Three color Ramsey numbers for graphs with at most 4 vertices

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Abstract

For given graphs H_1, H_2, H_3, the *3-color Ramsey number* $R(H_1, H_2, H_3)$ is the smallest integer n such that if we arbitrarily color the edges of the complete graph of order n with 3 colors, then it always contains a monochromatic copy of H_i colored with i, for some $1 \leqslant i \leqslant 3$.

We study the bounds on 3-color Ramsey numbers $R(H_1, H_2, H_3)$, where H_i is an isolate-free graph different from K_2 with at most four vertices, establishing that $R(P_4, C_4, K_4) = 14$, $R(C_4, K_3, K_4-e) = 17$, $R(C_4, K_3+e, K_4-e) = 17$, $R(C_4, K_4-e, K_4-e) = 19$, $28 \leqslant R(C_4, K_4-e, K_4) \leqslant 36$, $R(K_3, K_4-e, K_4) \leqslant 41$, $R(K_4-e, K_4-e, K_4) \leqslant 59$ and $R(K_4-e, K_4, K_4) \leqslant 113$. Also, we prove that $R(K_3+e, K_4-e, K_4-e) = R(K_3, K_4-e, K_4-e)$, $R(C_4, K_3+e, K_4) \leqslant \max\{R(C_4, K_3, K_4), 29\} \leqslant 32$, $R(K_3+e, K_4-e, K_4) \leqslant \max\{R(K_3, K_4-e, K_4), 33\} \leqslant 41$ and $R(K_3+e, K_4, K_4) \leqslant \max\{R(K_3, K_4, K_4), 2R(K_3, K_3, K_4)+2\} \leqslant 79$.

This paper is an extension of the article by Arste, Klamroth, Mengersen [*Utilitas Mathematica*, 1996].

1 Introduction

In this paper all graphs considered are undirected, finite and contain neither loops nor multiple edges. Let G be such a graph. The vertex set of G is denoted by $V(G)$, the edge set of G by $E(G)$, and the number of edges in G by $e(G)$. The degree of the vertex v and the number of the edge incident to v colored with color i are denoted with $d(v)$ and $d_i(v)$, respectively. By $\delta_i(G)$ and $\Delta_i(G)$ we denote the minimum and the maximum degree of vertices in G that are colored with color i, respectively. The open neighborhood of vertex v in color i in graph G is $N_i(v) = \{u \in V(G) | \{u,v\} \in E(G)$ and $\{u,v\}$ is colored with color $i\}$. Define $G[S]$ to be a subgraph of G induced by a set of vertices $S \subseteq V(G)$ and G^i to be a graph induced by the edges of G colored with color i. Let P_n (resp. C_n) be the path (resp. cycle) on n vertices. The cardinality of a set A is denoted by $|A|$.

For given graphs $G_1, G_2, \ldots, G_k$, $k \geqslant 2$, the *multicolor Ramsey number* $R(G_1, G_2, \ldots, G_k)$ is the smallest integer n such that if we arbitrarily color the edges of the complete graph of order n with k colors, then it always contains a monochromatic copy of G_i colored with the color i, for some $1 \leqslant i \leqslant k$. A coloring of the edges of n-vertex complete graph with m colors is called a $(G_1, G_2, \ldots, G_m; n)$-coloring, if it does not contain a subgraph isomorphic to G_i colored with the color i, for each i. The set of all non-isomorphic $(G_1, \ldots, G_m; n)$-colorings is denoted by $(G_1, \ldots, G_m; n)$. We refer to consecutive colors corresponding to the parameters of colorings as red, blue and green.

In this paper we consider isolate-free graphs different from K_2 with at most four vertices, improving some results from the article by Arste, Klamroth, Mengersen [1] from 1996 and [9].

Note that $R(H_1, H_2, H_3) = R(H_{\sigma(1)}, H_{\sigma(2)}, H_{\sigma(3)})$ for every permutation σ of $\{1, 2, 3\}$. The case K_2 is omitted here, because $R(K_2, H_2, H_3) = R(H_2, H_3)$ and these numbers were already determined in [6, 7, 12].

We use the following two formulas, which are very well known:

$R(H_1', H_2', H_3') \geqslant R(H_1, H_2, H_3)$, if H_i is a subgraph of H_i'. (1)

$R(H_1, H_2, H_3) \leqslant R(H_1 - v_1, H_2, H_3) + R(H_1, H_2 - v_2, H_3) + R(H_1, H_2, H_3 - v_3) - 1$, if $v_i \in V(H_i)$, with strict inequality when the right-hand-side and at least one of its terms are even. (2)

2 Results

The values $R(H_1, H_2, H_3)$ are known if:

- some H_i is P_3, $2K_2$, P_4 or $K_{1,3}$, except $R(P_4, C_4, K_4)$ [1, 4, 5, 10, 15, 17],
- some H_i is C_4, K_3, $K_3 + e$ or $K_4 - e$ and the other are C_4, K_3 or $K_3 + e$, except $R(K_3, C_4, K_4 - e)$ and $R(K_3 + e, C_4, K_4 - e)$ [1, 3, 9, 11, 12, 26, 27].

Also, bounds of other values are known [1, 9, 10, 16, 18, 22, 23, 27, 28].

In this paper we improve the lower bound $R(C_4, K_4 - e, K_4) \geqslant 27$ obtained from (1). Also, we improve the upper bounds $R(C_4, K_3, K_4 - e) \leqslant 25$, $R(C_4, K_3 + e, K_4 - e) \leqslant 25$, $R(C_4, K_3 + e, K_4) \leqslant 43$, $R(C_4, K_4 - e, K_4) \leqslant 54$, $R(K_3, K_4 - e, K_4) \leqslant 44$, $R(K_3 + e, K_4 - e, K_4) \leqslant 44$, $R(K_4 - e, K_4 - e, K_4) \leqslant 60$ and $R(K_4 - e, K_4, K_4) \leqslant 121$, obtained from (2), as well as $R(P_4, C_4, K_4) \leqslant 15$ [1] and $R(C_4, K_4 - e, K_4 - e) \leqslant 22$ [9].

We prove that $R(P_4, C_4, K_4) = 14$, $R(C_4, K_3, K_4 - e) = 17$, $R(C_4, K_3 + e, K_4 - e) = 17$, $R(C_4, K_4 - e, K_4 - e) = 19$, $28 \leqslant R(C_4, K_4 - e, K_4) \leqslant 36$, $R(K_3, K_4 - e, K_4) \leqslant 41$, $R(K_4 - e, K_4 - e, K_4) \leqslant 59$ and $R(K_4 - e, K_4, K_4) \leqslant 113$.

For some classes of G and H, it was known that $R(K_3+e, G, H) = R(K_3, G, H)$, for instance $R(K_3+e, K_3+e, K_4) = R(K_3, K_3+e, K_4) = R(K_3, K_3, K_4)$ [1] and $R(K_3+e, K_3+e, K_4-e) = R(K_3, K_3+e, K_4-e) = R(K_3, K_3, K_4-e)$ [27]. We prove that $R(K_3+e, K_4-e, K_4-e) = R(K_3, K_4-e, K_4-e)$. Also, we prove that $R(C_4, K_3+e, K_4) \leqslant \max\{R(C_4, K_3, K_4), 29\} \leqslant 32$, $R(K_3+e, K_4-e, K_4) \leqslant \max\{R(K_3, K_4-e, K_4), 33\} \leqslant 41$ and $R(K_3+e, K_4, K_4) \leqslant \max\{R(K_3, K_4, K_4), 2R(K_3, K_3, K_4)+2\} \leqslant 79$.

Now, we give the values and bounds of $R(H_1, H_2, H_3)$ in the following tables. Although the first two complete tables and parts of the remaining tables are shown in [1], for the sake of completeness, we repeat them below. In these tables, the lower bounds obtained from (1) are marked with a "*". Also, we will mark with "†" the upper bounds obtained using (2). We use bold style to denote the new values or bounds presented in this paper.

P_3	P_3	$2K_2$	P_4	$K_{1,3}$	C_4	K_3	K_3+e	K_4-e	K_4
P_3	5[5]								
$2K_2$	4[1]	5[1]							
P_4	5[1]	5[1]	5[1]						
$K_{1,3}$	5[5]	6[1]	7[1]	7[5]					
C_4	6[1]	6[1]	7[1]	7[17]	8[1]				
K_3	5[15]	6[1]	7[1]	9[15]	8[1]	11[4]			
K_3+e	5[1]	6[1]	7[1]	9[1]	8[1]	11[1]	11[1]		
K_4-e	7[1]	6[1]	7[1]	9[1]	9[1]	11[1]	11[1]	11[10]	
K_4	7[15]	8[1]	10[1]	13[15]	13[1]	17[4]	17[1]	17[1]	35[4]

Table 1: Values of $R(P_3, H_1, H_2)$.

$2K_2$	$2K_2$	P_4	$K_{1,3}$	C_4	K_3	K_3+e	K_4-e	K_4
$2K_2$	6[17]							
P_4	6[17]	6[17]						
$K_{1,3}$	6[17]	6[17]	7[17]					
C_4	6[17]	6[17]	7[17]	7[17]				
K_3	7[17]	8[17]	8[17]	8[15]	8[17]			
K_3+e	7[17]	8[17]	8[17]	8[17]	8[17]	8[17]		
K_4-e	7[17]	8[17]	8[17]	9[17]	8[17]	8[17]	11[17]	
K_4	8[17]	11[17]	11[17]	11[17]	11[17]	11[17]	13[17]	20[17]

Table 2: Values of $R(2K_2, H_1, H_2)$.

P_4	6[14]						
$K_{1,3}$	7[1]	7[1]					
C_4	7[1]	8[1]	9[1]				
K_3	9[1]	11[1]	9[1]	16[4]			
K_3+e	9[1]	11[1]	9[1]	16[1]	16[1]		
K_4-e	10[1]	11[1]	11[1]	16[1]	16[1]	16[1]	
K_4	13[1]	13[1]	**14**	25[4]	25[1]	25[1]	52[4]
P_4	P_4	$K_{1,3}$	C_4	K_3	K_3+e	K_4-e	K_4

Table 3: Values of $R(P_4, H_1, H_2)$.

$K_{1,3}$	8[5]					
C_4	8[1]	9[1]				
K_3	11[15]	11[1]	16[4]			
K_3+e	11[1]	11[1]	16[1]	16[1]		
K_4-e	11[1]	11[1]	16[1]	16[1]	16[1]	
K_4	16[15]	16[17]	25[4]	25[1]	25[1]	52[4]
$K_{1,3}$	$K_{1,3}$	C_4	K_3	K_3+e	K_4-e	K_4

Table 4: Values of $R(K_{1,3}, H_1, H_2)$.

C_4	11[3]				
K_3	12[26]	17[11]			
K_3+e	12[1]	17[1]	17[1]		
K_4-e	16[9]	**17**	**17**	**19**	
K_4	20[9]-22[28]	27[9]-32[28]	27*-**32**	**28-36**	52*-72[28]
C_4	C_4	K_3	K_3+e	K_4-e	K_4

Table 5: Values and bounds of $R(C_4, H_1, H_2)$.

K_3	17[12]			
K_3+e	17[1]	17[1]		
K_4-e	17[27]	17[27]	21[27]-27[27]	
K_4	30[16]-31[23]	$R(K_3, K_3, K_4)$[1]	30*-**41**	55[18]-79†
K_3	K_3	K_3+e	K_4-e	K_4

Table 6: Values and bounds of $R(K_3, H_1, H_2)$.

3 Proofs

3.1 $R(P_4, H_1, H_2)$

In [1] it is claimed that $14 \leqslant R(P_4, C_4, K_4) \leqslant 15$, but no $(P_4, C_4, K_4; 13)$-coloring is showed. A $(P_4, C_4, K_4; 13)$-coloring can be found in the Appendix. In this subsection we

K_3+e	17[29]		
K_4-e	17[27]	$\boldsymbol{R(K_3, K_4-e, K_4-e)}$	
K_4	$R(K_3,K_3,K_4)$[1]	30_*-**41**	55_*-$79_\dagger$
K_3+e	K_3+e	K_4-e	K_4

Table 7: Values and bounds of $R(K_3+e, H_1, H_2)$.

K_4-e	28[10]-30[22]	
K_4	33[27]-**59**	55_*-**113**
K_4-e	K_4-e	K_4

Table 8: Bounds on $R(K_4-e, H_1, H_2)$.

K_4	128[13]-$236_\dagger$
K_4	K_4

Table 9: Bounds on $R(K_4, K_4, K_4)$.

will prove that $R(P_4, C_4, K_4) = 14$.

Let F be a P_4-free graph and let $S(F)$ be the set of $(P_4, C_4, K_4; |V(F)|)$-colorings of G such that $G^1 = F$.

We note that $R(P_4, C_4, K_4) = 14$ if and only if for any P_4-free graph F of order 14 $S(F)$ is empty.

Let H and H' be two P_4-free graphs. It is easy to obtain $S(H')$ from $S(H)$ if one of the following two algorithms is applied:

Case a) H is an induced subgraph of H' such that $|V(H')| = |V(H)| + 1$.

Algorithm 1.
Input: coloring $G \in S(H)$.
Output: set of all one-vertex extensions of G which are in $S(H')$.

For instance, if we consider the coloring G belonging to $S(K_2 \cup K_1)$ with 3 vertices, such that the edge $\{1, 2\}$ is red and both $\{1, 3\}$ and $\{2, 3\}$ are blue, then we obtain two colorings of four vertices belonging to $S(2K_2)$, adding a new vertex, assigning red color to $\{3, 4\}$, green color to $\{1, 4\}$ and blue or green color to $\{2, 4\}$. The coloring such that $\{1, 4\}$ and $\{2, 4\}$ are blue, is not considered because it contains a blue C_4. The coloring such that $\{1, 4\}$ is blue and $\{2, 4\}$ is green, is not considered because it is isomorphic to coloring in which $\{1, 4\}$ is green and $\{2, 4\}$ is blue.

Case b) H' is a subgraph of H such that $|V(H')| = |V(H)|$.

Algorithm 2.
Input: coloring $G \in S(H)$.
Output: set of all colorings of $S(H')$ obtained assigning blue or green color to some edges of G.

For instance, if we consider the coloring G belonging to $S(2K_2 \cup K_1)$ with 5 vertices, such that the edges $\{1, 2\}$ and $\{3, 4\}$ are red, $\{1, 5\}$ is blue and the remaining seven edges are green, then we obtain three colorings of five vertices belonging to $S(K_2 \cup 3K_1)$, assigning blue color to $\{3,4\}$ or assigning blue or green color to $\{1,2\}$. The coloring obtained assigning green color to $\{3,4\}$ is not considered because it contains a green K_4.

The lists of graphs generated in order to prove that $S(F) = \emptyset$ for any P_4-free graph F of order 14 are not very large, thus, it is not necessary to utilize the program *nauty* to eliminate graph isomorphisms [20, 21]. To check isomorphisms of graphs, we use the *IsomormicQ* command of the *Combinatorica* package of the *Mathematica* 8.0 program [19].

We use the following result:

Lemma 3. *Let F_1, F_2 and F_3 be three graphs. If $S(F_1) = \emptyset$, F_3 is a subgraph of F_1 with $V(F_3) = V(F_1)$ and F_3 is an induced subgraph of F_2, then $S(F_2) = \emptyset$.*

Proof. Applying Algorithm 2 we obtain $S(F_3)$ from $S(F_1)$, thus $S(F_3) = \emptyset$, and applying Algorithm 1 we obtain $S(F_2)$ from $S(F_3)$, hence $S(F_2) = \emptyset$. □

In order to prove that for any P_4-free graph F of order 14, $S(F) = \emptyset$, without loss of generality we can assume that the components of F are K_2, K_3 or $K_{1,n}$, with $n \geqslant 3$, because if K_1 or P_3 are components of F, there exists a graph F_0 or order 14 such that F is a subgraph of F_0 and K_1 and P_3 are not components of F_0. Thus, if $S(F_0) = \emptyset$ then, applying Algorithm 2, we have that $S(F) = \emptyset$.

Since $R(C_4, K_4) = 10$ [7], F_0 has no independent set of order 10 and, since $R(P_3, C_4, K_4) = 13$, F_0 has at least two components different of K_2. Thus, it is easy to check that $F_0 \in \{4K_3 \cup K_2, 3K_3 \cup K_{1,4}, 2K_3 \cup K_{1,7}, 2K_3 \cup K_{1,5} \cup K_2, 2K_3 \cup 2K_{1,3}, 2K_3 \cup K_{1,3} \cup 2K_2, 2K_3 \cup 4K_2, K_3 \cup K_{1,6} \cup 2K_2, K_3 \cup K_{1,4} \cup K_{1,3} \cup K_2, K_3 \cup K_{1,4} \cup 3K_2, 2K_{1,3} \cup 3K_2\}$.

From Lemma 3, we obtain:

Corollary 4. *If $S(K_3 \cup 2K_2 \cup 5K_1) = \emptyset$ then $S(2K_3 \cup K_{1,7})$, $S(2K_3 \cup K_{1,5} \cup K_2)$, $S(K_3 \cup K_{1,6} \cup 2K_2)$, $S(K_3 \cup K_{1,4} \cup K_{1,3} \cup K_2)$ and $S(2K_{1,3} \cup 3K_2) = \emptyset$.*

Proof. Let $F_1 = K_3 \cup 2K_2 \cup 5K_1$. Considering $F_3 = K_3 \cup 2K_2 \cup 5K_1$ we have that $S(2K_3 \cup K_{1,5} \cup K_2)$, $S(K_3 \cup K_{1,6} \cup 2K_2) = \emptyset$. Considering $F_3 = K_3 \cup K_2 \cup 7K_1$ we obtain that $S(2K_3 \cup K_{1,7})$, $S(K_3 \cup K_{1,4} \cup K_{1,3} \cup K_2) = \emptyset$. Finally, considering $F_3 = 3K_2 \cup 6K_1$ we obtain that $S(2K_{1,3} \cup 3K_2) = \emptyset$. □

Corollary 5. *If $S(2K_3 \cup 3K_2 \cup K_1) = \emptyset$ then $S(2K_3 \cup K_{1,3} \cup 2K_2)$, $S(2K_3 \cup 4K_2)$, $S(K_3 \cup K_{1,4} \cup 3K_2) = \emptyset$.*

Proof. Let $F_1 = 2K_3 \cup 3K_2 \cup K_1$. Considering $F_3 = 2K_3 \cup 3K_2 \cup K_1$ we have that $S(2K_3 \cup 4K_2) = \emptyset$. Considering $F_3 = 2K_3 \cup 2K_2 \cup 3K_1$ we obtain that $S(2K_3 \cup K_{1,3} \cup 2K_2) = \emptyset$. Finally, considering $F_3 = K_3 \cup 3K_2 \cup 4K_1$ we obtain that $S(K_3 \cup K_{1,4} \cup 3K_2) = \emptyset$. □

Consequently, we have that if $S(F) = \emptyset$ for any $F \in \mathcal{F} = \{4K_3 \cup K_2, 3K_3 \cup K_{1,4}, 2K_3 \cup 2K_{1,3}, K_3 \cup 2K_2 \cup 5K_1, 2K_3 \cup 3K_2 \cup K_1\}$ then $R(P_4, C_4, K_4) = 14$.

Now, we are going to prove the main result of this subsection.

Theorem 6. $R(P_4, C_4, K_4) = 14$.

Proof. It is enough to prove that for any $F \in \mathcal{F}$ then $S(F) = \emptyset$.

There is a coloring in $S(5K_1)$ for every $(C_4, K_4; 5)$-coloring, thus $|S(5K_1)| = 13$. From this set, applying Algorithm 1, we obtain the sets $S(K_2 \cup 4K_1)$, $S(2K_2 \cup 3K_1)$, $S(3K_2 \cup 2K_1)$, $S(4K_2 \cup K_1)$, $S(5K_2)$, $S(K_3 \cup 4K_2)$, $S(2K_3 \cup 3K_2)$, the cardinalities of which are 122, 1012, 4808, 8569, 2676, 7466 and 968. From $S(2K_3 \cup 3K_2)$, applying Algorithm 2, we generate the sets $S(2K_3 \cup 2K_2 \cup 2K_1)$, $S(2K_3 \cup K_2 \cup 4K_1)$ and $S(2K_3 \cup 6K_1)$, the cardinalities of which are 944, 84 and 1. From $S(2K_3 \cup 6K_1)$, applying Algorithm 1, we obtain the sets $S(2K_3 \cup K_{1,3} \cup 3K_1)$, and $S(2K_3 \cup 2K_{1,3})$, the cardinalities of which are 5 and 0, respectively. Thus $S(2K_3 \cup 2K_{1,3}) = \emptyset$.

From $S(2K_3 \cup K_2 \cup 4K_1)$, applying Algorithm 2, we obtain that $S(K_3 \cup 2K_2 \cup 5K_1) = \emptyset$.

From $S(2K_3 \cup 3K_2)$, applying Algorithm 1, we generate the set $S(3K_3 \cup 2K_2)$. Its cardinality is 1. From $S(3K_3 \cup 2K_2)$, applying Algorithm 2, we obtain $S(3K_3 \cup K_2 \cup 2K_1)$ and $S(3K_3 \cup 4K_1)$. Their cardinalities are 2 and 1. From $S(3K_3 \cup 4K_1)$, applying Algorithm 1, we have that $S(3K_3 \cup K_{1,4}) = \emptyset$.

From $S(3K_3 \cup 2K_2)$, applying Algorithm 1, we obtain that $S(4K_3 \cup K_2) = \emptyset$ and, finally, from $S(3K_3 \cup 2K_2)$, applying Algorithm 2, we have that $S(2K_3 \cup 3K_2 \cup K_1) = \emptyset$. □

3.2 $R(C_4, H_1, H_2)$

To generate subfamilies of $(C_4, H_1, H_2; n)$, where $H_1, H_2 \in \{K_3, K_4 - e\}$ we used the following algorithm.

Algorithm 7. Extension
Input: coloring $G \in (C_4, H_1, H_2; n)$
Output: set of all one-vertex extensions of G which belong to $(C_4, H_1, H_2; n + 1)$

For $R(C_4, K_4 - e, K_3)$ we used the next algorithm. Let $H^- = K_2$ if $H = K_3$, and $H^- = P_3$ if $H = K_4 - e$.

Algorithm 8. Merge
Input: coloring $G_1 \in (C_4, H, P_3; n)$ and $G_2 \in (C_4, H^-, K_4 - e; m)$
Output: set of all colorings $G \in (C_4, H, K_4 - e; n + m + 1)$ such that $G_1 = G[N_2(v)]$ and $G_2 = G[N_3(v)]$

Algorithm 7 is a standard procedure in graph theoretical computations. In case of generated subfamilies of $(C_4, H_1, H_2; n)$, where $H_1, H_2 \in \{K_3, K_4 - e\}$ we cannot use it alone because we would have to keep collections of nonisomorphic colorings which are to large. Algorithm 7 is used to determine the collections of colorings $(C_4, P_3, K_3; n)$ for $n \leqslant 7$, $(C_4, K_4 - e, K_2; n)$ for $n \leqslant 6$ and $(C_4, P_3, K_4 - e; n)$ for $n \leqslant 8$. The colorings from these collections were used as the parameters of Algorithm 8. Both of these algorithms are often used to determine Ramsey numbers (see [2, 25]) therefore we do not discuss them in detail.

Let $t(n)$ denote the maximum number of edges of a graph with n vertices not containing a C_4 as a subgraph.

Theorem 9. $R(C_4, K_4 - e, K_3) = 17$.

Proof. Since $R(C_4, K_4 - e, K_3) \geqslant R(C_4, K_3, K_3) = 17$ [11], we obtain the lower bound. To obtain the upper bound we use the following computations. Since $R(C_4, P_3, K_3) = 8$ and $R(C_4, K_4 - e, K_2) = 7$ [1], then for every vertex u we have $d_2(u) \leqslant 7$ and $d_3(u) \leqslant 6$. Since $t(17) = 36$ [8], then every coloring of $(C_4, K_4 - e, K_3; 17)$ must contain a vertex v such that $d_1(v) \leqslant 4$. There are only 3 possibilities:

- There exists a vertex v such that $d_1(v) = 3$, $d_2(v) = 7$ and $d_3(v) = 6$.

  We use Algorithm 8 for every graph $G_1 \in (C_4, P_3, K_3; 7)$ and $G_2 \in (C_4, K_4 - e, K_2; 6)$ and find 8 colorings of $(C_4, K_4 - e, K_3; 14)$. Next we use Algorithm 7 to one-vertex extensions of these 8 colorings and obtain subfamilies of $(C_4, K_4 - e, K_3; n)$ for $n \in \{15, 16, 17\}$. Cardinalities of these sets are 6, 43, 0, respectively.

- There exists a vertex v such that $d_1(v) = 4$, $d_2(v) = 7$ and $d_3(v) = 5$.

  Similarly, for every $G_1 \in (C_4, P_3, K_3; 7)$ and $G_2 \in (C_4, K_4 - e, K_2; 5)$ we found 26355 colorings of $(C_4, K_4 - e, K_3; 13)$. Next, we computed subsets of $(C_4, K_4 - e, K_3; n)$ for $n \in \{14, 15, 16, 17\}$, the cardinalities of which are 470854, 515882, 3444, 0, respectively.

- There exists a vertex v such that $d_1(v) = 4$, $d_2(v) = 6$ and $d_3(v) = 6$.

  Again, for every $G_1 \in (C_4, P_3, K_3; 6)$ and $G_2 \in (C_4, K_4 - e, K_2; 6)$ we found 132266 colorings of $(C_4, K_4 - e, K_3; 13)$. Next, we computed subsets of $(C_4, K_4 - e, K_3; n)$ for $n \in \{14, 15, 16, 17\}$, the cardinalities of which are 4077662, 8109281, 56653, 0, respectively.

Finally, this means that $(C_4, K_4 - e, K_3; 17) = \emptyset$ and $R(C_4, K_4 - e, K_3) = 17$. □

In order to prove some Theorems 11 and 13, we use the following lemma:

Lemma 10. *Let F be a graph of order n and let $v_1, v_2, v_3 \in V(F)$, such that v_i, for $i = 1, 2, 3$, is adjacent to at least $\lfloor \frac{n}{3} \rfloor + 1$ vertices of $V(F) \backslash \{v_1, v_2, v_3\}$. Then C_4 is a subgraph of G.*

Proof. Let a_i be the number of vertices of $V(F) \backslash \{v_1, v_2, v_3\}$ adjacent to v_i and non-adjacent to the other two vertices of $\{v_1, v_2, v_3\}$, let $b_{i,j}$ be the number of vertices of $V(F) \backslash \{v_1, v_2, v_3\}$ adjacent to v_i and v_j and non-adjacent to the other one vertex of $\{v_1, v_2, v_3\}$ and let $c_{1,2,3}$ be the number of vertices of $V(F) \backslash \{v_1, v_2, v_3\}$ adjacent to v_1, v_2 and v_3. Then $a_1 + a_2 + a_3 + b_{1,2} + b_{1,3} + b_{2,3} + c_{1,2,3} \leqslant n - 3$.

Since v_i is adjacent at least to $\lfloor \frac{n}{3} \rfloor + 1$ vertices of $V(G) \backslash \{v_1, v_2, v_3\}$, we have that $a_1 + b_{1,2} + b_{1,3} + c_{1,2,3} \geqslant \lfloor \frac{n}{3} \rfloor + 1$, $a_2 + b_{1,2} + b_{2,3} + c_{1,2,3} \geqslant \lfloor \frac{n}{3} \rfloor + 1$ and $a_3 + b_{1,3} + b_{2,3} + c_{1,2,3} \geqslant \lfloor \frac{n}{3} \rfloor + 1$.

Consequently, $a_1 + a_2 + a_3 + 2b_{1,2} + 2b_{1,3} + 2b_{2,3} + 3c_{1,2,3} \geqslant 3\lfloor \frac{n}{3} \rfloor + 3 \geqslant n + 1$ and $b_{1,2} + b_{1,3} + b_{2,3} + 2c_{1,2,3} \geqslant 4$. Then there are i and j such that $b_{i,j} + c_{1,2,3} \geqslant 2$, v_i and v_j have at least two common neighbors belonging to $V(F) \backslash \{v_1, v_2, v_3\}$ and C_4 is a subgraph of F. □

We prove that adding an edge to K_3 leaves its Ramsey number unchanged, such as in the following theorem.

Theorem 11. $R(C_4, K_4 - e, K_3 + e) = R(C_4, K_4 - e, K_3) = 17$.

Proof. By Theorem 9 and by the monotonicity of Ramsey numbers we have that $17 = R(C_4, K_4 - e, K_3) \leqslant R(C_4, K_4 - e, K_3 + e)$. Assume, towards a contradiction, that $R(C_4, K_4 - e, K_3) < R(C_4, K_4 - e, K_3 + e)$. Let G be a $(C_4, K_4 - e, K_3 + e; 17)$-coloring. There is a green triangle in G. Let $\{v_1, v_2, v_3\}$ be the vertices of a green triangle of G. Since $R(C_4, P_3, K_3 + e) = 8$ [1], then $|N_2(v_i)| \leqslant 7$ and $|N_1(v_i)| \geqslant 7$ for $i \in \{1, 2, 3\}$. By Lemma 10, we obtain a red C_4, a contradiction. □

Theorem 12. $R(C_4, K_4 - e, K_4 - e) = 19$.

Proof. Lower bound $R(C_4, K_4 - e, K_4 - e) \geqslant 19$ is presented in [9]. To obtain the upper bound we use similar computations as in the proof of Theorem 9. Since $R(C_4, P_3, K_4 - e) = R(C_4, K_4 - e, P_3) = 9$ [1], then for every vertex u we have $d_2(u) \leqslant 8$ and $d_3(u) \leqslant 8$. Since $t(19) = 42$ [8], then every coloring of $(C_4, K_4 - e, K_3; 19)$ must contain vertex v such that $d_1(v) \leqslant 4$. There are only 4 possibilities:

- There is a vertex v such that $d_1(v) = 4$, $d_2(v) = 7$ and $d_3(v) = 7$.

  We use Algorithm 8 for every graph $G_1 \in (C_4, P_3, K_4 - e; 7)$ and $G_2 \in (C_4, K_4 - e, P_3; 7)$ and find 621308 colorings of $(C_4, K_4 - e, K_4 - e; 15)$. Next, we use Algorithm 7 to one-vertex extensions of these colorings and obtain subfamilies of $(C_4, K_4 - e, K_3; n)$ for $n \in \{16, 17, 18, 19\}$. Cardinalities of these sets are 731002, 18285, 7, 0, respectively.

- There is a vertex v such that $d_1(v) = 4$, $d_2(v) = 8$ and $d_3(v) = 6$, (a case in which $d_2(v) = 6$ and $d_3(v) = 8$ is symmetrical).

  Similarly, for every $G_1 \in (C_4, P_3, K_4 - e; 8)$ and $G_2 \in (C_4, K_4 - e, P_3; 6)$ we found 10488 colorings of $(C_4, K_4 - e, K_4 - e; 15)$. Next, we computed subsets of $(C_4, K_4 - e, K_4 - e; n)$ for $n \in \{16, 17, 18\}$, the cardinalities of which are 28733, 1807, 0, respectively.

- There is a vertex v such that $d_1(v) = 3$, $d_2(v) = 8$ and $d_3(v) = 7$, (a case in which $d_2(v) = 7$ and $d_3(v) = 8$ is symmetrical).

  In this case Algorithm 8 returns an empty set of colorings.

- There is a vertex v such that $d_1(v) = 2$, $d_2(v) = 8$ and $d_3(v) = 8$.

  In this case Algorithm 8 returns an empty set of colorings.

We state that set $(C_4, K_4 - e, K_4 - e; 19) = \emptyset$ and $R(C_4, K_4 - e, K_4 - e) = 19$. □

Also, we have:

Theorem 13. $R(C_4, K_3 + e, K_4) \leqslant \max\{R(C_4, K_3, K_4), 29\} \leqslant 32$.

Proof. We know that $27 \leqslant R(C_4, K_3, K_4) \leqslant 32$ [9, 28]. Let us suppose that there exists G a $(C_4, K_3+e, K_4; \max\{R(C_4, K_3, K_4), 29\})$-coloring. There is a blue triangle in G. Let v_1, v_2, v_3 be the vertices of a blue triangle. To avoid a blue K_4+e we have that $d_2(v_i) = 2$ and, since $R(C_4, K_3+e, K_3) = 17$, $d_3(v_i) \leqslant 16$. Thus $d_1(v_i) \geqslant \max\{R(C_4, K_3, K_4), 29\} - 19 \geqslant \lfloor \frac{\max\{R(C_4,K_3,K_4),29\}}{3} \rfloor + 1$. By Lemma 10, we have a red C_4, a contradiction. □

Theorem 14. $28 \leqslant R(C_4, K_4 - e, K_4) \leqslant 36$.

Proof. To prove the lower bound, we present a $(C_4, K_4 - e, K_4; 27)$-coloring in the Appendix. Let us suppose that there exits G, a $(C_4, K_4-e, K_4; 36)$-coloring. Since $R(C_4, P_3, K_4) = 13$ and $R(C_4, K_4 - e, K_3) = 17$ then $\Delta_2(G) \leqslant 12$, $\Delta_3(G) \leqslant 16$, $\delta_1(G) \geqslant 7$, and $|E(G^1)| \geqslant 126$. Since $t(36) \leqslant 115$ [8], there is a contradiction. □

3.3 $R(K_3, H_1, H_2)$ and $R(K_3 + e, H_1, H_2)$

Theorem 15. $R(K_3 + e, K_4 - e, K_4 - e) = R(K_3, K_4 - e, K_4 - e)$.

Proof. $R(K_3, K_4 - e, K_4 - e) \geqslant 21$ [27]. Let us suppose that $R(K_3 + e, K_4 - e, K_4 - e) > R(K_3, K_4-e, K_4-e)$, thus there exists G a $(K_3+e, K_4-e, K_4-e; R(K_3, K_4-e, K_4-e))$-coloring and there is a red triangle in G. Let v_1, v_2 and v_3 be the vertices of a red triangle of G. We may also assume that $d_2(v_1) \geqslant 9$. Clearly $G[N_2(v_1)]$ contains no blue P_3. Assume that w_i, $1 \leqslant i \leqslant 9$ are the 9 first vertices of $G[N_2(v_1)]$. Next, we consider the cases with respect to the number of blue edges in $G[N_2(v_1)]$ as follows.

- $G[N_2(v_1)]$ has 4 disjoint blue edges.

  Assume that $\{w_i, w_{i+1}\}$ are blue for all $i \in \{1, 3, 5, 7\}$. To avoid a blue $K_4 - e$, without the loss of generality we can assume that $\{v_2, w_i\}$ are green for all $i \in \{2, 4, 6, 8\}$. Without the loss of generality we can assume that $\{w_2, w_4\}$, $\{w_6, w_8\}$ are green, then $\{w_2, w_6\}$, $\{w_2, w_8\}$, $\{w_4, w_6\}$, $\{w_4, w_8\}$ must be red. To prevent $\{v_2, w_2, w_4, w_9\}$ from forming a green $K_4 - e$, for at least one $i \in \{2, 4\}$, the edge $\{w_i, w_9\}$ is red. We have that $\{w_6, w_9\}$, $\{w_8, w_9\}$ are green, then $\{v_2, w_6, w_8, w_9\}$ forms a green $K_4 - e$.

- $G[N_2(v_1)]$ has 3 disjoint blue edges.

  Clearly $G[N_2(v_1)]$ contains a $K_7 - e$ with only red and green edges. By Lemma 2 in [27], we have either a red $K_3 + e$ or a green $K_4 - e$.

- $G[N_2(v_1)]$ has at most 2 disjoint blue edges.

  Since $R(K_3 + e, K_4 - e) = 7$ [7], we immediately obtain a red $K_3 + e$ or a green $K_4 - e$. □

By using similar methods we obtain the following:

Theorem 16. $R(K_3, K_4 - e, K_4) \leqslant \max\{R(K_3 + e, K_4 - e, K_4), 33\} \leqslant 41$.

Proof. Let us suppose that $R(K_3+e, K_4-e, K_4) > \max\{R(K_3+e, K_4-e, K_4), 33\}$, thus there exists G be a $(K_3+e, K_4-e, K_4; \max\{R(K_3+e, K_4-e, K_4), 33\})$-coloring and there is a red triangle in G. Let v_1, v_2 and v_3 be the vertices of a red triangle of G. We may also assume that $d_2(v_1) \geqslant 14$. Clearly $G[N_2(v_1)]$ does not contain any blue P_3. Assume that $w_1, w_2, \ldots, w_{14}$ are the 14 first vertices of $G[N_2(v_1)]$. Next, we consider the cases with respect to the number of blue edges in $G[N_2(v_1)]$ as follows.

- $G[N_2(v_1)]$ has 7 disjoint blue edges.

  Assume that $\{w_i, w_{i+1}\}$ are blue for odd $1 \leqslant i \leqslant 13$. To avoid a blue $K_4 - e$, we can assume that $\{v_2, w_i\}$ are green for even $2 \leqslant i \leqslant 14$. Since $R(K_3 + e, K_3) = 7$ [7], we immediately obtain a red $K_3 + e$ or a green K_4.

- $G[N_2(v_1)]$ has 6 disjoint blue edges.

  Assume that $\{w_i, w_{i+1}\}$ are blue for odd $1 \leqslant i \leqslant 11$. To avoid a blue $K_4 - e$, we can assume that $\{v_2, w_i\}$ are green for even $2 \leqslant i \leqslant 12$. If $\{v_2, w_{13}\}$ is green, then by $R(K_3 + e, K_3) = 7$ [7], we have either a red $K_3 + e$ or a green K_4. This means that $\{v_2, w_{13}\}$ is blue. Since $R(K_3, K_3) = 6$, then we can assume there are two red triangles, let their vertices be $\{w_2, w_4, w_6\}$ and $\{w_8, w_{10}, w_{12}\}$. The remaining edges of a subgraph on $\{w_2, w_4, w_6, w_8, w_{10}, w_{12}\}$ are green. There is at least one green edge in the subgraph on vertices $\{w_1, w_3, w_5, w_7, w_9, w_{11}\}$, say $\{w_1, w_3\}$, then $\{w_1, w_3, w_6, w_8\}$ forms a green K_4.

- $G[N_2(v_1)]$ has 5 disjoint blue edges.

  Assume that $\{w_i, w_{i+1}\}$ are blue for all $i \in \{1, 3, 5, 7, 9\}$. Let us consider a subgraph H on vertices $\{w_2, w_4, w_6, w_8, w_{10}, w_{11}, w_{12}, w_{13}, w_{14}\}$. Since $R(K_3, K_4) = 9$, then H contain a red triangle. To avoid a red $K_3 + e$ and a green K_4 the remaining vertices of H contain a next red triangle. By the fact that a subgraph on vertices $\{w_1, w_3, w_5, w_7, w_9\}$ has at least one green edge, we immediately obtain a green K_4.

- $G[N_2(v_1)]$ has at most 4 disjoint blue edges.

  Since $R(K_3 + e, K_4) = 10$ [7], we immediately obtain a red $K_3 + e$ or a green K_4. □

In order to prove Theorems 20 and 21 we use some definitions and lemmas.

Lemma 17. *Let G be a $(K_3, K_4 - e, K_4; 41)$-coloring. For any $v \in V(G)$, $G^2[N_2(v)] \in \{7K_2, 6K_1 \cup 2K_1\}$.*

Proof. Let $v_0 \in V(G)$ such that $d_2(v_0) = \Delta_2(G)$. Then $d_2(v_0) \leqslant R(K_3, P_3, K_4) - 1 = 16$. If $d_2(v_0) = 16$, as $G[N_2(v_0)]$ is a $(K_3, P_3, K_4; 16)$-coloring and $R(K_3, K_4) = 9$ [12], we have that $G^2[N_2(v_0)] = 8K_2$. Let $x_1, \ldots, x_8$ be the edges of $G^2[N_2(v_0)]$ and let w be one of the 24 vertices of $N_1(v_0) \cup N_3(v_0)$. To prevent a blue $K_4 - e$, for at least one vertex w_i of x_i, with $1 \leqslant i \leqslant 8$, the edge $\{w, w_i\}$ is red or green. Then $G[\{w, w_1, \ldots, w_8\}]$ has not blue edges and there is a red K_3 or a green K_4. There is a contradiction, thus $\Delta_2(G) \leqslant 15$.

If $d_2(v_0) = 15$, as $G[N_2(v_0)]$ is a $(K_3, P_3, K_4; 15)$-coloring, then $G^2[N_2(v_0)] = 7K_2 \cup K_1$. Let $x_1, \ldots, x_7$ be the edges of $G^2[N_2(v_0)]$, let u be the isolated vertex of $G^2[N_2(v_0)]$ and let w be some of the at least 11 vertices of $(N_1(v_0) \cup N_3(v_0)) \backslash N_2(u)$. To prevent a blue $K_4 - e$, for at least one vertex w_i of x_i, with $1 \leqslant i \leqslant 7$, the edge $\{w, w_i\}$ is red or green. Then $G[\{w, u, w_1, \ldots, w_7\}]$ has not blue edges and there is a red K_3 or a green K_4. Thus $\Delta_2(G) \leqslant 14$.

Let $v \in V(G)$. Since $d_1(v) \leqslant R(K_2, K_4 - e, K_4) - 1 = 10$, $d_2(v) \leqslant 14$, $d_3(v) \leqslant R(K_3, K_4 - e, K_3) - 1 = 16$ and $d_1(v) + d_2(v) + d_3(v) = 40$, we have that $d_1(v) = 10$, $d_2(v) = 14$, $d_3(v) = 16$ and, as $G[N_2(v)]$ is a $(K_3, P_3, K_4; 14)$-coloring, $G^2[N_2(v)] \in \{7K_2, 6K_2 \cup 2K_1\}$. □

Let $A = \{v \in V(G) : N_2(v) = 6K_2 \cup 2K_1\}$. We have:

Lemma 18. *$A \neq \emptyset$.*

Proof. If $A = \emptyset$ then every vertex belongs to 7 blue triangles and G has $\frac{41 \cdot 7}{3}$ blue triangles. This number is not an integer and there is a contradiction. □

For any $v \in A$ we define $D_v = \{u \in N_2(v) : d_{G^2[N_2(v)]}(u) = 1\}$. Clearly $|N_2(v) \backslash D_v| = 2$ and $N_2(v) \backslash D_v \subseteq A$.

Lemma 19. *Let $v_1 \in A$ and $v_0, v_2 \in N_2(v_1) \backslash D_{v_1}$, then $N_2(v_0) \cap N_2(v_2) = \{v_1\}$.*

Proof. If $|N_2(v_0) \cap N_2(v_2) \backslash \{v_1\}| \geqslant 1$, since $|N_2(v_0) \backslash \{v_1\}|, |N_2(v_2) \backslash \{v_1\}| = 13$, $N_2(v_0) \backslash \{v_1\}$, $N_2(v_2) \backslash \{v_1\} \subset N_1(v_1) \cup N_3(v_1)$ and $|N_1(v_1) \cup N_3(v_1)| = 26$, we have that there exists $w \in (N_1(v_1) \cup N_3(v_1)) \backslash (N_2(v_0) \cup N_2(v_2))$. Let $x_1, \ldots, x_6$ be the edges of $G^2[N_2(v_1)]$. To prevent a blue $K_4 - e$, for at least one vertex w_i of x_i, with $1 \leqslant i \leqslant 6$, the edge $\{w, w_i\}$ is red or green. Then $G[\{w, v_0, v_2, w_1, \ldots, w_6\}]$ has not blue edges and, as $R(K_3, K_4) = 9$, there is a red K_3 or a green K_4, a contradiction. □

Now, we have:

Theorem 20. *$R(K_3, K_4 - e, K_4) \leqslant 41$.*

Proof. Let G be a $(K_3, K_4 - e, K_4; 41)$-coloring. Let $v_1 \in A$, $v_0, v_2 \in N_2(v_1) \backslash D_{v_1}$ and $v_3 \in (N_2(v_2) \backslash D_{v_2}) \backslash \{v_1\}$. By Lemma 19, $N_2(v_0) \cap N_2(v_2) = \{v_1\}$ and $N_2(v_1) \cap N_2(v_3) = \{v_2\}$. Let $B = V(G) \backslash \{v_0, v_1, v_2, v_3\} \backslash (D_{v_1} \cup D_{v_2})$. As $|B| = 13$, $|N_2(v_0) \backslash \{v_1, v_3\}|$, $|N_2(v_3) \backslash \{v_0, v_2\}| \geqslant 12$ and $N_2(v_0) \backslash \{v_1, v_3\}$, $N_2(v_3) \backslash \{v_0, v_2\} \subseteq B$, we have that $|(N_2(v_0) \backslash \{v_1, v_3\}) \cap (N_2(v_3) \backslash \{v_0, v_2\})| \geqslant 11$ and, as $R(K_3, K_4) = 9$, $G[(N_2(v_0) \backslash \{v_1, v_3\}) \cap (N_2(v_3) \backslash \{v_0, v_2\})]$ contains a blue edge and $G[((N_2(v_0) \backslash \{v_1, v_3\}) \cap (N_2(v_3) \backslash \{v_0, v_2\})) \cup \{v_0, v_3\}]$ contains a blue $K_4 - e$, a contradiction. □

Theorem 21. *$R(K_3 + e, K_4, K_4) \leqslant \max\{R(K_3, K_4, K_4), 2R(K_3, K_3, K_4) + 2\} \leqslant 79$.*

Proof. The result is obvious if $R(K_3 + e, K_4, K_4) = R(K_3, K_4, K_4) \leqslant 79$. Assume that $R(K_3 + e, K_4, K_4) > R(K_3, K_4, K_4)$, and let G be a $(K_3 + e, K_4, K_4; R(K_3 + e, K_4, K_4) - 1)$-coloring. As $R(K_3 + e, K_4, K_4) - 1 \geqslant R(K_3, K_3, K_4)$, there are at least a triangle in G. Let v be a vertex of a red triangle. To avoid a red $K_3 + e$, $d_1(v) = 2$, and $d_2(v), d_3(v) \leqslant R(K_3 + e, K_3, K_4) - 1 = R(K_3, K_3, K_4) - 1$. As $R(K_3 + e, K_4, K_4) - 1 = 1 + d_1(v) + d_2(v) + d_3(v)$, we have that $R(K_3 + e, K_4, K_4) \leqslant 2R(K_3, K_3, K_4) + 2 \leqslant 64$. □

3.4 $R(K_4 - e, H_1, H_2)$

In this subsection we show new upper bounds on $R(K_4-e, K_4-e, K_4)$ and $R(K_3, K_4, K_4)$. In order to obtain the main results of this subsection we use the following lemma, which has a computational proof.

Lemma 22. *Every $(P_3, K_4 - e, K_4; 16)$-coloring has 8 red edges.*

Theorem 23. $R(K_4 - e, K_4 - e, K_4) \leqslant 59$.

Proof. Let G be a $(K_4-e, K_4-e, K_4; 59)$-coloring and let $v \in V(G)$. It is straightforward that $d_1(v), d_2(v) \leqslant R(P_3, K_4-e, K_4)-1 = 16$ and $d_3(v) \leqslant R(K_4-e, K_4-e, K_3)-1 \leqslant 26$. Since $d_1(v) + d_2(v) + d_3(v) = 58$, we have $d_1(v) = 16$ and by Lemma 22, $G[N_1(v)]$ has 8 red edges, v belongs to 8 red triangles and G has $\frac{59 \cdot 8}{3}$ red triangles. This number is not an integer and we have a contradiction. □

In order to prove the last theorem we need the following lemma:

Lemma 24. *Let G be a $(K_4-e, K_4, K_4; 113)$-coloring and let $v \in V(G)$. Then $d_1(v) = 32$, $d_2(v), d_3(v) = 40$ and $G^1[N_1(v)] = 16K_2$.*

Proof. The proof of $\Delta_1(G) \leqslant 32$ is similar to the proof of Lemma 17.

Since $d_1(v) \leqslant 32$, $d_2(v), d_3(v) \leqslant R(K_4-e, K_3, K_4)-1 \leqslant 40$ and $d_1(v)+d_2(v)+d_3(v) = 112$, we have that $d_1(v) = 32$, $d_2(v) = d_3(v) = 40$ and, as $G[N_1(v)]$ is a $(P_3, K_4, K_4; 32)$-coloring and $R(K_4, K_4) = 18$ [12], we have that $G^1[N_1(v)] \in \{16K_2, 15K_2 \cup 2K_1\}$.

Let us suppose $G^1[N_1(v)] = 15K_2 \cup 2K_1$. Let $x_1, \ldots, x_{15}$ be the edges of $G^1[N_1(v)]$, let u_1 and u_2 be the isolated vertices of $G^1[N_1(v)]$ and let w be some of the at least 18 vertices of $(N_2(v) \cup N_3(v)) \backslash (N_1(u_1) \cup N_1(u_2))$. To prevent a red $K_4 - e$, for at least one vertex w_i of x_i, with $1 \leqslant i \leqslant 15$, the edge $\{w, w_i\}$ is blue or green. Then $G[\{w, u_1, u_2, w_1, \ldots, w_{15}\}]$ has not red edges and, as $R(K_4, K_4) = 18$ [12], there is a blue K_4 or a green K_4, thus there is a contradiction. □

Theorem 25. $R(K_4 - e, K_4, K_4) \leqslant 113$.

Proof. Let G be a $(K_4-e, K_4, K_4; 113)$-coloring. By Lemma 24, every vertex of G belongs to 16 blue triangles, thus G has $\frac{113 \cdot 16}{3}$ blue triangles. This number is not an integer and we have a contradiction. □

4 Open cases

The still open cases are the values of $R(H_1, K_4 - e, K_4 - e)$, with $H_1 \in \{K_3, K_3 + e, K_4 - e, K_4\}$, and $R(H_1, H_2, K_4)$, with $H_1, H_2 \in \{C_4, K_3, K_3 + e, K_4 - e, K_4\}$. In general, the difficulty of these open cases depends on the structure of the graphs and almost always grows with the number of edges of the graphs. On the one hand, we think that the following cases should be solved $R(C_4, C_4, K_4)$, $R(C_4, K_3, K_4)$, $R(K_3, K_4 - e, K_4 - e)$ or $R(K_3, K_4, K_4)$. On the other hand there is no progress in the bounds on $R(K_4, K_4, K_4)$

since 1982 [13] and the difference between its bounds are 108. Probably, this case can never be solved.

Also, we expect that the values of the open cases are closer to the lower bounds than the upper bounds and we conjecture that $R(K_3 + e, H, K_4) = R(K_3, H, K_4)$ for $H \in \{C_4, K_4 - e, K_4\}$.

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5 Appendix

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X022122211212
0X22221202212
22X2111022222
222X221220120
1212X12021222
22121X0122222
211120X222112
2202012X12021
10222221X2211
122012222X120
2221221021X21
11222212122X2
222022211012X
```

Figure 1: Matrix of $(P_4, C_4, K_4; 13)$-coloring.

```
X21211112222101200212202112
2X2211221110122220022011122
12X112222110210202222110200
221X22221022101122221021110
1112X0220221222111101021202
11220X222221012211121222021
122222X01102120110201211222
1222220X1112022111200211222
21210211X221202021202120221
211022112X01221222121002212
2112220120X1220222111202211
20021122111X122222022110122
112120102221X12201120222021
0210212202221X1122221221010
12012202210221X211222112001
222112110222212X20210120221
0202111122220212X0212122112
00221101122212100X212022112
202211222110122222X20112122
1222020002122221112X1211222
22211110211201202201X202120
201002221021221110122X21202
0112221120012212221102X2212
21011211022021202221212X221
112120222221000211121222X21
1201022221122102112220122X1
22002122121210112222022111X
```

Figure 2: Matrix of $(C_4, K_4 - e, K_4; 27)$-coloring.