n$, by Part (1) of the theorem, we obtain that $\Lambda_\tau'$ is invariant under $\Phi$, where $\tau' = \tau_1 \tau_2 \cdots \tau_{n-1}$. Therefore, by the induction hypothesis, we deduce that

$$\tau_1 \tau_2 \cdots \tau_{n-1} = \begin{cases} (n-1)(n-2) \cdots (i+1)(i-1) \cdots 2 1 i, & \text{if } \tau_{n-1} = i > 1; \\ (n-1)(n-2) \cdots 2 1, & \text{if } \tau_{n-1} = 1. \end{cases}$$

Now, we claim that $\tau_{n-1} = 1$. Suppose to the contrary that $\tau_{n-1} = i > 1$. Then we have

$$\tau = (n-1)(n-2) \cdots (i+1)(i-1) \cdots 2 1 i n,$$

and so

$$\sigma = n(n-2) \cdots (i+1)(i-1) \cdots 2 1 i(n-1).$$
Consider the permutation
\[ \pi = (n-2) \cdots (i+1) i (i-1) \cdots 2 1 n (n-1). \]

Clearly,
\begin{align*}
\pi & \leq (n-2) \cdots (i+1) i (i-1) \cdots 2 1 n (n-1) \\
& \leq n (n-2) \cdots (i+1) (i-1) \cdots 2 1 i (n-1) \\
& = \sigma.
\end{align*}

However,
\[ \Phi(\pi) = n (n-2) \cdots (i+1) i (i-1) \cdots 2 1 (n-1) \not\leq \sigma, \]
which contradicts the assumption that \( \Lambda_\pi \) is invariant under \( \Phi \). This completes the proof of the above claim that \( \tau_{n-1} = 1 \).

We now arrive at the conclusion that \( \tau = (n-1) (n-2) \cdots 2 1 n \). Hence
\[ \sigma = (n-1, n) \tau = n (n-2) \cdots 2 1 (n-1), \]
as required.

Case 2: \( \sigma_n = k < n-1 \). The proof is by induction on \( \sigma_n \). Assume that the assertion is true for \( \sigma_n > k \). We now consider the case \( \sigma_n = k \).

To apply the induction hypothesis, we need to consider the principal order ideal generated by
\[ w = M(\sigma, k+1) = (k, k+1)\sigma. \]

We first show that the principal order ideal \( \Lambda_w \) has the following form
\[ \Lambda_w = \{ \pi \in \Lambda_\sigma : \pi_n \geq k+1 \}. \tag{8} \]

It is clear that \( \Lambda_w \subseteq \{ \pi \in \Lambda_\sigma : \pi_n \geq k+1 \} \). It remains to show that
\[ \{ \pi \in \Lambda_\sigma : \pi_n \geq k+1 \} \subseteq \Lambda_w. \tag{9} \]

To prove (9), we need the permutation \( \gamma(w) \). Keep in mind that the last element of \( \gamma(w) \) is equal to \( w_n \). This implies that \( \gamma(w) \in \Lambda_\sigma(k+1) \). It follows that \( \Phi(\gamma(w)) \in \Lambda_\sigma(k+1) \). By Theorem 3, we see that \( \Phi(\gamma(w)) \leq M(\sigma, k+1) = w \). Using the arguments in the proof of Lemma 4, we deduce that \( w \) is 132-avoiding. Hence we conclude \( \sigma^{-1}(i) < \sigma^{-1}(k+1) \) for \( i > k+1 \).

In view of the construction of \( M(\sigma, i) \), for any \( i > k+1 \), there exist integers \( i_1 > i_2 > \cdots > i_{s-2} > k+1 \) such that
\begin{align*}
M(\sigma, i) &= (i, k, k+1)(i, i_{s-2}) \cdots (i, i_1)\sigma \\
& \leq (i, k+1)M(\sigma, i) \\
& = (k, k+1)(i, i_{s-2}) \cdots (i, i_1)\sigma \\
& \leq (k, k+1)\sigma.
\end{align*}
Thus, for \( i \geq k + 1 \) and \( \pi \in \Lambda_\sigma(i) \), we have
\[
\pi \leq M(\sigma, i) \leq (k, k + 1)\sigma = w,
\]
which implies the relation (9). Hence the proof of (8) is complete.

From (8), we see that \( \Lambda_w \) is invariant under \( \Phi \). By the induction hypothesis, we get
\[
w = n (n-1) \cdots (k+2) (k-1) \cdots 2 \cdot 1 (k+1),
\]
which yields that
\[
\sigma = n (n-1) \cdots (k+2) (k+1) (k-1) \cdots 2 \cdot 1 \cdot k.
\]
This completes the proof.

Theorem 5 has the following consequence.

**Corollary 6.** There are \( \binom{n}{2} + 1 \) permutations \( \sigma \) in \( S_n \) such that the principal order ideals \( \Lambda_\sigma \) are invariant under Foata’s transformation \( \Phi \).

**Proof.** Let \( a_n \) denote the number of principal order ideals in \( S_n \) that are invariant under \( \Phi \). By Theorem 5, it is easy to derive the following recurrence relation
\[
a_n = a_{n-1} + n - 1.
\]
Since \( a_1 = 1 \), the formula for \( a_n \) is easily verified. This completes the proof.

For example, for \( n = 3 \), there are four permutations \( \sigma \) for which \( \Lambda_\sigma \) is invariant under \( \Phi \): 123, 213, 312, 321. The following two figures are the Hasse diagrams of \( (S_3, \leq) \) and \( (S_3, \leq_w) \). The permutations \( \sigma \) such that \( \Lambda_\sigma \) is \( \Phi \)-invariant are written in boldface in Figure 1, and the permutations \( \sigma \) such that \( \tilde{\Lambda}_\sigma \) is \( \Phi \)-invariant are written in boldface as well in Figure 2.

![Figure 1: \( (S_3, \leq) \)](image1)

![Figure 2: \( (S_3, \leq_w) \)](image2)
3 Invariant principal ideals under Han’s map

In this section, we show that in the Bruhat order, Han’s bijection \( H \) has the same invariant principal order ideals as Foata’s transformation \( \Phi \).

**Theorem 7.** Let \( \sigma \) be a permutation in \( S_n \). Then \( \Lambda_\sigma \) is invariant under \( H \) if and only if it is invariant under \( \Phi \).

**Proof.** We first show that if \( \Lambda_\sigma \) is invariant under \( \Phi \) then it is invariant under \( H \). We have the following two cases.

Case 1: \( \sigma_n = n \). We use induction on \( n \). Assume that the assertion is true for \( S_{n-1} \). By Theorem 5, we see that \( \Lambda_{\sigma'} \) is invariant under \( \Phi \), where \( \sigma' = \sigma_1 \sigma_2 \cdots \sigma_{n-1} \). Thus, by the induction hypothesis, we obtain that \( \Lambda_{\sigma'} \) is invariant under \( H \).

Notice that
\[
\Lambda_\sigma = \{ \tau n : \tau \in \Lambda_{\sigma'} \}.
\]

Since \( H(\pi n) = H(\pi)n \) for any permutation \( \pi \in S_{n-1} \), we deduce that \( \Lambda_\sigma \) is invariant under \( H \).

Case 2: \( \sigma_n = k < n \). Again, by Theorem 5, we see that
\[
\sigma = n (n - 1) \cdots (k + 1)(k - 1) \cdots 21 \cdot k,
\]

which implies that \( \Lambda_\sigma = \{ \pi \in S_n : \pi_n \geq k \} \). Since the map \( H \) preserves the last element of a permutation in \( S_n \), we see that \( \Lambda_\sigma \) is invariant under \( H \).

Conversely, assume that \( \Lambda_\sigma \) is invariant under \( H \). We wish to show that \( \Lambda_\sigma \) is invariant under \( \Phi \). We also have the following two cases.

Case 1: \( \sigma_n = n \). We proceed to use induction on \( n \). Assume that the assertion is true for \( S_{n-1} \). It is easy to see that
\[
\Lambda_\sigma = \{ \tau n : \tau \in \Lambda_{\sigma'} \},
\]

where \( \sigma' = \sigma_1 \sigma_2 \cdots \sigma_{n-1} \). So we have
\[
H(\Lambda_\sigma) = \{ H(\tau n) : \tau \in \Lambda_{\sigma'} \} = \{ H(\tau) n : \tau \in \Lambda_{\sigma'} \}.
\]

By the assumption that \( \Lambda_\sigma \) is invariant under \( H \), we see that \( \Lambda_{\sigma'} \) is invariant under \( H \). Thus, by the induction hypothesis, we find that \( \Lambda_{\sigma'} \) is invariant under \( \Phi \). Therefore,
\[
\Phi(\Lambda_\sigma) = \{ \Phi(\tau n) : \tau \in \Lambda_{\sigma'} \} = \{ \Phi(\tau) n : \tau \in \Lambda_{\sigma'} \} = \{ \tau n : \tau \in \Lambda_{\sigma'} \} = \Lambda_\sigma.
\]

So we arrive at the assertion that \( \Lambda_\sigma \) is invariant under \( \Phi \).

Case 2: \( \sigma_n = k < n \). We claim that
\[
\sigma = n (n - 1) \cdots (k + 1)(k - 1) \cdots 1 \cdot k.
\]
To prove (10), we use induction on $k$. We first verify that (10) is valid for $k = n - 1$. In this case, by an argument similar to the proof of Lemma 4, we may deduce that $\sigma$ is $132$-avoiding. This implies that $\sigma_1 = n$. Assume that $\sigma_{n-1} = i$. It is easy to check that

$$H(\sigma)_{n-1} = i + 1, \quad H^2(\sigma)_{n-1} = i + 2, \quad \ldots, \quad H^{n-i-1}(\sigma)_{n-1} = n, \quad H^{n-i}(\sigma)_{n-1} = 1.$$ 

Since $\Lambda_\sigma$ is invariant under $H$, we see that $H^{n-i}(\sigma) \leq \sigma$, which implies $i = 1$. Thus we have

$$\sigma = n \sigma_2 \cdots \sigma_{n-2} 1 (n-1).$$

To show that $\sigma$ is a permutation of form (10), we notice that the last element of the permutation $\tau = (n, n-1)\sigma$ is $n$. Clearly, $\Lambda_\tau = \{\pi: \pi \in \Lambda_\sigma, \pi_n = n\}$. So we deduce that $\Lambda_\tau$ is invariant under $H$. Let $\tau' = \tau_1 \cdots \tau_{n-1}$. Since $\tau_n = n$, by the assertion in Case 1, we see that $\Lambda_{\tau'}$ is invariant under $H$. Therefore, by the induction hypothesis, we conclude that $\Lambda_{\tau'}$ is invariant under $\Phi$. In view of Theorem 5, we have

$$\tau' = (n-1) (n-2) \cdots 2 1,$$

and hence

$$\tau = \tau' n = (n-1) (n-2) \cdots 2 1 n,$$

which yields that $\sigma = n (n-2) \cdots 2 1 (n-1)$. This completes the proof of (10) in the case $k = n - 1$.

We next consider the case $\pi_n = k < n - 1$. Let

$$\tau = (k, k+1)\sigma.$$

It is clear that $\tau = M(\sigma, k+1)$. By an argument analogous to the proof of relation (8), we deduce that

$$\Lambda_\tau = \{\pi: \pi \leq \sigma, \pi_n \geq k + 1\},$$

from which it is easily seen that $\Lambda_\tau$ is invariant under $H$. Since $\tau_n > k$, by the induction hypothesis, we get

$$\tau = n (n-1) \cdots (k+2) k (k-1) \cdots 2 1 (k+1).$$

It follows that

$$\sigma = (k, k+1)\tau = n (n-1) \cdots (k+2) (k+1) (k-1) \cdots 2 1 k.$$

Thus the proof of (10) is complete.

By Theorem 5, we see that $\Lambda_\sigma$ is invariant under $\Phi$. This completes the proof.

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References


