The structure of colored complete graphs free of proper cycles

Vincent Coll
Department of Mathematics
Lehigh University
Bethlehem, PA, U.S.A.
vec208@lehigh.edu

Colton Magnant
Department of Mathematical Sciences
Georgia Southern University
Statesboro, GA, U.S.A.
cmagnant@georgiasouthern.edu

Kathleen Ryan
Department of Mathematics
Lehigh University
Bethlehem, PA, U.S.A.
kmr207@lehigh.edu

Submitted: May 2, 2012; Accepted: Nov 26, 2012; Published: Dec 6, 2012
Mathematics Subject Classifications: 05C15, 05C38

Abstract

For a fixed integer m, we consider edge colorings of complete graphs which contain no properly edge colored cycle C_m as a subgraph. Within colorings free of these subgraphs, we establish a global structure by bounding the number of colors that can induce a spanning and connected subgraph. In the case of small cycles, namely C_4, C_5, and C_6, we show that our bounds are sharp.

Keywords: proper coloring; forbidden subgraph; monochromatic subgraph

1 Introduction

This work considers edge colorings of complete graphs K_n on n vertices which contain no properly edge colored cycle of length m as a subgraph, where $m \geqslant 3$ is an integer. Within edge colorings free of a properly colored C_m, we establish a global structure on the coloring by bounding the number of colors that can induce a spanning and connected subgraph. For small m, namely for C_4, C_5, and C_6, we show that our bounds are the best possible. We tacitly assume that by *coloring*, we mean a partition of $E(K_n)$ into parts called *color classes*. A subgraph is called *rainbow* if all of its edges are colored distinctly while a subgraph is called *proper* (elsewhere called "alternating" [1]) if no two adjacent edges receive the same color.

This project is primarily motivated by the following result of Gyárfás et al, who translated the work of Gallai, recasting his work on oriented graphs to be sensible in the context of graph coloring (see [6] and [7]).

Theorem 1 (Gallai [6], Gyárfás et al. [7])**.** *A coloring of K_n is rainbow triangle free if and only if there exists a partition of the vertices into at least two non-empty parts such that between each pair of parts, all edges have a single color, between the parts in general, the edges come from only two colors and within each part, the edges are colored to avoid rainbow triangles.*

This theorem is a strong structural result demonstrating how restricted the structure of an edge colored complete graph K_n is if its coloring is known to be rainbow triangle free. Theorem 1 has naturally led to an investigation of the structure of colorings of K_n which are free of rainbow subgraphs other than triangles (see [2] and [3]).

In this work, we generalize the rainbow triangle free results, but do so by investigating colorings of K_n which are free of proper cycles of length longer than three. For a coloring G of K_n, we let G^i denote the subgraph of G induced on color i, and we say that color i is *spanning* and *connected* if G^i is a spanning and connected subgraph of K_n. The following theorem suggests the form of the structural results that we seek.

Theorem 2 (Gallai [6], Gyárfás et al. [7])**.** *If a coloring of K_n is rainbow triangle free, then there are at most two colors which are spanning and connected.*

This theorem shows that a coloring of K_n with three or more spanning and connected colors contains a rainbow triangle. In the translation of Gallai's results, Theorem 2 was the main tool in the proof of Theorem 1. Along these lines, our goal is to obtain analogs of Theorem 2 for proper-C_m free colorings of K_n so as to find structural results as strong as Theorem 1 for such graphs. In Sections 2, 3, and 4, we determine the number of spanning and connected colors which respectively force the existence of a proper C_4, C_5, or C_6. In Section 5, we obtain more general results by finding upper and lower bounds for the number of spanning and connected colors required to ensure the existence of a proper C_m, for $m \geqslant 4$.

Note that a survey of Gallai's results, as well as many others related to rainbow subgraphs of complete graphs, can be found in [4] with an updated version maintained at [5].

When it is convenient, we use names of colors like "red" or "green".

2 Proper-C_4 free Colorings

The first result of this section provides some structure to colorings of K_n that are free of proper even cycles C_{2m}. When $m = 2$, this result provides a first insight into the structure of proper-C_4 free colorings.

Theorem 3. *For $m \geqslant 2$, if G is a proper-C_{2m} free coloring of K_n, then the sum of the diameters of the components of G^i is at most $2(m-1)$. In particular, when $m = 2$, if G*

is a proper-C_4 free coloring of K_n, then with the exception of isolated vertices, each color in G is connected with diameter at most two.

Proof. This proof is by contradiction. Consider a coloring of K_n with no proper C_{2m} and suppose that G^1 is spanning and connected but that there exist vertices a and b at distance at least $2m-1$ in G^1. Without loss of generality, choose a and b so that the distance between them is exactly $2m-1$. Let P be a shortest path from a to b. Since the path P is a shortest $a-b$ path in color 1, all chords of this path must not have color 1. Label the vertices of P in order as $a = a_1, a_2, \ldots, a_{2m} = b$. Indices will be considered modulo $2k$.

Now, suppose m is even. Let C be a proper cycle constructed from P as follows: For $i \geqslant 0$, consider the segment $a_{4i+1}a_{4i+2}a_{4i+3}a_{4i+4}a_{4i+5}$. This segment is replaced by the path segment $P_i = a_{4i+1}a_{4i+2}a_{4i+4}a_{4i+3}a_{4i+5}$. By construction, this path segment P_i is proper. If this process is repeated for all $i \leqslant m/2-1$, it produces a proper C_{2m} for a contradiction. Note that the case when $i = m/2-1$ produces the proper segment $a_{2m-3}a_{2m-2}a_{2m}a_{2m-1}a_1$.

If m is odd, we employ a similar process: For $i \leqslant (m-1)/2-1$, we make the same switches. For the final segment, when $i = (m-1)/2$, we use is $a_{2m-1}a_{2m}a_1$ to complete a proper C_{2m} for a contradiction.

A similar argument within and between components shows the result holds when G^i is disconnected but the sum of the diameters of the components is greater than $2(m-1)$. □

Note that a result like Theorem 3 does not hold in general for proper-C_{2m+1} free colorings of K_n. Consider a red path on $n-2$ vertices and a single isolated edge colored in green. Color all other edges in this graph blue to yield a coloring of K_n. This coloring contains no properly colored odd cycle of any length since every other edge in any proper cycle C must be blue and thus C must be even. On the other hand, the subgraph induced on the red edges in this coloring has diameter $n-3$, thus demonstrating that we can find colorings of K_n with no proper cycles of odd length and with at least one color with large diameter. Applying Theorem 3, we obtain the analog of Theorem 2 for proper-C_4 free colorings of K_n. The next theorem implies that any coloring containing two or more spanning and connected colors forces the existence of a proper C_4.

Theorem 4. *For $n \geqslant 4$, if G is a proper-C_4 free coloring of K_n, then G has at most one spanning and connected color.*

Proof. The proof is by induction on n. First, suppose $n = 4$. The only way for two colors to be spanning and connected in a colored K_4 is to have a P_4 in color 1 and a P_4 (the complement of the first P_4) in color 2. Then neither color induces a graph of diameter at most two, contradicting Theorem 3.

Now, suppose the result holds for colorings of K_{n-1}, and let G be a coloring of K_n. Suppose colors 1 and 2 are both spanning and connected in G. By induction, if we remove any vertex v, we are left with at most one spanning and connected color, say color 1. Then, in G^2, the vertex v is a cut vertex. Since the diameter of G^2 is at most two by Theorem 3, the vertex v must be adjacent to all of $G \setminus \{v\}$ in color 2. This means that color 2 is the only spanning and connected color in G. □

This theorem establishes that, in a proper-C_4 free coloring of a complete graph, there is at most one spanning and connected color. On the other hand, there may not even be one such color, as seen in the following example: Consider three sets A_1, A_2, and A_3, each of order $n/3$ (see Figure 1). Color all edges contained in each set A_i with color i and all edges from A_i to A_{i+1} with color i where indices are taken modulo 3. This example is proper-C_4 free but has no spanning and connected color.

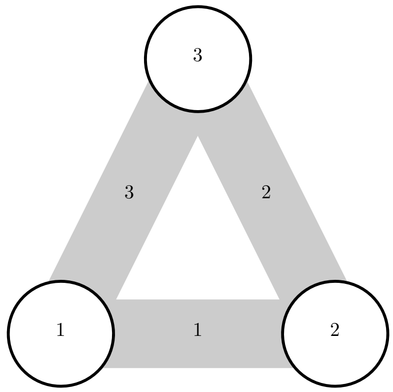

Figure 1: Coloring with no proper C_4 and no spanning and connected color

3 Proper-C_5 free Colorings

We now obtain the analog of Theorem 2 for proper-C_5 free colorings of K_n. Theorem 5 shows that any coloring containing three or more spanning and connected colors forces the existence of a proper C_5.

Theorem 5. *For $n \geqslant 5$, if G is a proper-C_5 free coloring of K_n, then G has at most two spanning and connected colors.*

Proof. The proof is by induction on n. Suppose G is a proper-C_5 free coloring of K_n for $n \geqslant 5$ in which at least 3 colors, say colors 1, 2, and 3, are spanning and connected. For a base case, if $n = 5$, each spanning and connected color needs at least $n - 1 = 4$ edges so there cannot be three spanning and connected colors on $e(K_5) = 10$ edges. Thus, we may assume $n \geqslant 6$. First, we show that the graph induced on each of colors 1, 2, and 3 has small diameter.

Claim 6. *The diameter of G^i is at most three for all i.*

Proof. This proof is by contradiction. Suppose the diameter of G^1 is at least four. Then there exists an induced path $P = v_1 v_2 \ldots v_5$ on 5 vertices where the distance, in G^1, between v_1 and v_5 is four. By combining colors, we may assume that there are at most two other colors, suppose 2 and 3, on the edges. Our first goal is to show that there is only one color on the remaining edges in $G' = G[V(P)]$.

First, suppose color 3 has only one edge e in G' and all remaining edges within $G[V(P)]$ have color 2. Regardless of the position of e, it is easy to construct a proper C_5 in this structure so each color 2 and 3 must have at least two edges in G'. At least one of these colors, say color 2, must be present on two disjoint edges. Suppose v_1v_3 and v_2v_4 both have color 2. The case where v_1v_3 and v_2v_5 have color 2 is similar and all other cases can be argued similarly. Now, the edge v_1v_5 must also have color 2 since otherwise $v_1v_3v_2v_4v_5v_1$ forms a proper C_5. Also v_3v_5 must have color 2 since otherwise $v_1v_2v_4v_3v_5v_1$ gives a proper C_5. Finally, if $c(v_1v_4) = c(v_2v_5) = 3$, then $v_1v_4v_5v_2v_3v_1$ yields a proper C_5 (here $c(e)$ denotes the color of the edge e). Hence, there is at most one edge of color 3, for a contradiction. Thus, there are only two colors, say 1 and 2, on the edges of G'.

Since color 3 is spanning and connected, there exists a vertex $v \in G \setminus P$ such that $c(vv_1) = 3$. It is easy to see that in order to avoid a proper C_5, all edges from v to P must have color 3. Now since color 1 is spanning and connected, there is an edge of color 1 from v to a vertex $w \in G \setminus P$. In particular, this means n must be at least 7. In order to avoid easily constructing a proper C_5, all edges from w to P must have color 1, but this contradicts the assumption that the distance from v_1 to v_5 in G^1 is four. □

Continuing with the proof of Theorem 5, suppose there exists a vertex v such that $G \setminus \{v\}$ still has three spanning and connected colors. Then, by induction on n, there exists a proper C_5 in $G \setminus \{v\}$ and so also in G. Thus, every vertex of G is a cut vertex of G^i, for some $1 \leqslant i \leqslant 3$. Let S_i be the set of cut vertices of G^i for all i. For all i, since G_i has diameter at most 3 by Claim 6, it can be shown that S_i induces a complete graph in color i.

Suppose for a moment that $n \geqslant 7$. Then $|S_i| \geqslant 3$ for some i, and without loss of generality, suppose $i = 1$. Let $S_1 = \{s_1, s_2, s_3, \dots\}$ and let V_i be the set of vertices in $G \setminus S_1$ with only one edge in color 1 to S_1, to s_i, for all i. Note that $V_i \neq \emptyset$ for all i since every vertex of S_1 must be a cut vertex of G^1. The subgraph H induced by $S_1 \cup V_1 \cup V_2 \cup \dots$ is shown Figure 2.

Claim 7. *All edges in H have color 1 or color $c \neq 1$.*

Proof. This proof is also by contradiction. In this proof, v_i indicates any vertex in V_i. Let i, j, k be distinct indices. Edges s_iv_j and v_jv_k have the same color because otherwise, $s_iv_jv_ks_kv_is_i$ is a proper C_5. This forces that all edges between s_i, V_j, and V_k have the same color but this color may depend on i, j, k. Call this Fact (1). To show all edges between s_i, V_j, and V_k have the same color c regardless of i, j, k, it suffices to show all edges between V_i and V_j and between V_j and V_k have the same color. Assume this is not true. There there are edges v_iv_j and v_jv_k which have different colors. Then by Fact (1), s_iv_j and v_js_k have different colors as well and so $v_is_iv_js_kv_kv_i$ is a proper C_5. Thus, all edges between V_i and V_j and between V_j and V_k have the same color c and we have just shown that all edges which are not color 1 and are not contained in some V_i have the same color c. Finally, we show all edges within V_i have colors 1 or c as well. Assume some edge xy within V_i has a color that is not 1 or c. Then for any s_j where $j \neq i$ and $v_js_jxys_iv_j$ is a proper C_5, a contradiction. □

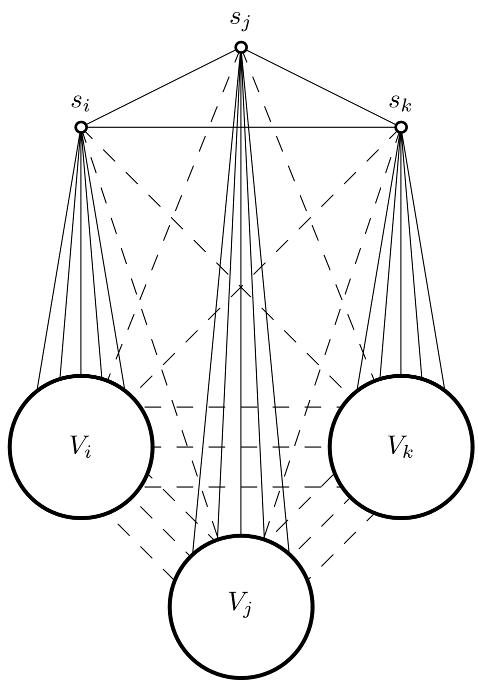

Figure 2: Graphs induced by each color

Again continuing with the proof of Theorem 5, assume, without loss of generality, that $c = 2$. Now, let $v_1 \in V_1$. We know that v_1 has an edge of color 3 to a vertex u but by Claim 7, we must have $u \in G \setminus H$. Since G^1 has diameter at most 3, u has an edge in color 1 to S_1 but since $u \notin V_i$ for any i, u must have at least two edges in color 1 to S_1. That means u must have at least one edge in color 1 to s_i where $i \neq 1$. Without loss of generality, suppose $i = 2$. Then $v_1 u s_2 v_3 s_3 v_1$ is a proper C_5 for some vertex $v_3 \in V_3$, a contradiction. This completes the proof for the case when $n \geqslant 7$.

Finally, suppose $n = 6$. In a coloring of K_6 with three spanning and connected colors, each color must use exactly five edges and form a tree. By Claim 6, we know that each color has diameter at most three. If one color has diameter 2, then since it is also a tree, it is a spanning star and no other color is connected to the center of this star. Thus, we may assume each color induces a tree of diameter exactly three. This means that each color induces one of the graphs in Figure 3.

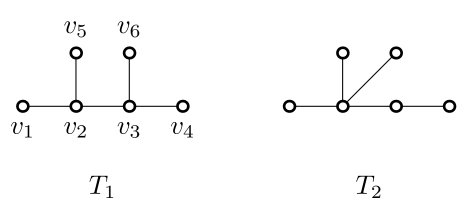

Figure 3: Graphs induced by each color

In fact, since each vertex of K_6 has degree five and all three colors are spanning, every color must induce precisely the tree T_1. Without loss of generality, we may assume color 1 induces the tree T_1 shown and that v_1v_5 has color 2. Then v_3v_5 also has color 2 or else $v_1v_2v_4v_3v_5v_1$ forms a proper C_5. Furthermore, v_1v_3 must also have color 2 since otherwise $v_1v_5v_2v_6v_3v_1$ gives a proper C_5. Now there is a triangle in color 2, contradicting the fact that it must induce a copy of T_1. This completes the proof of Theorem 5. □

4 Proper-C_6 free Colorings

In this section, we establish an analog of Theorem 2 for proper-C_6 free colorings of K_n. As in the case of proper-C_5 free colorings, Theorem 8 similarly shows that any coloring with three or more spanning and connected colors forces the existence of a proper C_6.

Theorem 8. *For $n \geqslant 6$, if G is a proper-C_6 free coloring of K_n, then G has at most two spanning and connected colors.*

Proof. We show the claim is true for $n = 6$ at the end of our proof. For $n > 6$, our proof uses a minimal counterexample but we first need some claims that will be extremely important in the proof. Let G be a proper-C_6 free coloring of K_n with $n > 6$ and suppose, for a contradiction, that G has at least three colors which induce spanning and connected graphs but the result holds for smaller complete graphs. This leads to the following immediate fact.

Fact 9. *For every vertex $v \in G$, $G \setminus \{v\}$ contains at most two spanning and connected colors.*

In particular if G has $\ell \geqslant 3$ spanning and connected colors, this means that every vertex is a cut vertex of at least $\ell - 2 \geqslant 1$ colors.

Claim 10. *Within the subgraph induced on each spanning and connected color, the graph induced on the cut vertices must be connected.*

Proof. The proof is by contradiction. Suppose a monochromatic subgraph, say red, is spanning and connected and contains two cut vertices u and v that are not adjacent in red (and not connected by a sequence of cut vertices). Then the red subgraph, after removal of these two vertices, must have at least three components A, B, and C. One such component, say B, must have at least one edge to both u and v while at least one other component, say A, must be adjacent to u and another, say C, adjacent to v. Since $n \geqslant 7$, at least one of these components must contain at least two vertices and so an edge e. Suppose, without loss of generality, that $e \in A$ (the case where $e \in B$ is handled identically). Then let f be an edge from B to u and let g be an edge from C to v. Then using the edges e, f, g, it is easy to produce a properly colored C_6 for a contradiction. □

For the statement of the next claim, we define a *slim caterpillar* to be a graph consisting of two vertex disjoint stars each containing at least one edge with the addition of a single vertex adjacent only to the centers of the two stars. The centers of the stars along with the added vertex are called the *body* of the caterpillar.

Claim 11. *The graph induced on each spanning and connected color has at most three cut vertices. Moreover, if the graph induced on a color has three cut vertices, it must induce a slim caterpillar.*

Proof. Suppose the red subgraph has at least three cut vertices. Within the red subgraph, by Claim 10, the cut vertices must induce a connected graph. If there is a set T of three cut vertices which induce a red connected graph, each with at least one red edge to a vertex that is not in T (since vertices in T are cut vertices, these neighbors must all be distinct), then using these three disjoint red edges, one may easily produce a proper C_6. Thus, if there are three cut vertices of the red subgraph, one must only have red edges to the other two, thereby forcing the graph induced on the red edges to be a slim caterpillar.

Next suppose that there are at least four cut vertices of the subgraph induced on the red edges. Let $C = \{u, v, w, x\}$ be four of the cut vertices which themselves induce a red connected graph. Thus, we may assume two of the vertices, say v and w, only have edges to u and/or x. Since each of these vertices is a cut vertex of the red subgraph and they induce a connected subgraph themselves, this means C must induce a red path, say $uvwx$. Let e be an edge from u to $G \setminus C$ and f be an edge from x to $G \setminus C$. Such edges exist since these are cut vertices. Then using e, f, and the edge vw we may again easily produce the desired properly colored C_6. □

First, suppose $n \geqslant 10$. By Fact 9, every vertex of G is a cut vertex of some spanning and connected color but by Claim 11, each spanning and connected color has at most three cut vertices. This means that there must be at least $\lceil \frac{n}{3} \rceil \geqslant 4$ spanning and connected colors in G. By Fact 9 again, every vertex of G is a cut vertex for at least $\lceil \frac{n}{3} \rceil - 2$ colors but since each color still has at most three cut vertices, there must be at least

$$\left\lceil \frac{n\left(\lceil \frac{n}{3} \rceil - 2\right)}{3} \right\rceil \geqslant \frac{2n}{3}$$

spanning and connected colors. This is clearly a contradiction since each spanning and connected color requires at least $n-1$ edges to be connected. This concludes the proof when $n \geqslant 10$ so it remains to show that the result holds for $6 \leqslant n \leqslant 9$.

If $7 \leqslant n \leqslant 9$ and there are at least four spanning and connected colors, then the above argument holds for a contradiction. Since each color has at most three cut vertices and every vertex is a cut, there must then be exactly three spanning and connected colors.

We break the remainder of the proof into cases based on the value of n.

Case 1. $8 \leqslant n \leqslant 9$.

By Claim 11, two of these colors induce body-disjoint slim caterpillars. Up to relabeling, these must induce the graph in Figure 4 where $\{v_1, v_2, v_3\}$ is the body of the caterpillar on the thin edges representing color 1, and $\{v_4, v_5, v_6\}$ is the body of the caterpillar on the thick edges representing color 2. Note that the dotted edges may not all be the same color but certainly must not be either color 1 or 2. Regardless of the colors of the dotted edges, the cycle $v_5v_6v_4v_1v_3v_2v_5$ must be a proper C_6 for a contradiction.

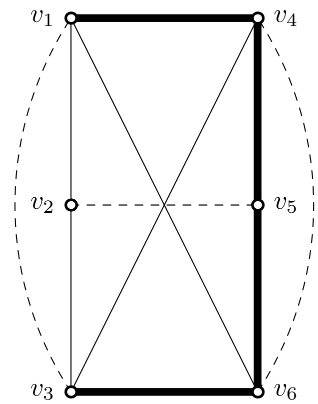

Figure 4: If there are two slim caterpillars

Case 2. $n = 7$.

This case is proven by case analysis similar to classical forbidden subgraph arguments. As observed above, there can only be three colors that are spanning and connected and one of these colors, say red, must induce a slim caterpillar. Then suppose blue and green are the other two spanning and connected colors, each having at least two cut vertices. Let v_1, v_2, v_3 be the body of the red caterpillar.

First, suppose each of v_1 and v_3 has exactly two red neighbors other than v_2. Let v_4, v_5 be the red neighbors of v_1 and let v_6, v_7 be the red neighbors of v_2. Since we have assumed the cycle $v_1v_3v_av_bv_iv_jv_1$ where $\{a, b\} = \{6, 7\}$ and $\{i, j\} = \{4, 5\}$ is not proper, we may assume that the edge v_bv_i must be the same color as either v_av_b or v_iv_j. On the other hand, if all edges within $\{v_4, v_5, v_6, v_7\}$ have the same color, say blue, then by Claim 10, it is impossible for two of these vertices to be cut vertices of the subgraph induced on green edges. Thus, we may assume, without loss of generality, that v_4v_5 is green and that v_5v_6 and v_6v_7 are both blue.

By looking at the cycle $v_4v_5v_6v_3v_1v_2v_4$, the edge v_2v_4 must be green to avoid a proper C_6. Also using the cycle $v_4v_5v_1v_3v_7v_2v_4$, the edge v_7v_2 must be green. From the cycle $v_6v_7v_2v_3v_5v_1v_6$, the edge v_1v_6 must be blue. From the cycle $v_4v_1v_3v_3v_7v_5v_4$, we get v_7v_5 must be green. Finally, from the cycle $v_6v_8v_5v_1v_3v_2v_6$, the edge v_2v_6 is blue so, since green is spanning and connected, v_4v_6 must be green. This gives a proper C_6 on the vertices $v_4v_1v_3v_2v_7v_6v_4$ for a contradiction.

Thus, we may assume one of v_1 or v_3, say v_1, has three red neighbors, say v_4, v_5, v_6. Since three colors are spanning and connected, v_1 must also have a blue edge and a green edge so suppose v_1v_3 is blue and v_1v_7 is green. The vertex v_3 must have another blue edge since the blue graph is spanning and connected so, without loss of generality, say v_3v_5 is blue. Also v_3 needs a green edge so say v_3v_6 is green. By considering the cycle $v_1v_4v_2v_3v_6v_7v_1$, we see that v_6v_7 must be green to avoid a proper C_6.

Suppose first that the edge v_5v_7 is blue. Then the cycle $v_1v_5v_7v_6v_2v_3v_1$ must not be proper so v_2v_6 must be green. Also the cycle $v_1v_4v_3v_2v_5v_7v_1$ must not be proper so v_2v_5 is blue. If v_4v_6 is blue, then the cycle $v_1v_4v_6v_5v_3v_7v_1$ implies v_6v_5 is blue and since green is spanning and connected, v_5v_4 must be green so $v_1v_4v_5v_6v_7v_3v_1$ is a proper C_6. Thus, v_4v_6 must be green. Since blue is spanning and connected, v_5v_6 must be blue which means v_5v_4 must be green - again making $v_1v_4v_5v_6v_7v_3v_1$ a proper C_6.

Finally, suppose the edge v_5v_7 is green. First, we will also assume v_4v_6 is green. The cycle $v_1v_4v_6v_5v_7v_3v_1$ cannot be proper so v_5v_6 must also be green. Since v_6 must have a blue edge, v_6v_2 must be blue. Since the cycle $v_1v_3v_7v_2v_6v_5v_1$ is not proper, this shows that v_7v_2 must be blue but the cycle $v_1v_4v_2v_6v_7v_3v_1$ must also not be proper, showing that v_4v_2 must also be blue. This implies that v_5v_2 must be green for v_2 to have a green edge but then $v_1v_5v_2v_6v_7v_3v_1$ is a proper C_6.

Thus, we may also assume v_4v_6 is blue. The cycle $v_1v_4v_6v_7v_2v_3v_1$ shows that v_2v_7 must be green which immediately implies v_7v_4 is blue since v_7 needs a blue edge. The cycle $v_1v_4v_7v_5v_2v_3v_1$ implies v_5v_2 must be green while the cycle $v_1v_4v_7v_6v_2v_3v_1$ implies v_6v_2 is also green. Since v_2 needs a blue edge, v_4v_2 must be blue. Then $v_1v_4v_2v_5v_3v_7v_1$ is a proper C_6 to complete the proof in this case.

Case 3. *$n = 6$.*

As in the previous cases, there must be three colors that are spanning and connected. By Theorem 3, the diameter of each color can be at most four. Recall that in the base case of Theorem 5, we argue that each G^i has exactly five edges and that if each G^i has a diameter of at most three, then each G^i is a T_1. Thus, if no color has diameter four, then up to relabeling of colors, there is exactly one coloring of G in which each color induces a T_1. This coloring is shown in Figure 5 below.

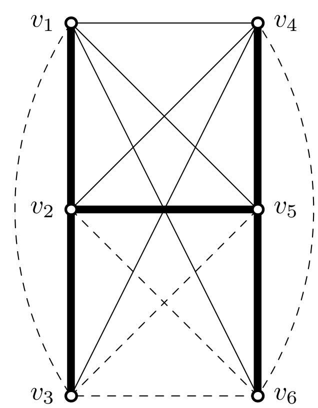

Figure 5: Coloring of K_6

In Figure 5, we see that $v_5v_6v_4v_2v_3v_1v_5$ is a proper C_6. We now assume that there is a color, say color 1, with diameter exactly four. For the duration of the proof, we rely on the following claim.

Claim 12. *In any coloring G of K_6 with three spanning and connected colors, G does not contain a monochromatic triangle.*

Proof. This proof is by contradiction. Assume G has a monochromatic triangle in color i. Since G has three spanning and connected colors, G^i has exactly five edges. Since G^i contains a triangle $v_1v_2v_3$, G^i has exactly two other edges that must connect the remaining three vertices $v_4v_5v_6$ to the triangle, a contradiction. $\square$

Since color 1 has diameter exactly four, G^1 has a path $v_1v_2v_3v_4v_5$ with no chords of color 1. If edge v_3v_6 is color 1, then G^1 has three disjoint color 1 edges and we can easily find a C_6 in G. Since G^1 is spanning, this forces that either edge v_2v_6 or v_4v_6 is color 1. Without loss of generality, assume v_2v_6 is color 1. We also know that edges v_1v_6 and v_5v_6 are not color 1 because otherwise the diameter of G^1 is not four. We now consider two subcases.

Subcase 3.1. *v_1v_6 and v_5v_6 have the same color.*

Assume v_1v_6 and v_5v_6 have color 2. Then v_1v_5 must be color 3 or triangle $v_1v_5v_6$ forms a monochromatic triangle, thus contradicting Claim 12. If v_3v_5 is color 2, then $v_1v_6v_2v_4v_3v_5v_1$ is a proper C_6, so assume v_3v_5 is instead color 3. If v_1v_3 is color 3, then triangle $v_1v_3v_5$ is a monochromatic triangle, again contradicting Claim 12. Thus, v_1v_3 is color 2. Finally, this implies that v_3v_6 must be color 3 or otherwise $v_1v_3v_6$ is a monochromatic triangle in color 2, another contradiction of Claim 12. In this structure, we see that $v_1v_2v_4v_5v_6v_3v_1$ is a proper C_6.

Subcase 3.2. *v_1v_6 and v_5v_6 have different colors.*

If edge v_1v_3 also has a different color from edge v_1v_6, then $v_1v_6v_5v_4v_2v_3v_1$ is a proper C_6. Thus, v_1v_3 and v_1v_6 have the same color and similarly v_3v_5 and v_5v_6 have the same color. This forces either $v_1v_3v_6$ or $v_3v_5v_6$ to be a monochromatic triangle, which contradicts Claim 12. $\square$

5 Proper-C_m free Colorings

The results contained in the previous sections establish that two or more spanning and connected colors imply the existence of a proper C_4, while three or more spanning and connected colors imply the existence of a proper C_5 and a proper C_6. The next result shows that as the desired cycle length gets longer, we need more and more spanning and connected colors to force the existence of a properly colored cycle. Specifically, Theorem 13 ensures that we need at least $k+1$ spanning and connected colors to force the existence of a proper C_m, for $m > 2k$.

Theorem 13. *For $n \geqslant 2k$, there exists a coloring of K_n which contains k spanning and connected colors but no proper cycles of length greater than $2k$.*

Proof. The construction is by induction on n. Suppose there exists a coloring G of K_{n-2k} which contains no proper cycle of length greater than $2k$. If $n \leqslant 4k$, this is trivial so we may suppose $n > 4k$. Add $2k$ new vertices $\{u_1, \ldots, u_k, v_1, \ldots, v_k\}$ to G. For all $i \geqslant j$, color the edges u_iv_j and v_iu_j with color j. For $i < j$, color the edges u_iu_j and v_iv_j with color j. This provides a coloring of all the edges among pairs of the added vertices. All edges between u_i and G and between v_i and G receive color i. In this coloring of K_n, colors $1, 2, \ldots, k$ are spanning and connected.

Suppose there exists a properly colored cycle $C = C_m$ in this graph for some $m > 2k$. Since G was assumed to have no properly colored C_m, we know $C \nsubseteq G$. Conversely, since $|C| = m > 2k$, we also know that $C \nsubseteq \{u_1, \ldots, u_k, v_1, \ldots, v_k\}$. Consider an edge e of C from G to u_i (or identically v_i). By construction, e has color i. Following C from u_i, we cannot take an edge of color i, so must take an edge of color j to a vertex u_j for $j \geqslant i$ or v_j for $j < i$. Similar statements hold when leaving u_j or v_j, and we see that we can never return to G. This is a contradiction. Note that all colors are spanning and connected. □

Theorem 13 yields a lower bound for the number of spanning and connected colors required to force the existence of a properly colored cycle. Theorem 14 yields a general upper bound and shows that any coloring with $2m-1$ or more spanning and connected colors forces the existence of a proper C_m, for $m \geqslant 3$.

Theorem 14. *Let $m \geqslant 3$ and $n \geqslant m$. If G is a proper-C_m free coloring of K_n, then G has at most $2m-2$ spanning and connected colors.*

Proof. This proof is by induction on m. If $m \leqslant 6$, the result follows from Theorems 2, 4, 5, and 8 so suppose $m \geqslant 7$. Let G be a coloring of K_n with at least $2m-1$ colors spanning connected graphs and suppose G contains no proper C_m. Note that this implies $n \geqslant 2m$. By induction, we may assume there is a properly colored C_{m-1} in G. Call this cycle C. We now prove a sequence of claims which lead to the construction of a proper C_m.

Claim 15. *If $w \in G \setminus C$ has an edge e to C of color i where i is not used in C, then w has only edges in color i to C.*

Proof. Let c_1 be the vertex of C contained in e and let $c_2, c_3, \ldots, c_{m-1}$ be the remaining vertices of C in order around C. Let $i_1, i_2, \ldots, i_{m-1}$ be the colors of the edges in C such that the color of $c_j c_{j+1}$ is i_j (where the indices are taken modulo $m-1$). In order to avoid creating a properly colored C_m, the edge wc_2 must either have color i or i_2. Let $j_2 \in \{i, i_2\}$ be the color of wc_2. Similarly, the edge wc_3 must either have color j_2 or i_3. Let $j_3 \in \{j_2, i_3\}$ be the color of wc_3. Following this pattern, let j_t denote the color of the edge wc_t where $j_t \in \{j_{t-1}, i_t\}$ for all $2 \leqslant t \leqslant m-1$. Since the edge wc_1 has color i, it must be that i is in the set $\{j_{m-1}, i_1\}$. Since i is unused in C, we must have $j_{m-1} = i$. This, in turn, implies that $i \in \{j_{m-2}, i_{m-1}\}$, so again $j_{m-2} = i$. This argument can be repeated all the way around C to conclude that the color of wc_j is i for all j. □

Without loss of generality, suppose C uses colors $m+1, m+2, \ldots, m+s$ where $s \leqslant m-1$. Since $2m-1$ colors are spanning and connected, we may assume colors $1, 2, \ldots, m$ are all spanning and connected and not present in C. For all $i \geqslant 1$, let V_i be the set of vertices in $G \setminus C$ with all edges to C in color i. Since colors $1, 2, \ldots, m$ are all spanning and connected, applying Claim 15, we see that $V_i \neq \emptyset$ for all $i \leqslant m$.

Claim 16. *If $w \in G \setminus (\cup_{i \geqslant 1} V_i)$ and $v \in V_i$, then wv has color i.*

Proof. This proof is by contradiction. By the definition of V_i, all edges between V_i and C have color i so if $H = C$, we're finished. Thus, we may assume $H \setminus C \neq \emptyset$. Let $w \in H \setminus C$ and suppose $v \in V_i$ and vw has color $j \neq i$ where j is not used in C. Since $w \notin \cup V_i$, there exists an edge wc_1 with color $k \neq j$ for some $c_1 \in C$. Again let c_i denote the vertices of C in order around the cycle. Since either $c_{m-1}c_1$ or c_1c_2 has a color other than k, we will assume the color of $c_{m-1}c_1$ is $\ell \neq k$ (see Figure 6). Now, by considering the cycle $wc_1c_{m-1}\cdots c_3vw$, we see that the color of c_3c_4 must be i since otherwise this produces a proper C_m, a contradiction. □

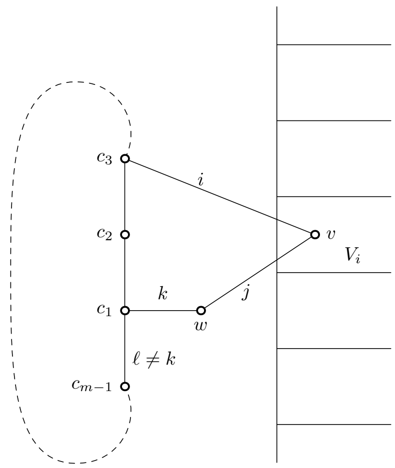

Figure 6: Structure of G

Continuing with the proof of Theorem 14, consider edges between sets V_i and V_j for $1 \leqslant i, j \leqslant m$. Suppose $v_i \in V_i$ and $v_j \in V_j$ and the color of v_iv_j is $k \notin \{i, j\}$. Then $v_ic_1c_2\cdots c_{m-2}v_jv_i$ is the desired proper C_m. Note that this holds even if $i = j$. The next fact is immediate.

Fact 17. *Let $v_i \in V_i$ and $v_j \in V_j$ for some $1 \leqslant i, j \leqslant m$. Then the color of v_iv_j is either i or j.*

Note that if $i = j$, this means that the color of v_iv_j is i.

Suppose $1 \leqslant i, j \leqslant m$ with $i \neq j$. By Fact 17 and Claim 16, in order for a vertex in V_i to have an edge of color j, it must have an edge in color j to a vertex of V_j. Since each of these colors is spanning and connected, we get the following easy fact.

Fact 18. *Every vertex in V_i must have an edge of color j to a vertex of V_j for all i and j with $1 \leqslant i, j \leqslant m$.*

Let $v_{1,1} \in V_1$. By Fact 18, there exists a vertex $v_{2,1} \in V_2$ such that the color of the edge $v_{1,1}v_{2,1}$ is 2. Similarly, there exists a vertex $v_{3,1} \in V_3$ such that the color of the edge $v_{2,1}v_{3,1}$ is 3. This process can be continued to create vertices $v_{i,1}$ for all $1 \leqslant i \leqslant m$. Then there exists a vertex $v_{1,2} \in V_1$ such that $v_{m,1}v_{1,2}$ has color 1. Note that $v_{1,2} \neq v_{1,1}$ since otherwise we would have created a proper (in fact rainbow) C_m. Thus, we may continue to create a long proper path using vertices $v_{i,j} \in V_i$ where j denotes the number of times we pass through set V_i. Since G is finite, this path must repeat a vertex at some point, creating a proper cycle of order a multiple of m. Let C' be the shortest such cycle created by this process which has order a multiple of m. Without loss of generality, suppose $C' = v_{1,1}v_{2,1}\cdots v_{m,r}v_{1,1}$.

Now for all i, let $W_i \subseteq V_i$ denote the vertices of V_i which are also in C'. Without loss of generality, consider consecutive sets W_1 and W_2. The edges $v_{1,k}v_{2,k}$ must all have color 2 by construction but all other edges between W_1 and W_2 must have color 1 since otherwise we could create a shorter proper cycle which still has length a multiple of m, contradicting the choice of C' (see Figure 7). Using the edges between W_i and W_{i+1} which are not in C', we may easily construct a proper C_m unless $r = 2$. In this case, the cycle must look like $v_{1,1}v_{2,2}v_{3,1}v_{4,2}\cdots v_{m,1}v_{1,1}$ but this works only when m is odd. Thus, we may assume $r = 2$ and m is even. Consider the edge $v_{1,1}v_{4,1}$. If this edge has color 4, then $C'' = v_{1,1}v_{4,1}v_{3,2}v_{4,2}\cdots v_{m,2}v_{1,1}$ is a proper C_m. By this argument, all edges of the form $v_{i,1}v_{i+3,1}$ and $v_{i,2}v_{i+3,2}$ have color i. Then we can create all even cycles with length $2t$ a multiple of four from length eight up to $2m$ as follows. The cycle

$$v_{1,1}v_{2,1}\cdots v_{t,1}v_{t-1,2}v_{t,2}v_{t-3,2}v_{t-2,2}v_{t-5,2}v_{t-4,2}\cdots v_{1,2}v_{2,2}v_{1,1}$$

is properly colored and has length $2t$.

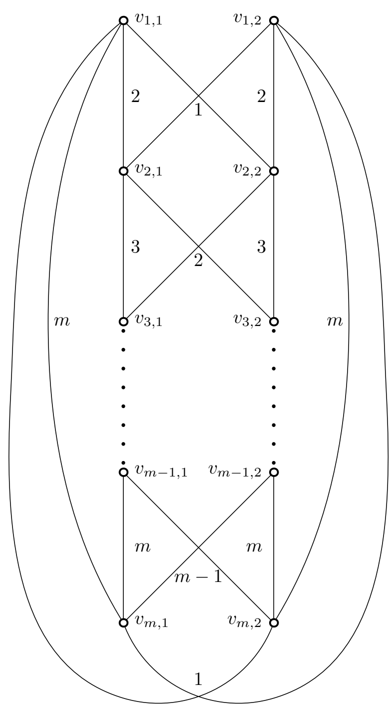

Figure 7: Structure of C'

Finally, it remains to produce a proper C_{2t} when t is odd. If $v_{1,1}v_{3,1}$ has color 3 and

$v_{3,1}v_{5,1}$ has color 5, then

$$v_{1,1}v_{3,1}v_{5,1}v_{6,1}\cdots v_{t+1,1}v_{t,2}v_{t+1,2}v_{t-2,2}v_{t-1,2}v_{t-4,2}v_{t-3,2}\cdots v_{1,2}v_{2,2}v_{1,1}$$

is a properly colored C_{2t} for $t \geqslant 5$ while $v_{1,1}v_{3,1}v_{5,1}v_{4,2}v_{1,2}v_{2,2}v_{1,1}$ suffices when $t = 3$. By symmetry, we cannot have both $v_{1,1}v_{3,1}$ in color 1 and $v_{3,1}v_{5,1}$ in color 3 (by considering the relabeling $v'_{1,1} = v_{7,1}$, $v'_{2,1} = v_{6,2}$, $v'_{3,1} = v_{5,1}$, $v'_{4,1} = v_{4,2}$, $\ldots$, $v'_{7,1} = v_{1,1}$, $v'_{8,1} = v_{m,1}$, $v'_{9,1} = v_{m-1,2}$, $\ldots$ and so on). By the same argument, we cannot have both $v_{i,j}v_{i+2,j}$ in color i (or $i+2$) and $v_{i+2,j}v_{i+4,j}$ in color $i+2$ (or respectively, $i+4$). Thus, if $m \geqslant 8$, we get, either $v_{1,1}v_{3,1}$ in color 3 and $v_{5,1}v_{7,1}$ in color 7 or $v_{1,1}v_{3,1}$ in color 1 and $v_{5,1}v_{7,1}$ in color 5. In the first case, the desired proper C_m is

$$v_{1,1}v_{3,1}v_{4,1}v_{5,1}v_{7,1}v_{8,1}\cdots v_{m/2+1,1}v_{m/2,2}v_{m/2+1,2}v_{m/2-2,2}v_{m/2-1,2}v_{m/2-4,2}\cdots v_{1,2}v_{2,2}v_{1,1}$$

while, in the second case, we use

$$v'_{1,1}v'_{3,1}v'_{4,1}v'_{5,1}v'_{7,1}v'_{8,1}\cdots v'_{m/2+1,1}v'_{m/2,2}v'_{m/2+1,2}v'_{m/2-2,2}v'_{m/2-1,2}v'_{m/2-4,2}\cdots v'_{1,2}v'_{2,2}v'_{1,1}.$$

This completes the proof of Theorem 14. □

We end this section with a corollary summarizing the lower and upper bounds given by Theorems 13 and 14.

Corollary 19. *Given integers m and n with $n \geqslant m$, let $M(m,n)$ be the maximum number of spanning and connected colors in a coloring of K_n containing no proper C_m. Then $\frac{m-1}{2} \leqslant M(m,n) \leqslant 2m-2$.*

6 Conclusion

Future work in the area of proper-cycle free colorings includes tightening the gap between the upper and lower bounds given in Corollary 19. The results of Theorems 4, 5, and 8 indicate that the lower bound of Corollary 19 may be sharp, but in general, this may be difficult to show. The proofs of Theorems 4, 5, and 8 do not suggest a way to easily extend the results of proper-C_4 free, C_5 free, and C_6 free colorings to general C_m free colorings. Since we believe the lower bound in Corollary 19 is sharp (see Conjecture 20), this indicates that the sharp results for even cycles C_m may be more difficult to prove since the cycle is longer than the corresponding odd cycle C_{m-1}. Even so, we conjecture the result is the same.

Conjecture 20. Given integers m and n with $n \geqslant m$, the maximum number of spanning and connected colors in a coloring of K_n containing no proper C_m is $\left\lfloor \frac{m-1}{2} \right\rfloor$.

Finding analogs to Theorem 1 would also be a welcome addition to the literature; in particular, proving partition results as described in Problem 21 for proper-C_m free colorings of complete graphs. We pose the following problem.

Problem 21. Given an integer m, find the smallest number $\lambda(m)$ such that every proper-C_m free coloring of a complete graph has a non-trivial partition with at most $\lambda(m)$ colors on the edges between the parts.

Some other related results concerning colorings free of proper cycles bear mentioning. Many such results are contained in [1]. In particular, the following important structural result of Yeo [8].

Theorem 22 (Yeo [8])**.** *If G is a proper cycle free coloring of a (not necessarily complete) graph, then there is a vertex $z \in G$ such that no connected component of $G \setminus \{z\}$ is joined to z by more than one color.*

Note that if G is a coloring of K_n, Theorem 22 yields the following corollary almost immediately.

Corollary 23. *If G is a rainbow triangle free and proper-C_4 free coloring of K_n, then there is a vertex $z \in G$ incident only to edges of a single color.*

Therefore, to gain more structural information about proper-C_4 free colorings, one may assume the existence of a rainbow triangle.

References

[1] J. Bang-Jensen and G. Gutin. *Digraphs*. Springer Monographs in Mathematics. Springer-Verlag London Ltd., London, second edition, 2009. Theory, algorithms and applications.

[2] S. Fujita, A. Gyárfás, C. Magnant, and A. Seress. Disconnected colors in generalized Gallai colorings. *J. Graph Theory*, To appear.

[3] S. Fujita and C. Magnant. Extensions of Gallai-Ramsey results. *J. Graph Theory*, 70(4):404–426, 2012.

[4] S. Fujita, C. Magnant, and K. Ozeki. Rainbow generalizations of Ramsey theory: a survey. *Graphs and Combin.*, 1:1–30, 2010.

[5] S. Fujita, C. Magnant, and K. Ozeki. Rainbow generalizations of Ramsey theory: a survey (upated). http://math.georgiasouthern.edu/~cmagnant/, 2011.

[6] T. Gallai. Transitiv orientierbare Graphen. *Acta Math. Acad. Sci. Hungar*, 18:25–66, 1967.

[7] A. Gyárfás and G. Simonyi. Edge colorings of complete graphs without tricolored triangles. *J. Graph Theory*, 46(3):211–216, 2004.

[8] A. Yeo. A note on alternating cycles in edge-coloured graphs. *J. Combin. Theory Ser. B*, 69(2):222–225, 1997.