On Extensions of the Alon-Tarsi Latin Square Conjecture

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Abstract

Expressions involving the product of the permanent with the $(n-1)^{\text{st}}$ power of the determinant of a matrix of indeterminates, and of (0,1)-matrices, are shown to be related to an extension to odd dimensions of the Alon-Tarsi Latin Square Conjecture, first stated by Zappa. These yield an alternative proof of a theorem of Drisko, stating that the extended conjecture holds for Latin squares of odd prime order. An identity involving an alternating sum of permanents of (0,1)-matrices is obtained.

Keywords: Latin square, Alon-Tarsi Latin Square conjecture, Parity of a Latin square, adjacency matrix, permanent of (0,1)-matrix.

1 Introduction

A *Latin square* of order n is an $n \times n$ array of numbers in $[n] := \{1, \ldots, n\}$ so that each number appears exactly once in each row and each column. Let L_n be the number of Latin squares of order n. Let $\mathrm{Sym}(n)$ be the symmetric group of permutations of $[n]$. For a permutation $\pi \in \mathrm{Sym}(n)$ we denote its sign by $\epsilon(\pi)$. Viewing the rows and columns of a Latin square L as elements of $\mathrm{Sym}(n)$, the row-sign (column-sign) of L is defined to be the product of the signs of the rows (columns) of L. The sign of L, denoted $\epsilon(L)$, is the product of the row-sign and the column-sign of L. The *parity* of a Latin square is even (resp. odd) if its sign is 1 (resp. -1). The *row parity* and *column parity* of a Latin square are defined analogously. We denote by L_n^{EVEN} (L_n^{ODD}) the number of even (odd) Latin

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squares of order n. The Alon-Tarsi Latin Square Conjecture [1] asserts that for even n, $L_n^{\mathrm{EVEN}} - L_n^{\mathrm{ODD}} \neq 0$. Values of $L_n^{\mathrm{EVEN}} - L_n^{\mathrm{ODD}}$ for small n can be found in [10]. Drisko [3] proved the conjecture for $n = p + 1$, where p is an odd prime, and Glynn [5] proved it for $n = p - 1$. Since for odd n it holds that $L_n^{\mathrm{EVEN}} = L_n^{\mathrm{ODD}}$, some extensions of this conjecture that are applicable to odd n were proposed, as will be described shortly.

A Latin square is called *normalized* if its first row is the identity permutation. A Latin square is called *unipotent* if all the elements of its main diagonal are equal. Let U_n^{E} (resp. U_n^{O}) be the numbers of even (resp. odd) Latin squares of order n which are both normalized and unipotent. Zappa [12] defined the Alon-Tarsi constant $AT(n) := U_n^{\mathrm{E}} - U_n^{\mathrm{O}}$ and proposed the following extension of the Alon-Tarsi conjecture:

Conjecture 1. For all n, $AT(n) \neq 0$.

A Latin square is called *reduced* if its first row and first column are both equal to the identity permutation. Let R_n^{E} and R_n^{O} denote the numbers of even and odd reduced Latin squares of order n, respectively. Another possible extension of the Alon-Tarsi conjecture was recently stated in [10]:

Conjecture 2. For all n, $R_n^{\mathrm{E}} - R_n^{\mathrm{O}} \neq 0$.

If n is even these two conjectures are equivalent to the Alon-Tarsi conjecture. However, despite the existence of a bijection between reduced Latin squares and normalized unipotent Latin squares of order n (see [12]), it is not clear whether for odd n the two conjectures are equivalent. Drisko [4] proved Conjecture 1 in the case that n is an odd prime. Conjecture 2 is only known to be true for small values of n (see [10]).

A Latin square L of order n determines n permutation matrices P_s, $s \in [n]$, defined by $(P_s)_{ij} = 1$ if and only if $L_{ij} = s$. Let S_n be the collection of all $n \times n$ permutation matrices. For $P \in S_n$ let α_P be the corresponding permutation in $\mathrm{Sym}(n)$. The *symbol-sign* of L, denoted by $\epsilon_{\mathrm{sym}}(L)$, is the product of all the $\epsilon(\alpha_{P_s})$, $s = 1, \ldots, n$. A Latin square L is *symbol-even* if $\epsilon_{\mathrm{sym}}(L) = 1$ and *symbol-odd* if $\epsilon_{\mathrm{sym}}(L) = -1$.

Let $X = (X_{ij})$ be the $n \times n$ matrix of indeterminates. The following theorem is due to MacMahon [7]:

Theorem 1. *L_n is the coefficient of $\prod_{i=1}^n \prod_{j=1}^n X_{ij}$ in $\mathrm{per}(X)^n$.*

Here $\mathrm{per}(A)$ denotes the permanent of A. Stones [9] showed that if we replace permanent by determinant in the expression in Theorem 1, an expression for the Alon-Tarsi conjecture is obtained:

Theorem 2. *$L_n^{\mathrm{EVEN}} - L_n^{\mathrm{ODD}}$ is the coefficient of $(-1)^{n(n-1)/2} \prod_{i=1}^n \prod_{j=1}^n X_{ij}$ in $\det(X)^n$.*

The idea of taking the n^{th} power of the determinant was used by Stones [9] to obtain another expression for $L_n^{\mathrm{EVEN}} - L_n^{\mathrm{ODD}}$:

Theorem 3. *Let B_n be the set of all $n \times n$ $(0,1)$-matrices. For $A \in B_n$ let $\sigma_0(A)$ be the number of zero elements in A. Then*

$$L_n^{\mathrm{EVEN}} - L_n^{\mathrm{ODD}} = (-1)^{\frac{n(n-1)}{2}} \sum_{A \in B_n} (-1)^{\sigma_0(A)} \det(A)^n. \tag{1.1}$$

It will be shown in Section 2 that when n is odd "hybrid" expressions involving one permanent and $n-1$ determinants yield analogous results for $AT(n)$. Section 3 contains an alternative proof of Drisko's result [4], that $AT(p) \neq 0$ for all odd primes p. In Section 4 a formula linking Conjectures 1 and 2 is obtained. Section 5 introduces a formula relating the permanents of all distinct regular $p \times p$ adjacency matrices of bipartite graphs (up to renaming the vertices of one of the sides).

2 Formulae for $AT(n)$

For $\alpha \in \mathrm{Sym}(n)$ let $L_n^{\mathrm{SE}}(\alpha)$ (resp. $L_n^{\mathrm{SO}}(\alpha)$) be the number of symbol-even (resp. symbol-odd) Latin squares with $\alpha = \alpha_{P_1}$. Let $L_n^{\mathrm{CE}}(\alpha)$ (resp. $L_n^{\mathrm{CO}}(\alpha)$) be the number of column-even (resp. column-odd) Latin squares with α as the first column. Let $L_n^{\mathrm{CE}}(\alpha,\beta)$ (resp. $L_n^{\mathrm{CO}}(\alpha,\beta)$) be the number of column-even (resp. column-odd) Latin squares with α as the first row and β as the first column. We have

Lemma 1. *If n is odd then*

$$\sum_{\pi \in \mathrm{Sym}(n)} \epsilon(\pi)(L_n^{\mathrm{SE}}(\pi) - L_n^{\mathrm{SO}}(\pi)) = (-1)^{\frac{n(n-1)}{2}} n!(n-1)!AT(n).$$

Proof. Viewing a Latin squares as a set of n^2 triples (i,j,k), such that $L_{ij} = k$, and applying the mapping $\tau : (i,j,k) \to (i,k,j)$, the k^{th} column of $\tau(L)$ is the permutation α_{P_k} corresponding to the permutation matrix P_k in L. Thus $L_n^{\mathrm{SE}}(\alpha) = L_n^{\mathrm{CE}}(\alpha)$ and $L_n^{\mathrm{SO}}(\alpha) = L_n^{\mathrm{CO}}(\alpha)$. We have

$$\sum_{\pi \in \mathrm{Sym}(n)} \epsilon(\pi)(L_n^{\mathrm{SE}}(\pi) - L_n^{\mathrm{SO}}(\pi)) = \sum_{\pi \in \mathrm{Sym}(n)} \epsilon(\pi)(L_n^{\mathrm{CE}}(\pi) - L_n^{\mathrm{CO}}(\pi)).$$

By applying π^{-1} to the columns of each Latin squares with π as its first column we see that if n is odd then $\epsilon(\pi)(L_n^{\mathrm{CE}}(\pi) - L_n^{\mathrm{CO}}(\pi)) = L_n^{\mathrm{CE}}(\mathrm{id}) - L_n^{\mathrm{CO}}(\mathrm{id})$. Thus,

$$\sum_{\pi \in \mathrm{Sym}(n)} \epsilon(\pi)(L_n^{\mathrm{SE}}(\pi) - L_n^{\mathrm{SO}}(\pi)) = n!(L_n^{\mathrm{CE}}(\mathrm{id}) - L_n^{\mathrm{CO}}(\mathrm{id})).$$

Since exchanging columns of a Latin square does not alter the column parity we have that for each $\beta \in \mathrm{Sym}(n)$ such that $\beta(1) = 1$, $L_n^{\mathrm{CE}}(\beta,\mathrm{id}) - L_n^{\mathrm{CO}}(\beta,\mathrm{id}) = L_n^{\mathrm{CE}}(\mathrm{id},\mathrm{id}) - L_n^{\mathrm{CO}}(\mathrm{id},\mathrm{id})$. Thus,

$$\begin{aligned}\sum_{\pi \in \mathrm{Sym}(n)} \epsilon(\pi)(L_n^{\mathrm{SE}}(\pi) - L_n^{\mathrm{SO}}(\pi)) &= n!(L_n^{\mathrm{CE}}(\mathrm{id}) - L_n^{\mathrm{CO}}(\mathrm{id})) \\ &= n! \sum_{\substack{\beta \in \mathrm{Sym}(n) \\ \beta(1)=1}} L_n^{\mathrm{CE}}(\beta,\mathrm{id}) - L_n^{\mathrm{CO}}(\beta,\mathrm{id}) \\ &= n!(n-1)!(L_n^{\mathrm{CE}}(\mathrm{id},\mathrm{id}) - L_n^{\mathrm{CO}}(\mathrm{id},\mathrm{id})).\end{aligned}$$

We use the notation $R_n^{(+,-)}$ for the number of reduced Latin squares with even row parity and odd column parity ($R_n^{(+,+)}$, $R_n^{(-,+)}$ and $R_n^{(-,-)}$ are defined accordingly). Since $L_n^{\mathrm{CE}}(\mathrm{id},\mathrm{id})$ is the number of column-even reduced Latin squares, we have:

$$\begin{aligned} L_n^{\mathrm{CE}}(\mathrm{id},\mathrm{id}) - L_n^{\mathrm{CO}}(\mathrm{id},\mathrm{id}) &= R_n^{(+,+)} + R_n^{(-,+)} - R_n^{(+,-)} - R_n^{(-,-)} \\ &= R_n^{(+,+)} - R_n^{(-,-)}. \end{aligned}$$

Since

$$AT(n) = \begin{cases} R_n^{(+,+)} - R_n^{(-,-)}, & \text{if } n \equiv 0,1 \pmod 4 \\ R_n^{(-,-)} - R_n^{(+,+)}, & \text{if } n \equiv 2,3 \pmod 4, \end{cases}$$

by Section 5 in [12], the result follows. □

We now have a result, analogous to Theorem 2, for $AT(n)$:

Theorem 4. *Let n be odd and let $X = (X_{ij})$ be the $n \times n$ matrix of indeterminates. Then $AT(n)$ is the coefficient of $(-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^n \prod_{j=1}^n X_{ij}$ in $\frac{1}{n!(n-1)!}\mathrm{per}(X)\det(X)^{n-1}$.*

Proof. For $\mathcal{P} \in (S_n)^n$ let $\mathcal{P} = (P_1, P_2, \ldots, P_n)$ and for $s = 1, \ldots, n$ let $\alpha_s = \alpha_{P_s}$. Expanding $\mathrm{per}(X)$ and $\det(X)$ we obtain

$$\mathrm{per}(X)\det(X)^{n-1} = \sum_{\pi \in \mathrm{Sym}(n)} \prod X_{i\pi(i)} \sum_{\substack{\mathcal{P} \in (S_n)^n \\ \pi = \alpha_1}} \prod_{s=2}^{n} \epsilon(\alpha_s) \prod_{k=1}^{n} X_{k\alpha_s(k)}. \tag{2.1}$$

Now, for each $\pi \in \mathrm{Sym}(n)$ the coefficient of $\prod_{i=1}^n \prod_{j=1}^n X_{ij}$ in

$$\prod X_{i\pi(i)} \sum_{\substack{\mathcal{P} \in (S_n)^n \\ \pi = \alpha_1}} \prod_{j=2}^{n} \epsilon(\alpha_j) \prod_{i=1}^{n} X_{i\alpha_j(i)}$$

is equal to $\epsilon(\pi)(L_n^{\mathrm{SE}}(\pi) - L_n^{\mathrm{SO}}(\pi))$. Hence, by (2.1), the coefficient of $\prod_{i=1}^n \prod_{j=1}^n X_{ij}$ in $\mathrm{per}(X)\det(X)^{n-1}$ is

$$\sum_{\pi \in \mathrm{Sym}(n)} \epsilon(\pi)(L_n^{\mathrm{SE}}(\pi) - L_n^{\mathrm{SO}}(\pi)),$$

and the result follows from Lemma 1. □

We also have an analogue of Theorem 3 for $AT(n)$:

Theorem 5. *Let B_n be the set of all $n \times n$ $(0,1)$-matrices. For $A \in B_n$ let $\sigma_0(A)$ be the number of zero elements in A. If n is odd then*

$$AT(n) = \frac{(-1)^{\frac{n(n-1)}{2}}}{n!(n-1)!} \sum_{A \in B_n} (-1)^{\sigma_0(A)} \mathrm{per}(A)\det(A)^{n-1} \tag{2.2}$$

Proof. Most of the proof follows Stones' proof of Theorem 3. By (2.1),

$$\sum_{A\in B_n}(-1)^{\sigma_0(A)}\mathrm{per}(A)\det(A)^{n-1}=\sum_{(A,\mathcal{P})\in B_n\times(S_n)^n}Z(A,\mathcal{P}),\tag{2.3}$$

where

$$Z(A,\mathcal{P})=(-1)^{\sigma_0(A)}\prod_{i=1}^{n}A_{i\alpha_1(i)}\prod_{s=2}^{n}\epsilon(\alpha_s)\prod_{k=1}^{n}A_{k\alpha_s(k)}.$$

If for $(A,\mathcal{P})$ there exists $i,j\in[n]$ such that $(P_s)_{ij}=0$ for all $s=1,\dots,n$, then let A^c be the matrix formed by toggling A_{ij} in the lexicographically first such coordinate ij. Thus $Z(A,\mathcal{P})=-Z(A^c,\mathcal{P})$ and these two terms cancel in the sum in (2.3). So, on the right hand side of (2.3) we are left only with $\sum_{\mathcal{P}\in S^*}\prod_{s=2}^{n}\epsilon(P_s)$, where $S^*=\{(P_1,\dots,P_n):\sum_{s=1}^{n}sP_s$ is a Latin square$\}$ and A is the all-1 matrix. Now,

$$\begin{aligned}\sum_{\mathcal{P}\in S^*}\prod_{s=2}^{n}\epsilon(\alpha_s)&=\sum_{\pi\in\mathrm{Sym}(n)}\epsilon(\pi)\sum_{\substack{\mathcal{P}\in S^*\\ \alpha_{P_1}=\pi}}\prod_{s=1}^{n}\epsilon(\alpha_s)\\&=\sum_{\pi\in\mathrm{Sym}(n)}\epsilon(\pi)\sum_{\substack{\mathcal{P}\in S^*\\ \alpha_{P_1}=\pi}}\epsilon_{\mathrm{sym}}\left(\sum_{s=1}^{n}sP_s\right)\\&=\sum_{\pi\in\mathrm{Sym}(n)}\epsilon(\pi)(L_n^{\mathrm{SE}}(\pi)-L_n^{\mathrm{SO}}(\pi)),\end{aligned}$$

and the result follows from Lemma 1. □

3 An alternative proof of Drisko's theorem

The main result of this section (Corollary 1) was first proved by Drisko [4]. An alternative proof, based on the results of Section 2, is presented here. I am indebted to an anonymous reviewer for suggesting this proof.

In this section the rows and columns of an $n\times n$ matrix will be indexed by the numbers $0,1,\dots,n-1$.

Definition 1. Let A be an $n\times n$ matrix and Let B be a subset of cells of A. Let k be an integer. The k-left shift of B is the set of cells $\{b_{i,(j-k)\bmod n}:b_{i,j}\in B\}$. The k-down shift of B is the set of cells $\{b_{(i+k)\bmod n,j}:b_{i,j}\in B\}$.

Definition 2. An $n\times n$ matrix A will be said to be k-left row shifted, for some k, $0<k<n$, if for all $i=1,\dots,n-1$, the i^{th} row of A is equal to the k-left shift of the $(i-1)^{\mathrm{st}}$ row, and the 0^{th} row is equal to the k-left shift of the $(n-1)^{\mathrm{st}}$ row.

Remark 6. If p is an odd prime and A is a $p\times p$ k-left row shifted matrix, then the set of cells of A is the disjoint union of p diagonals, where the elements of each diagonal are all equal. These diagonals will be referred to as the *k-broken diagonals* of A.

Lemma 2. *Let A be a $p \times p$ k-left row shifted (0,1)-matrix, where p is an odd prime. Let* $\mathbf{b}$ *be the first row of A and let $|\mathbf{b}|$ be the number of 1's in* $\mathbf{b}$. *Then*

(i) $\operatorname{per}(A) \equiv |\mathbf{b}| \pmod p$

(ii) $\det(A) \equiv \pm|\mathbf{b}| \pmod p$

Proof. Part (i) can be easily obtained from Ryser's permanent formula ([8], see also http://mathworld.wolfram.com/RyserFormula.html). However, a different approach, that will also apply to Part (ii), is used here. We define a mapping s_k on the set of diagonals of A as follows: For a diagonal d in A, $s_k(d)$ is obtained by taking the k-left shift of d and then taking the 1-down shift of the result. Note that the fixed points of s_k are exactly the k-broken diagonals defined in Remark 6. The mapping s_k is a bijection and, since A is k-left row shifted, $s_k(d)$ contain the same set of values as d. In particular, if d consists only of 1's, so does $s_k(d)$. Also note that $s_k^p(d) = d$ for all d and thus, since p is prime, each orbit under s_k is of size 1 or p. As mentioned above, the orbits of size 1 are those containing the k-broken diagonals. Thus, $\operatorname{per}(A) \bmod p$ is equal to the number of k-broken diagonals consisting only of 1's, and since there are $|\mathbf{b}|$ such diagonal Part (i) follows.

For Part (ii), we need to show that s_k preserves the parity of the permutation corresponding to the diagonal acted upon, and that all k-broken diagonals correspond to permutations of the same parity. Let d_1 and d_2 be two diagonals. Suppose that d_1 is the l-left shift of d_2 for some l. This means that if π_1 and π_2 are the corresponding permutations, then $\pi_2 = \nu^l \circ \pi_1$ (application from right to left), where $\nu = (12 \dots p)$. Since p is odd, ν is an even permutation, and thus d_1 and d_2 correspond to permutations of the same parity. If d_1 is the l-down shift of d_2, then the corresponding permutations satisfy $\pi_1 = \pi_2 \circ \nu^l$. Since s_k consists of a left shift and a down shift, s_k preserves the parity. Now suppose d_1 and d_2 are k-broken diagonals. Then d_1 is the l-left shift of d_2 for some l. As shown above, d_1 and d_2 correspond to permutations of the same parity. It follows that all fixed diagonals correspond to permutations of the same parity. This proves (ii). □

Theorem 7. *Let B_p be the set of all $p \times p$ (0,1)-matrices, where p is an odd prime. Then*

$$\frac{1}{p} \sum_{A \in B_p} (-1)^{\sigma_0(A)} \operatorname{per}(A) \det(A)^{p-1} \equiv 1 \pmod p. \tag{3.1}$$

Proof. Define the group $G = \langle \nu \rangle \times \langle \nu \rangle$, where $\nu = (12 \cdots p)$. The group G acts on B_p by permuting the rows and columns, so that for each element of G, its first component permutes the order of the rows and the second component permutes the order of the columns. By The Orbit-Stabilizer Theorem, an orbit has size $|G| = p^2$ unless each of its elements has a non-trivial stabilizer in G. If $g = (\nu^i, \nu^j)$ is a stabilizer of $A \in B_p$, so is any of its powers, including (ν, ν^k) for some k, since p is prime. Thus, an orbit has size smaller than p^2 if and only if for each matrix A in that orbit there exists some $0 < k < p$ for which $(\nu, \nu^k)A = A$. Let

$$D = \{A \in B_p | (\nu, \nu^k)A = A \text{ for some } 0 < k < p\}.$$

The action of G preserves σ_0 and, since ν is an even permutation, it also preserves the permanent and the determinant. We have

$$\frac{1}{p}\sum_{A\in B_p}(-1)^{\sigma_0(A)}\mathrm{per}(A)\det(A)^{p-1} \equiv \frac{1}{p}\sum_{A\in D}(-1)^{\sigma_0(A)}\mathrm{per}(A)\det(A)^{p-1} \pmod p.$$

Hence, it suffices to prove (3.1) with "B_p" replaced by "D".

Suppose $(\nu,\nu^k)A = A$. Then, after applying ν^k to the i^{th} row the $(i+1)^{\text{st}}$ is obtained, for $i = 0,\dots,p-2$ and applying ν^k to the $(p-1)^{\text{st}}$ row yields the 0^{th} row. This implies that A is a $(p-k)$-left row shifted matrix. Thus, A is uniquely determined by its first row $\mathbf{b}$ and the number k. We denote this by $A = A(\mathbf{b},k)$.

Now, suppose $A = A(\mathbf{b},k)$ is not the all-1 matrix and let $a = |\mathbf{b}|$. Since p is odd, $\sigma_0(A) \equiv a+1 \pmod 2$. Then, by Lemma 2 and Fermat's Little Theorem, $(-1)^{\sigma_0(A)}\mathrm{per}(A)\det(A)^{p-1} \equiv -((-1)^a a) \pmod p$. For a fixed $a \in \{1,\dots,p-1\}$, the number of distinct matrices $A(\mathbf{b},k)$ with $|\mathbf{b}| = a$ is $\binom{p}{a}(p-1)$. Therefore,

$$\frac{1}{p}\sum_{A\in D}(-1)^{\sigma_0(A)}\mathrm{per}(A)\det(A)^{p-1} \equiv -\frac{1}{p}\sum_{a=1}^{p-1}\binom{p}{a}(p-1)(-1)^a a \pmod p,$$

where the cases that $a \in \{0,p\}$ have been discarded since they correspond to the all-0 and all-1 matrices, which have zero determinant. The result now follows from the binomial identity

$$\sum_{a=0}^{p}\binom{p}{a}(-1)^a a = 0$$

(see http://en.wikipedia.org/wiki/Binomial_coefficient). □

The following result was first proved by Drisko [4]:

Corollary 1. *If p is an odd prime, then*

$$AT(p) \equiv (-1)^{\frac{p-1}{2}} \pmod p.$$

Proof. When $n = p$ is an odd prime we can rearrange (2.2) to obtain

$$\begin{aligned} AT(p) &= (-1)^{\frac{p-1}{2}} \times \frac{1}{(p-1)!^2} \times \frac{1}{p}\sum_{A\in B_p}(-1)^{\sigma_0(A)}\mathrm{per}(A)\det(A)^{p-1} \\ &\equiv (-1)^{\frac{p-1}{2}} \pmod p, \end{aligned}$$

by Wilson's theorem and Theorem 7. The result follows. □

4 Linking Conjectures 1 and 2

The following statement is obtained as part of a proof in [6]:

Proposition 1. *Let n be odd and let $A_1, A_2, \ldots, A_n$ be $n \times n$ matrices over a field. Then*

$$\sum_{\substack{\rho,\sigma\in \mathrm{Sym}(n)^n \\ \rho_1=\mathrm{id}}} \epsilon(\sigma_1)\epsilon(\sigma)\epsilon(\rho) \prod_{i,j=1}^{n} (A_j)_{\sigma_i(j),\rho_j(i)} = (n-1)! \cdot (R_n^{\mathrm{E}} - R_n^{\mathrm{O}}) \mathrm{per}(A_1) \prod_{j=2}^{n} \det(A_j). \quad (4.1)$$

Here ρ_1 and σ_1 are the first components in ρ and σ respectively. Combining Proposition 1 with Theorem 4 yields the following identity, linking $AT(n)$ and $R_n^{\mathrm{E}} - R_n^{\mathrm{O}}$:

Theorem 8. *Let $X = (X_{ij})$ be an $n\times n$ matrix of indeterminates. Then $AT(n)\cdot(R_n^{\mathrm{E}} - R_n^{\mathrm{O}})$ is the coefficient of $(-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^{n}\prod_{j=1}^{n} X_{ij}$ in*

$$\frac{1}{n!(n-1)!^2} \sum_{\substack{\rho,\sigma\in \mathrm{Sym}(n)^n \\ \rho_1=\mathrm{id}}} \epsilon(\sigma_1)\epsilon(\sigma)\epsilon(\rho) \prod_{i,j=1}^{n} X_{\sigma_i(j),\rho_j(i)}.$$

Proof. This follows by taking $A_1 = A_2 = \cdots = A_n = X$ in (4.1) and applying Theorem 4. □

Thus, showing that the above coefficient is nonzero would prove Conjectures 1 and 2.

5 On the permanent of adjacency matrices

The evaluation of the permanent of a (0,1)-matrix is of special significance, since it was the first proven $\#P$-complete problem. This was shown by Valiant in a landmark paper ([11], see also [2]). Theorem 5 leads to an interesting identity involving the permanents of certain (0,1)-matrices:

Theorem 9. *Let p be an odd prime, let B_p be the set of $p \times p$ $(0,1)$-matrices, and let $B_p^* = \{A \in B_p : \det(A) \not\equiv 0 \pmod{p}\}$. Let $B_p^{\dagger}$ be a set of representatives in B_p of the row permutation classes. Then*

$$\sum_{A\in B_p^{\dagger}\cap B_p^*} (-1)^{\sigma_0(A)} \mathrm{per}(A) \equiv -1 \pmod{p}.$$

Proof. Let B_p^r be the subset of B_p containing the regular matrices. From (2.2) we have:

$$AT(p) = \frac{(-1)^{\frac{p-1}{2}}}{p!(p-1)!} \sum_{A\in B_p^r} (-1)^{\sigma_0(A)} \mathrm{per}(A) \det(A)^{p-1}$$

If A' can be obtained from A by permuting the rows, then $\mathrm{per}(A') = \mathrm{per}(A)$ and $\det(A')^{p-1} = \det(A)^{p-1}$ (since $p-1$ is even). Since the rows of each $A \in B_p^r$ are all distinct, each row permutation class in B_p^r contains exactly $p!$ matrices. Let $B_p^\dagger$ be a set of representatives of the row permutation classes in B_p. Then

$$AT(p) = \frac{(-1)^{\frac{p-1}{2}}}{(p-1)!} \sum_{A \in B_p^\dagger \cap B_p^r} (-1)^{\sigma_0(A)} \mathrm{per}(A) \det(A)^{p-1}.$$

By Fermat's little theorem and Wilson's theorem we have

$$AT(p) \equiv (-1)(-1)^{\frac{p-1}{2}} \sum_{A \in B_p^\dagger \cap B_p^*} (-1)^{\sigma_0(A)} \mathrm{per}(A) \pmod p.$$

The result follows from Corollary 1. □

Remark 10. If we view an $n \times n$ (0,1)-matrix A as the adjacency matrix of a bipartite graph G_A, having two parts of identical size n, then per(A) is the number of perfect matchings in G_A. A set $B_p^\dagger$, as in Theorem 9, represents all possible such graphs, up to renaming the vertices of one of the parts.

References

[1] N. Alon and M. Tarsi, *Colorings and orientations of graphs*, Combinatorica **12** (1992), no. 2, 125–134.

[2] A. Ben-Dor and S. Halevi, *Zero-one permanent is #p-complete, a simpler proof*, Theory and Computing Systems, 1993., Proceedings of the 2nd Israel Symposium on the, IEEE, 1993, pp. 108–117.

[3] A. A. Drisko, *On the number of even and odd Latin squares of order* $p+1$, Advances in Math. **128** (1997), 20–35.

[4] A. A. Drisko, *Proof of the Alon-Tarsi conjecture for* $n = 2^r p$, The Electronic Journal of Combinatorics (1998), no. R28.

[5] D. Glynn, *The conjectures of Alon-Tarsi and Rota in dimension prime minus one*, SIAM J. Discrete Math. **24** (2010), 394–399.

[6] D. Kotlar, *A weak case of Rota's basis conjecture for odd dimensions*, manuscript (arXiv:1110.1830v3).

[7] P. A. MacMahon, *A new method in combinatory analysis, with applications to Latin squares and associated questions*, Trans. Cambridge Phil. Soc. **16** (1898), 262–290.

[8] H. J. Ryser, *Combinatorial mathematics*, The Mathematical Association of America, 1963.

[9] D. S. Stones, *Formulae for the Alon-Tarsi conjecture*, SIAM J. Discrete Math. **26** (2012), no. 1, 65–70.

[10] D. S. Stones and I. M. Wanless, *How not to prove the Alon-Tarsi conjecture*, Nagoya Math. J. **205** (2012), 1–24.

[11] L.G. Valiant, *The complexity of computing the permanent*, Theoretical Computer Science **8** (1979), no. 2, 189–201.

[12] P. Zappa, *The Cayley determinant of the determinant tensor and the Alon-Tarsi conjecture*, Advances in Applied Mathematics **19** (1997), 31–44.