

Orientations, semiorders, arrangements, and parking functions

Sam Hopkins

Department of Mathematics
Reed College
Portland, OR, U.S.A.
hopkins.reed.edu

David Perkinson

Department of Mathematics
Reed College
Portland, OR, U.S.A.
davidp@reed.edu

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Abstract

It is known that the Pak-Stanley labeling of the Shi hyperplane arrangement provides a bijection between the regions of the arrangement and parking functions. For any graph G , we define the G -semiorder arrangement and show that the Pak-Stanley labeling of its regions produces all G -parking functions.

In his study of Kazhdan-Lusztig cells of the affine Weyl group of type A_{n-1} , [15], J.-Y. Shi introduced the arrangement of hyperplanes in \mathbb{R}^n now known as the Shi arrangement:

$$x_i - x_j = 0, 1 \quad 1 \leq i < j \leq n.$$

Among other things, he proved that the number of regions in the complement of this set of hyperplanes is $(n+1)^{(n-1)}$, Cayley's formula for the number of trees on $n+1$ labeled vertices. The first bijective proof of this fact is due to Pak and Stanley, [16], who provide a method for labeling the regions with parking functions of size n . Given a graph G , Postnikov and Shapiro, [14], introduced the notion of a G -parking function. In the case where there exists a vertex q connected by edges to every other vertex of G , Duval, Klivans, and Martin, [7], have defined the G -Shi arrangement, and conjecture that when its regions are labeled by the method of Pak and Stanley, the resulting labels are exactly the G -parking functions with respect to q . In this case, however, there may be duplicates among the labels. Letting G be the complete graph on $n+1$ vertices recaptures the original result of Pak and Stanley. Our work was motivated by this conjecture.

Figure 1 serves as a guide to our paper. The four corners of the square in Figure 1 are labeled by structures we associate to a graph G , which we now describe.

Parking functions and superstable configurations. Chip-firing is a key tool for us. In the abelian sandpile model for G , one first chooses a vertex of G to serve as the

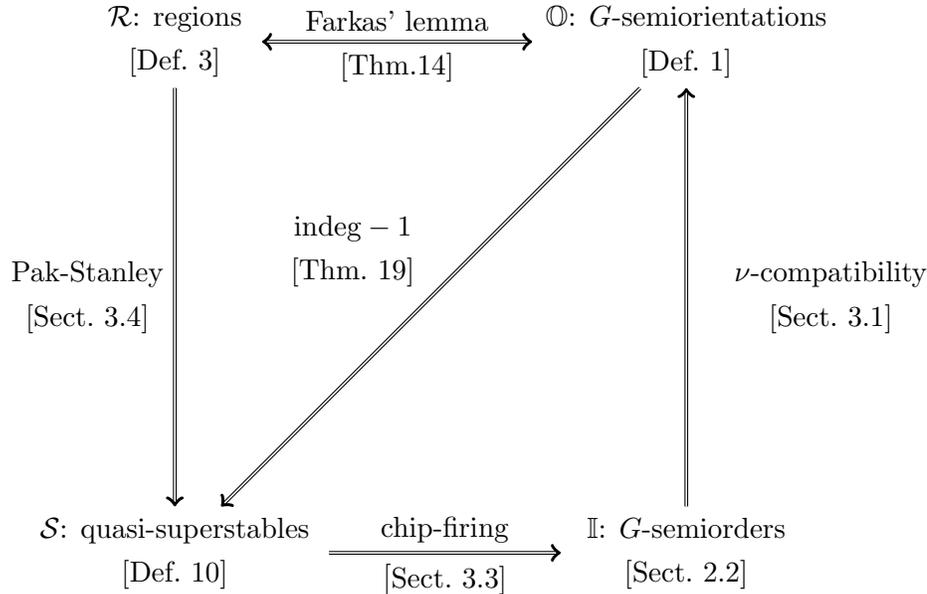


Figure 1: Schematic diagram of results.

“sink.” Then placing grains of sand (or chips) on each of the nonsink vertices defines a *configuration*, c , on G . A vertex v is *unstable* in c if it has at least as many grains of sand as its degree. In that case, one may *fire* (or *topple*) v by sending one grain of sand from v to each of its neighbors. In this process neighbors of v may become unstable, themselves. Sand is not necessarily conserved under firing: grains of sand that are sent to the sink vertex disappear. For undirected, connected graphs—to which we limit ourselves here—by repeatedly firing vertices, c is eventually transformed into a stable configuration, i.e., one with no unstable vertices.

Instead of firing one vertex at a time, one may consider a firing rule that allows sets of vertices to be fired simultaneously. This gives rise to a stronger type of stability and a corresponding set of *superstable* configurations (see Section 2.4). These superstable configurations serve to define G -parking functions. A G -parking function with respect to the chosen sink is a function from the vertices of G to the integers whose value at the sink is -1 and whose values at the nonsink vertices are the numbers of grains of sand for some superstable configuration on G . Thus, there is only a slight difference between superstable configurations and G -parking functions, and at any rate, they are in one-to-one correspondence. Parking functions are more widely-known, and they originally arose independently (for instance in [11, p. 545]). This accounts for why they, and not superstable configurations, are mentioned in the title of this paper. However, given the importance of chip-firing to our work, we will mainly refer to superstable configurations outside of this introduction.

Although G -parking functions and superstable configurations are defined with respect to a chosen sink vertex, we have found it natural and convenient to first prove our results in a sinkless context, akin to working in projective space rather than affine space. To this

end, we define generalized G -parking functions, which we call *quasi-superstable divisors* on G . These appear in the bottom left corner of Figure 1, denoted by \mathcal{S} . As indicated by the figure, quasi-superstables are formally defined in Definition 10.

Semiorders and the semiorder arrangement. The *semiorder arrangement*, [17], is the set of $n(n - 1)$ hyperplanes in \mathbb{R}^n given by

$$x_i - x_j = 1, \quad i, j \in \{1, \dots, n\}, \quad i \neq j.$$

Its regions are in bijection with certain n -element posets called *semiorders*.

A G -*semiorder* is a semiorder whose elements are the vertices of G . The collection of G -semiorders, denoted by \mathbb{I} , appears in the bottom right corner of Figure 1. In the same way that a Shi arrangement is modified by Duval, Klivans, and Martin in [7] to take into account the structure of a graph, we modify a semiorder arrangement to produce the G -*semiorder arrangement* associated with G . The regions of the G -semiorder arrangement are denoted by \mathcal{R} in the top left corner of Figure 1.

Fixing a sink vertex, v , we then refine the definition of a G -semiorder arrangement to get the (G, v) -*semiorder arrangement* (Definition 28).

Orientations. A *partial orientation* of G consists of orienting some, not necessarily all, of the edges of G . The G -*semiorientations*, denoted by \mathbb{O} in the top right of Figure 1, are the partial orientations satisfying an extra condition relative to the cycles of the underlying graph, G .

Relations among the structures. Using the method of Pak and Stanley, we label each region of the (G, v) -semiorder arrangement with what amounts to a function from the vertices of G to the integers taking the value -1 at v . The **main goal of this paper** is Theorem 32, which, along with Theorem 34, shows that the set of Pak-Stanley labels that are negative only at the sink are exactly the set of G -parking functions. The set of Pak-Stanley labels consists of only G -parking functions exactly when the sink is adjacent to every nonsink vertex.

Our main goal, just described, is a consequence of the analogous result in the nonsink context, represented by the left-most vertical arrow in Figure 1. This arrow represents the fact that the Pak-Stanley labels for the regions of the G -semiorder arrangement form the set of quasi-superstables, which is proved in Theorem 23. This theorem is proved by establishing the relations represented by the other arrows in Figure 1.

In working on the G -Shi conjecture of Duval, Klivans, and Martin, we were led to labeling the regions of the G -Shi arrangement with partial orientations of G . We used Farkas' lemma to develop a rule determining which partial orientations would appear as labels. However, it seemed that the criterion we developed was awkward, and that it would become much simpler and symmetric if we altered the hyperplane arrangement somewhat. This is how we were led to consider G -semiorder arrangements. The resulting correspondence between the regions of the G -semiorder arrangement and their labels with partial orientations (the G -semiorientations) is represented by the top horizontal arrow in Figure 1 and is established in Theorem 14.

The correspondence between maximal superstable configurations—those containing the most amount of sand—and acyclic orientations of the edges of G has been noted several

times, in different forms ([3], [4], [8], [9]). In [3], the correspondence is a consequence of an extended version of Dhar’s burning algorithm, a tool from the chip-firing literature, [6]. The input to their extended algorithm is a maximal superstable configuration, c . The configuration is transformed into an unstable configuration \tilde{c} . The sequence of vertex firings that stabilizes \tilde{c} is then used to orient the edges of G . We modify this algorithm in Section 3.3 to apply to all superstable configurations, not just the maximal ones. This algorithm is encoded (in its nonsink version) as the bottom horizontal and right vertical arrows in Figure 1.

Ultimately, we label each region of the G -semiorder arrangement with (i) a G -semiorientation, (ii) semiorders on the vertices of G , and (iii) a quasi-superstable divisor on G . Our aim—to show that all quasi-superstables appear as labels—is achieved by showing that the upper-left and the lower-right triangles in Figure 1 commute and that the mapping referred to as “indeg $- 1$ ” along the diagonal in Figure 1 is surjective, (cf. Theorem 23, Corollary 20, and Theorem 19, respectively).

Organization. Following this introduction, the paper is organized into four main sections and a conclusion. Section 1 is an extended example that illustrates our main results and may serve as a foundation for understanding them. The rest of the paper may be considered a justification of the claims made there. Section 2 defines our four main graphical structures. Section 3 contains the central results describing the correspondences between these structures, as discussed above. Section 4 explains how to transfer the results of Section 3 to the context in which a sink vertex is chosen. The concluding section presents a conjecture and suggests further lines of inquiry. Imagine starting with the G -semiorder arrangement but then perpendicularly displacing the hyperplanes (replacing the original hyperplanes with parallel ones) before applying the labeling method of Pak and Stanley. As one slides the hyperplanes, some regions disappear and new regions form. We conjecture that as long as a “central region” is preserved, the resulting set of labels does not change. A special case implies the G -Shi conjecture of Duval, Klivans, and Martin.

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1 Introductory example

We introduce our main results with an example, beginning with an explanation of the construction of Figure 2. Consider the arrangement of six planes in \mathbb{R}^3 ,

$$x_i - x_j = 1, \quad i, j \in \{1, 2, 3\}, \quad i \neq j.$$

The complement of these planes in \mathbb{R}^3 consists of 19 connected components. The planes are all parallel to the vector $(1, 1, 1)$, so for our purposes it suffices to intersect the arrangement

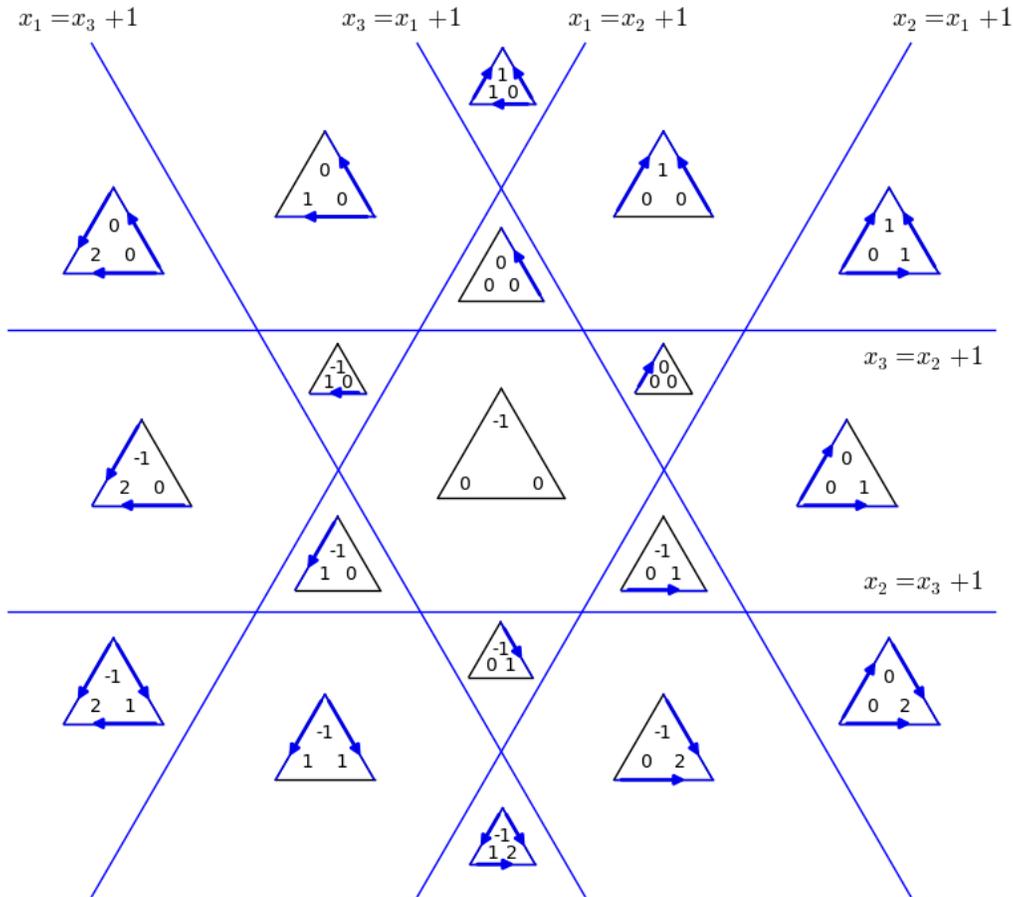


Figure 2: Labeled (G, q) -semiorder arrangement.

with the perpendicular plane, $x_1 + x_2 + x_3 = 0$, as pictured in Figure 2. Our goal here is to explain the labeling of the 19 regions by vertex-labeled, partially oriented graphs.

To start, each region contains a copy of the graph, G , pictured in Figure 3 after removal of the sink vertex, q . The vertices of the remaining triangle are labeled by integers, and some of the edges of the triangle are oriented. We refer to the vertex labels as vectors, (c_1, c_2, c_3) , where c_i is the label for v_i , and designate oriented edges as ordered pairs of vertices, (u, v) , where u is the tail and v is the head. If there were more room, each region would be labeled by the full graph, G . In each region, the label on q would be -1 , and each edge incident on q would be oriented with tail at q .

The labeling of the regions is done inductively, starting at the center. In our example, the central region of the arrangement is a hexagon, and its triangle has vertex labels $(0, 0, -1)$ and no oriented edges (besides the assumed ones from the sink). The rule for the central region is that a vertex v_i has a label of 0 if $\{q, v_i\}$ is an edge; otherwise the label is -1 . More generally, the rule for any region is that the vertex v_i is labeled by one less than the number of oriented edges pointing into v_i .

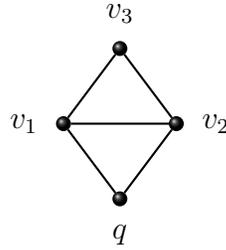


Figure 3: Graph G .

The central region shares an edge with six bordering regions. Let r denote one of these regions, and say that $x_j - x_i = 1$ is the border between it and the central region. Moving from the central region to r , one may think of the value of x_j as increasing at the expense of x_i . We record this fact by orienting the edge $\{v_i, v_j\}$ as (v_i, v_j) and increasing the vertex label for v_j by one. Label the five remaining bordering regions similarly. For example, moving into the compact region directly above the central region, we cross into the region where $x_3 > x_2 + 1$. Thus, the label for this compact region adds an oriented edge (v_2, v_3) , and the label for v_3 increases from -1 to 0 .

At this point in the labeling process, there would be six unlabeled regions bordering the six that were just labeled. Let r be one of these unlabeled regions, and let $x_\ell - x_k = 1$ be the edge it shares with a region r' that was just labeled. In our case, there are two choices for r' , but this choice does not affect the eventual label for r . To label r , start with the label for r' , add the oriented edge (x_k, x_ℓ) , and increase the vertex label for v_ℓ by 1. After labeling these six regions, there are six remaining unlabeled regions, and these are labeled by continuing the procedure just described.

In the end, each region is labeled with a copy of G having labeled vertices and a partial orientation of its edges. The resulting partial orientations have a special property. Call an unoriented edge “blank.” Consider a collection of edges, C , forming a cycle in G . Under any partial orientation, some of the edges in C will have an orientation. If it is possible to orient the remaining blank edges in C to get a directed cycle, then the special property is that there must be a greater number of blank edges in C than oriented edges: “more blanks than arrows for potential directed cycles.” For example, consider the label for the region just above the central region. It has one oriented edge, (v_2, v_3) . By orienting the two blank edges as (v_1, v_2) and (v_3, v_1) , we would get a directed cycle, but this potential cycle has two blanks and only one arrow. Theorem 14 guarantees that the 19 regions are in bijection with partial orientations of G having this special property.

As mentioned above, the vertex labels are given as one less than the indegree at each vertex. In our example, there are eight distinct vertex labels with nonnegative entries:

$$(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (2, 0, 0), (1, 0, 1), (0, 2, 0), (0, 1, 1).$$

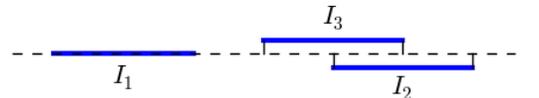
The surjectivity of the map ψ in Theorem 19 ultimately implies, through Theorem 32, that appending -1 for the sink, q , to each of these labels yields the set of G -parking functions (with respect to q). As they are, these labels are exactly the superstable configurations for

the abelian sandpile model on G . Subtracting each from the maximal stable configuration, $(2, 2, 1)$, gives the recurrent configurations, i.e., the elements of the sandpile group:

$$(2, 2, 1), (1, 2, 1), (2, 1, 1), (2, 2, 0), (0, 2, 1), (1, 2, 0), (2, 0, 1), (2, 1, 0).$$

Now pick any region r and a point $t = (t_1, t_2, t_3) \in r$. The point determines a collection of closed intervals, $I_i = [t_i, t_i + 1]$. Turn the set of intervals into a poset, P , by saying $I_i < I_j$ if I_i lies completely to the left of I_j with no overlap, i.e., if $t_i + 1 < t_j$. Overlapping intervals are not comparable in P . Posets arising from finite sets of intervals, ordered in this way, are called *semiorders*. Identifying interval I_i with vertex v_i gives a poset on the nonsink vertices, and we then set q to be the unique minimal element to get a poset on all of the vertices.

For instance, the point $p = (1, 3, 2.5)$ is in the unbounded region directly to the right of the central region in Figure 2, giving the collection of intervals:



The Hasse diagram for the resulting semiorder on the vertices of G is shown in Figure 4.

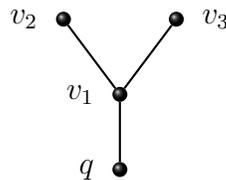


Figure 4: Semiorder determined by $(1, 3, 2.5) \in \mathbb{R}^3$.

The partial orientation of the label for region r can then be read from the semiorder: (v_i, v_j) appears as an oriented edge if and only if $\{v_i, v_j\}$ is an edge of G and $v_i < v_j$ in P . In general, varying the point p selected in r may result in different semiorders: they may disagree for pairs of vertices that do not determine an edge of G . (In our example, each pair of the v_i form an edge, so only one poset on the vertices arises from each region.)

How does this example generalize to an arbitrary graph, G ? Suppose G has designated sink q and has nonsink vertices v_1, \dots, v_n . Form the arrangement of hyperplanes, $x_i - x_j = 1$ for all $i \neq j$ such that $\{v_i, v_j\}$ is an edge of G . Thus, for each edge, $\{v_i, v_j\}$, there is a “stripe” consisting of the two hyperplanes $x_i - x_j = \pm 1$. The partial orientations in our labeling record on which side of each stripe a given region lies.

Label the central region, for which $|x_i - x_j| < 1$ for all edges $\{v_i, v_j\}$, by the graph G , orienting the edges incident with q so that they point away from q . Label the v_i by the number of oriented edges pointing into v_i minus 1 (hence, by 0 or -1), and label q with -1 . Then proceed inductively to label all of the regions, as in the example. In the end, the regions will be in bijection with those partial orientations satisfying the property that for each potential directed cycle, there are more blanks than arrows. The collection of vertex

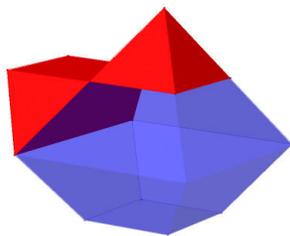


Figure 5: The central region and two bordering regions for G .

labels with nonnegative values at each v_i are the G -parking functions, which correspond to the elements of the sandpile group. Picking a point in any region determines a semiorder on the vertices of G , from which one may reconstruct the labeling of G for the region.

We now describe a version of the above construction that avoids an initial choice of a sink. Let G be a graph with vertices $\{v_0, \dots, v_n\}$. Form the hyperplane arrangement as described above: $x_i - x_j = 1$ for each $i \neq j$ such that $\{v_i, v_j\}$ is an edge of G . Thus, there are two hyperplanes for each edge. Label the central region—given by $|x_i - x_j| < 1$ for all i, j such that $\{i, j\}$ is an edge of G —with a copy of G having no oriented edges and with a -1 at each vertex. Proceed as before, labeling each region. By Theorem 26, those regions for which v_i has label -1 , and all other vertices have nonnegative labels, are the G -parking functions with respect to v_i .

For example, again take G to be the graph in Figure 3 but with no vertex chosen as sink (take $q = v_0$). The corresponding hyperplane arrangement consists of ten hyperplanes in \mathbb{R}^4 . Each hyperplane is parallel to the vector $(1, 1, 1, 1)$, so we project onto the hyperplane given by $\sum_{i=0}^3 x_i = 0$, which we identify with \mathbb{R}^3 , to get an arrangement whose 109 regions are in bijection with those of the original arrangement. The central region is a polytope with ten faces. It is pictured in Figure 5 along with two of its bordering regions. The bordering region forming a pyramid on top of the central region is labeled by a copy of G with one directed edge, (v_2, v_1) , and with vertex label $(-1, 0, -1, -1)$. The other bordering region in the figure is labeled by a copy of G with one directed edge, (v_3, v_1) , and with the same vertex label, $(-1, 0, -1, -1)$. Figure 6 depicts those (unbounded) regions of the hyperplane arrangement for G that satisfy $x_i > x_0 + 1$ for $i = 1, 2, 3$. They are in bijection with the regions in Figure 2, and corresponding regions would have the same labels. (Recall that for convenience the drawing of the sink vertex and its edges is suppressed in Figure 2.)

2 Four structures associated with G

From now on, we take G to be a finite, connected, undirected graph, with vertices $V = \{v_0, \dots, v_n\}$ and edges E . Loops and multiple edges are disallowed (the former for convenience of notation).

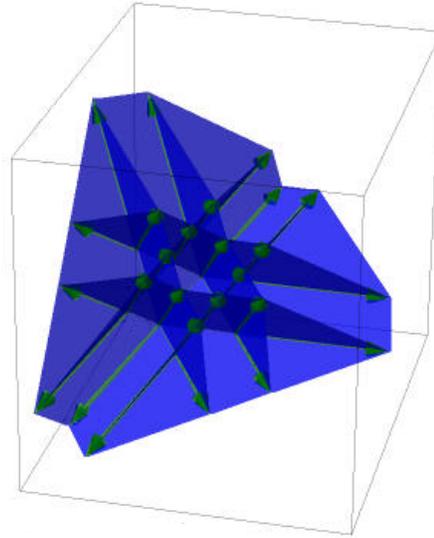


Figure 6: Regions for G corresponding to choosing a sink vertex.

2.1 G -semiorientations.

In this section we define \mathbb{O} , the collection of G -semiorientations. A *partial orientation* of G is a choice of directions for a subset of the edges of G . Formally, a partial orientation is a subset $\mathcal{O} \subset V \times V$ with the property that if $(u, v) \in \mathcal{O}$, then $\{u, v\} \in E$ and $(v, u) \notin \mathcal{O}$. Let \mathcal{O} be a partial orientation. If $e = \{u, v\} \in E$ and $(u, v) \in \mathcal{O}$, then despite the ambiguity, we write $e \in \mathcal{O}$ and say e is oriented. In that case, we think of e as an arrow from u to v and write $e^- = u$ and $e^+ = v$. If neither (u, v) nor (v, u) is in \mathcal{O} , we write $e \notin \mathcal{O}$ and say that e is an unoriented or *blank* edge. The *outdegree* of the vertex $u \in V$ relative to \mathcal{O} , denoted $\text{outdeg}_{\mathcal{O}}(u)$, is the number of edges $e \in \mathcal{O}$ such that $e^- = u$. Similarly, the *indegree* of $u \in V$ relative to \mathcal{O} , denoted $\text{indeg}_{\mathcal{O}}(u)$, is the number of edges $e \in \mathcal{O}$ such that $e^+ = u$. We use the notation $\text{deg}(u)$ to denote the ordinary degree of u , i.e., the number of $e \in E$ containing u . Fixing \mathcal{O} , some of the edges of any cycle $C \subseteq E$ of G will be oriented and others will be blank. If it is possible to assign directions to the blank edges so that C would become a directed cycle, then we call C a *potential cycle* for \mathcal{O} .

Definition 1. A G -semiorientation is a partial orientation, \mathcal{O} , such that each potential cycle for \mathcal{O} has more blank edges than oriented edges. The set of G -semiorientations of G is denoted \mathbb{O} .

2.2 G -semiorders.

A reference for ordinary semiorders is [17]. Let k be any nonnegative integer, and consider a collection of unit length closed intervals, $P = \{I_1, \dots, I_k\}$, of the real line. Order the elements of P by $I_i < I_j$ if I_i lies strictly to the left of I_j , i.e., if $I_i = [a_i, a_i + 1]$ and $I_j = [a_j, a_j + 1]$, then $a_i + 1 < a_j$. Any poset isomorphic to a poset P , constructed as

above, is called a *semiorder*. The number of non-isomorphic semiorders with k elements is the k -th Catalan number, C_k , and there is a corresponding generating function

$$C(x) = \sum_{k \geq 0} C_k x^k = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

If f_k denotes the number of *labeled* semiorders, there is the exponential generating function

$$\sum_{k \geq 0} f_k \frac{x^k}{k!} = C(1 - e^{-x}).$$

Definition 2. A G -*semiorder* is a semiorder on the vertices of G . The set of G -semiorders is denoted \mathbb{I} .

2.3 The G -semiorder arrangement.

Definition 3. The G -*semiorder arrangement*, denoted \mathcal{S} , is the set of $2|E|$ hyperplanes in \mathbb{R}^{n+1} given by

$$x_i - x_j = 1,$$

for all $i \neq j$ such that $\{v_i, v_j\} \in E$. The *regions* of \mathcal{S} , denoted \mathcal{R} , are the connected components of $\mathbb{R}^{n+1} \setminus \mathcal{S}$.

If G were the complete graph on $n + 1$ vertices, the G -semiorder arrangement would be the ordinary semiorder arrangement discussed in [17], whose regions are in bijection with labeled semiorders on $n + 1$ elements. For general G , each region of the G -semiorder arrangement is a union of regions from the ordinary semiorder arrangement.

2.4 Quasi-superstables on G .

Now designate $q := v_0$ as the *sink* vertex. We recall the basic facts about the abelian sandpile model on G , including G -parking functions. A reference for the sandpile results stated here is [10]. (More references for the abelian sandpile model: [2] and [5] are seminal; [6] and [13] are general references; and [12] is a quick overview.) We then proceed to define \mathcal{S} , the set of quasi-superstable divisors of G .

Let $D = \text{diag}(\deg v_0, \dots, \deg v_n)$, and let A be the adjacency matrix for G , defined by

$$A_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The *Laplacian* matrix for G is

$$\Delta = D - A.$$

The *reduced Laplacian* is the matrix $\tilde{\Delta}$ obtained by removing the first row and column of Δ , i.e., the row and column corresponding to the sink vertex.

A *configuration* on G is an element of the free abelian group on the nonsink vertices. Having ordered the nonsink vertices, as above, we identify the set of configurations with \mathbb{Z}^n in the natural way: $c = \sum_{i=1}^n c_i v_i \leftrightarrow (c_1, \dots, c_n)$. A *divisor* on G is an element of the free abelian group on all of the vertices of G , which we similarly identify with \mathbb{Z}^{n+1} . Once a sink is chosen, we may consider configurations as those divisors whose sink coefficient is 0.

Given two configurations or two divisors c and c' , we write $c \geq c'$ if $c_i \geq c'_i$ for all i . We say c is *nonnegative* if $c \geq 0$, i.e., if each component of c is nonnegative.

If X is a subset of the nonsink vertices, write 1_X for the configuration whose i -th component is 1 if $v_i \in X$ and 0 otherwise. *Firing* X from a configuration c results in the configuration $c - \tilde{\Delta} 1_X$. If $X = \{v_i\}$, we call this operation *firing* v_i . One may speak of *firing the sink vertex*, which adds 1 to each vertex connected to the sink. Since the sum of the columns of the Laplacian matrix is zero, firing the set of all nonsink vertices is the same as *reverse-firing* the sink, subtracting 1 from each vertex attached to the sink.

Definition 4. Let c be a nonnegative configuration on G . We say c is *stable* if there is no i such that firing v_i from c results in a nonnegative configuration. We say that c is *superstable* if there is no nonempty subset X of the nonsink vertices such that firing X from c results in a nonnegative configuration.

The notion of a superstable configuration is essentially the same as that of a parking function.

Definition 5. A function $f: V \rightarrow \mathbb{Z}$ is a *G -parking function* (with respect to q) if $f(q) = -1$ and $(f(v_1), \dots, f(v_n))$ is a superstable configuration on G . We identify a G -parking function f with the divisor $\sum_{v \in V} f(v) v$.

Suppose c is a nonnegative configuration. Then c is stable exactly when $c_i < \deg v_i$ for all i . A vertex v_i such that $c_i < \deg v_i$ is said to be *stable* in c ; otherwise it is *unstable*. Firing a set of vertices is *legal* from c if the resulting configuration is nonnegative. In particular, firing a single unstable vertex of c is legal. A sequence of vertices is called a *legal firing sequence* for c if each vertex in the sequence is unstable after firing the previous vertices in the sequence. Since there is a path from each nonsink vertex to the sink in G , there is a legal firing sequence leading to a stable configuration c° called the *stabilization* of c . This process is called *stabilizing* c . It turns out that c° is independent of the order in which unstable vertices are fired, as is the number of times each vertex is fired in reaching c° .

Definition 6. A stable configuration $c \geq 0$ is *recurrent* if given any nonnegative configuration a , there exists a nonnegative configuration b such that $c = (a + b)^\circ$. The recurrent elements with the operation of (vertex-wise) addition followed by stabilization is called the *sandpile group* of G (with respect to q), denoted $\text{Sand}(G)$.

It is well-known that the sandpile group actually is a group and the mapping

$$\begin{aligned} \text{Sand}(G) &\rightarrow \mathbb{Z}^n / \text{image}(\tilde{\Delta}) \\ c &\mapsto c \end{aligned}$$

is an isomorphism. Each equivalence class of \mathbb{Z}^n modulo the image of the reduced Laplacian contains a unique recurrent element. It is also known that each equivalence class contains a unique superstable element. Define the *maximal stable configuration* to be

$$c_{\max} = \sum_{i=1}^n (\deg v_i - 1) v_i.$$

The next two propositions are well-known.

Proposition 7 ([10, Theorem 4.4]). *The configuration c is recurrent if and only if $c_{\max} - c$ is superstable.*

Proposition 8 (Dhar's burning algorithm, [6], [10, Lemma 4.1]). *Let $b \geq 0$ be a stable configuration on G , and let \tilde{b} be the configuration obtained from b by firing the sink. The following are equivalent:*

1. b is recurrent,
2. $(\tilde{b})^\circ = b$, i.e., the stabilization of \tilde{b} is b ,
3. in stabilizing \tilde{b} , each nonsink vertex fires exactly once.

Remark. We refer to Proposition 8 as Dhar's burning algorithm, although it would more properly be called the theoretical underpinning of the algorithm.

Definition 9. Let $K(G)$ be the graph G with the addition of a new vertex \tilde{q} and edges $\{\tilde{q}, v_i\}$ for all vertices v_i , including $q = v_0$. Set \tilde{q} as the sink vertex of $K(G)$.

Thus, the set of nonsink vertices of $K(G)$ is V , and each of these is connected to the sink, \tilde{q} , by an edge. The divisors on G are exactly the configurations on $K(G)$.

Definition 10. A divisor $c \in \mathbb{Z}^{n+1}$ on G is called *quasi-superstable* if $c = \tilde{c} - 1_V$ for some superstable \tilde{c} on $K(G)$. The collection of quasi-superstable divisors is denoted \mathcal{S} .

For the relation between the superstables and quasi-superstables of G , see Theorem 26 (2).

3 Correspondences between the structures

So far, we have defined the following structures on G :

- \mathbb{O} : G -semiorientations,
- \mathbb{I} : G -semiorders,
- \mathcal{S} : the G -semiorder arrangement,
- \mathcal{S} : quasi-superstable divisors on G .

In this section, we describe relations among these structures, culminating in Theorems 19 and 23.

3.1 Semiorders and semiorientations

Definition 11. A G -semiorder P and a G -semiorientation \mathcal{O} are *compatible* if for each edge $e = \{u, v\}$ of G , we have that $u < v$ if and only if $(u, v) \in \mathcal{O}$. Thus, if $e \notin \mathcal{O}$, then u and v are not comparable in P .

Given a G -semiorder, P , define

$$\mathcal{O}_P = \{(u, v) : \{u, v\} \in E \text{ and } u < v \text{ in } P\}.$$

Theorem 12. Let P be a G -semiorder. Then $\mathcal{O}_P \in \mathbb{O}$ and \mathcal{O}_P is the unique element of \mathbb{O} compatible with P .

Proof. The only part of this theorem that is not immediate from the definitions is the fact that every potential cycle for \mathcal{O}_P has more blank edges than oriented edges, which we now prove. Let $\alpha = \{e_1, \dots, e_k\}$ be a potential cycle. We may assume that $e_i = \{u_i, u_{i+1}\}$ where $u_{k+1} = u_1$, and that for each i , either (i) $(u_i, u_{i+1}) \in \mathcal{O}_P$, in which case $u_i < u_{i+1}$, or (ii) e_i is a blank edge, in which case u_i and u_{i+1} are not comparable. Since P is a semiorder, it is isomorphic to a semiorder of intervals, allowing us to identify each u_i with an interval $I_i = [a_i, a_i + 1]$. If $e_i \in \mathcal{O}_P$, we have $I_i < I_{i+1}$, in which case $a_i < a_{i+1} - 1$; and if $e_i \notin \mathcal{O}_P$, then I_i and I_{i+1} overlap, so in particular, $a_i \leq a_{i+1} + 1$. Thus, if there are δ oriented edges and β blank edges in α , since $u_{k+1} = u_1$,

$$a_1 + \delta - \beta \leq a_1,$$

with equality if and only if $\beta = \delta = 0$. However, since α has at least one edge, the inequality must be strict, and $\beta > \delta$ as required. \square

Thus, we can refer to *the* semiorientation determined by a G -semiorder and define the mapping

$$\begin{aligned} \nu : \mathbb{I} &\rightarrow \mathbb{O} \\ P &\mapsto \mathcal{O}_P. \end{aligned}$$

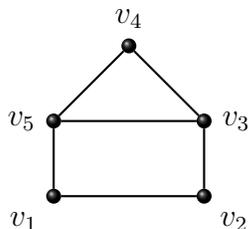


Figure 7: The house graph.

Example 13. Let G be the *house graph* of Figure 7. Figure 8 depicts a G -semiorder and its corresponding compatible G -semiorientation.

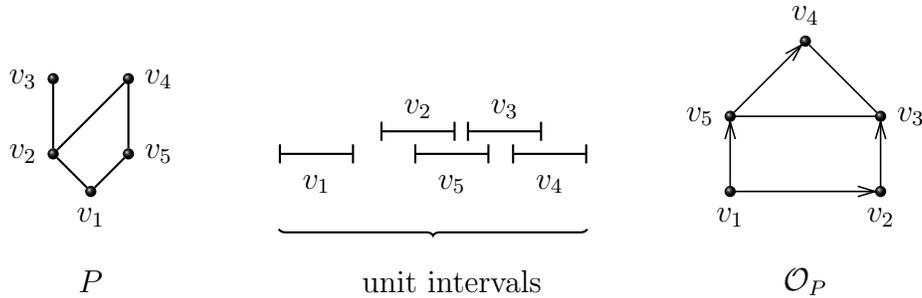


Figure 8: A G -semiorder P on the vertices of the house graph, G ; a collection of intervals realizing the semiorder; and the unique G -semiorientation, $\nu(P) = \mathcal{O}_P$, compatible with P .

3.2 A bijection between semiorientations and hyperplane regions

We now define a mapping

$$\rho: \mathbb{O} \rightarrow \mathcal{R}.$$

If $\mathcal{O} \in \mathbb{O}$, let $\rho(\mathcal{O})$ be the region defined by the following inequalities: for each edge e of G :

- if $e = (v_i, v_j) \in \mathcal{O}$, then $x_j > x_i + 1$,
- if $e \notin \mathcal{O}$, then $|x_i - x_j| < 1$.

The reader may find it helpful to skim the example below the proof of the following theorem before reading the proof itself.

Theorem 14. *The mapping ρ is a well-defined bijection.*

Proof. Let $\mathcal{O} \in \mathbb{O}$, and let $r = \rho(\mathcal{O})$. The system of inequalities defining r indicates on which side of each hyperplane of \mathcal{S} the region sits. To show r is a region of \mathcal{S} , it suffices to show that r is nonempty.

We define a directed, weighted graph G' with the same vertices as G in the following manner: for each edge e between vertices of G ,

- if $e = (v_i, v_j) \in \mathcal{O}$, then $(v_i, v_j) \in G'$ and the weight of (v_i, v_j) in G' is -1 ,
- if $e = \{v_i, v_j\} \notin \mathcal{O}$, then $(v_i, v_j), (v_j, v_i) \in G'$ and the weight of both (v_i, v_j) and (v_j, v_i) in G' is 1 .

Choose an ordering of the edges of G' : e'_1, e'_2, \dots, e'_k . Define a $k \times (n+1)$ edge-vertex adjacency matrix with rows r_1, \dots, r_k as follows: if $e'_\ell = (v_i, v_j)$, let r_ℓ be the vector having 1 in the i th entry, -1 in the j th entry, and 0 s elsewhere. Let b be a column vector in \mathbb{R}^k where b_ℓ is the weight of e'_ℓ .

Thus, the inequalities of $\rho(\mathcal{O})$ are encoded as $Ax < b$. By Farkas' lemma the insolvability of $Ax < b$ is equivalent to the existence of a row vector $y = (y_1, \dots, y_k)$ satisfying:

$$y_i \geq 0 \quad \forall i, \quad y \neq 0, \quad yA = 0, \quad y \cdot b \leq 0. \quad (1)$$

For sake of contradiction, suppose such a y exists. The *support* of y is

$$\text{supp}(y) = \{i : y_i \neq 0\}.$$

Among all row vectors satisfying condition (1), suppose y has been chosen so that the cardinality of its support is minimal. Say $\ell_1 \in \text{supp}(y)$ and $e'_{\ell_1} = (v_i, v_j)$. Hence, r_{ℓ_1} has a -1 in its v_j -th entry. Since

$$yA = y_1 r_1 \cdots + y_k r_k = 0,$$

and the components of y are nonnegative, there must be some $\ell_2 \in \text{supp}(y)$ such that r_{ℓ_2} has a 1 in its v_j -th entry. This row will have a -1 in some other entry, forcing the existence of some $\ell_3 \in \text{supp}(y)$ such that r_{ℓ_3} has a 1 in that entry, and so on. Since the support of y is finite, the sequence ℓ_1, ℓ_2, \dots , eventually has a repeat. Thus, there is a sequence of elements $j_1 := \ell_{m+1}, j_2 := \ell_{m+2}, \dots, j_t := \ell_{m+t}$ in the support of y for some t corresponding to a directed cycle of edges $e'_{j_1}, \dots, e'_{j_t}$ in G' .

Let $z = (z_1, \dots, z_k)$ be the row vector with $z_\ell = 1$ if $\ell \in \{j_1, \dots, j_t\}$ and $z_\ell = 0$, otherwise. Since the support of z corresponds to a directed cycle of edges in G' , we have $zA = 0$. Furthermore, since any potential cycle in $\rho(\mathcal{O})$ has more blank edges than oriented edges, we have $z \cdot b > 0$. Let $a = \min\{y_{j_1}, \dots, y_{j_t}\}$ and define $y' = y - az$. Then y' satisfies condition (1) but its support is strictly contained in the support of y , yielding a contradiction. So there must be some solution to $Ax < b$, which means that $r = \rho(\mathcal{O}) \in \mathcal{R}$.

We now define a mapping

$$\tau : \mathcal{R} \rightarrow \mathbb{O}.$$

If $r \in \mathcal{R}$ let $\tau(r)$ be the partial orientation of G obtained by (i) $(v_i, v_j) \in \tau(r)$ if $\{v_i, v_j\} \in E$ and $x_j > x_i + 1$ in r , and (ii) all other edges of G are blank. Once we show τ is well-defined, it is immediate that it is the inverse of ρ .

Let $r \in \mathcal{R}$. To see that $\tau(r) \in \mathbb{O}$, suppose $\tau(r)$ has a potential cycle α having at least as many oriented edges as blank edges. Define A and b as above to encode the system of inequalities that defines the region r as $Ax < b$. Let y be the row vector with 1 s in the entries corresponding to the (oriented and blank) edges of α and 0 s elsewhere. We have $y \geq 0$, $y \neq 0$, $yA = 0$, and $y \cdot b \leq 0$. But by Farkas' lemma this means $Ax < b$ has no solutions, contradicting the fact that $r \in \mathcal{R}$. Thus, $\tau(r) \in \mathbb{O}$. \square

Example 15. This example is intended to be read in conjunction with the proof of Theorem 14 and illustrates Farkas' lemma at work. Consider the partial orientation, \mathcal{O} , of a triangle, pictured in Figure 9. Naively attempting to apply the region-labeling function, ρ ,

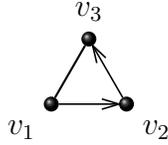


Figure 9: Partial orientation, \mathcal{O} .

to \mathcal{O} results in the system of inequalities $Ax < b$ where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

The rows of A correspond to directed edges, (v_1, v_2) , (v_2, v_3) , (v_1, v_3) , and (v_3, v_1) , respectively, of the underlying triangle, G . The last two inequalities combine to describe the “sandwich” $|x_1 - x_3| < 1$, corresponding to the undirected edge, $\{v_1, v_3\}$.

The partial orientation, \mathcal{O} , has a potential cycle with more oriented edges than blanks, and hence is not a G -semiorientation. By the way in which A is defined in the proof of Theorem 14, we know that adding the rows of A corresponding to the directed cycle (v_1, v_2) , (v_2, v_3) , (v_3, v_1) gives the zero vector. Our system of inequalities is inconsistent since adding the corresponding entries of b gives -1 (since there are more oriented edges than blanks in \mathcal{O} for this cycle).

Let r be a region in the image of ρ . If $t = (t_0, \dots, t_n) \in r$, define the unit intervals $I_i = [t_i, t_i + 1]$ for $i = 0, \dots, n$. Define P_t to be the G -semiorder determined by these intervals, labeled by the vertices of G by identifying I_i with v_i .

Theorem 16. *Let $\mathcal{O} \in \mathbb{O}$, and let $r = \rho(\mathcal{O})$. The semiorders P_t as t ranges over points in r are exactly the G -semiororders compatible with \mathcal{O} .*

Proof. Choose any $t \in r$. Say $(v_i, v_j) \in \mathcal{O}$. Then $x_j > x_i + 1$ in r ; so $t_j > t_i + 1$, and hence, $v_i < v_j$ in P_t . Now suppose $e = \{v_i, v_j\} \in E$ but $e \notin \mathcal{O}$. Then $|t_i - t_j| < 1$, which means that I_i and I_j overlap, and hence, v_i and v_j are not comparable in P_t . This shows that P_t is compatible with \mathcal{O} .

Now let P be a G -semiorder compatible with \mathcal{O} . The semiorder P is isomorphic to the semiorder on a set of unit intervals, $\{I_i\}_{i=0}^n$, where I_i corresponds to v_i . Say $I_i = [t_i, t_i + 1]$ for each i , and let $t = (t_0, \dots, t_n)$. So $P = P_t$, but we must show that $t \in r$. Suppose $e = \{v_i, v_j\} \in E$. If $x_j > x_i + 1$ in r , then since $\mathcal{O} = \tau(r)$, we have $(v_i, v_j) \in \mathcal{O}$, and thus $t_j > t_i + 1$. If $|x_i - x_j| < 1$, then $e \notin \mathcal{O}$ and the intervals I_i and I_j overlap, i.e., $|t_i - t_j| < 1$. Hence, $t \in r$. \square

Example 17. Let G be the house graph pictured in Figure 7, and let \mathcal{O} be the G -semiorientation pictured on the right in Figure 8. The region $\rho(\mathcal{O})$ corresponding to \mathcal{O} is

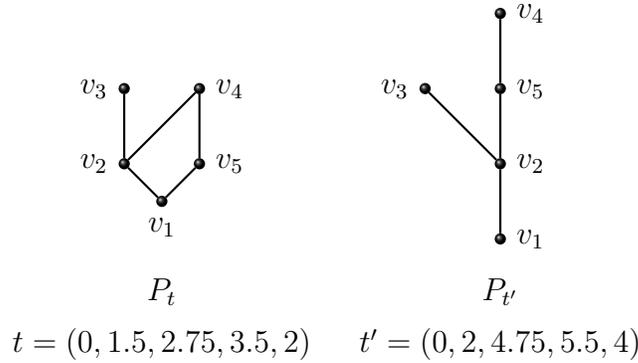


Figure 10: Two semiorders.

determined by the system of inequalities:

$$\begin{aligned}
 x_2 &> x_1 + 1, & x_3 &> x_2 + 1, \\
 x_4 &> x_5 + 1, & x_5 &> x_1 + 1, \\
 |x_3 - x_4| &< 1, & |x_3 - x_5| &< 1.
 \end{aligned}$$

Figure 10 displays points $t, t' \in \rho(\mathcal{O})$ whose corresponding semiorders, P_t and $P_{t'}$, are the two G -semiorders compatible with \mathcal{O} .

3.3 Superstables algorithm

So far, we have described relations among three of our four structures: G -semiorders, G -semiorientations, and the regions of the G -semiorder arrangement. We would now like to forge a connection between these and the quasi-superstables on G (ultimately relating all of this, in Section 4, to ordinary superstables).

Dhar's burning algorithm says that starting from a recurrent configuration, then firing the sink, each non-sink vertex will fire exactly once in the stabilization process. We use these vertex firings to build a partial orientation starting from the unoriented graph. To sketch the idea, suppose that at some point in the stabilization process there are nonsink vertices v and w such that $e = \{v, w\}$ is an edge and v is unstable. Suppose e has not already been oriented or marked as a blank edge. When v fires, if w is stable, orient e as (v, w) ; otherwise mark e blank. Since all vertices fire, each edge is visited.

Starting with a quasi-superstable, c , on G , Proposition 7 provides a corresponding recurrent configuration on $K(G)$. We use Dhar's algorithm as above to create a G -semiorientation, \mathcal{O} , of G . Using the construction of \mathcal{O} as a guide, one may further refine the procedure to simultaneously create a G -semiorder compatible with \mathcal{O} . This is the idea behind the intervals labeled by J in the algorithm, below. Theorem 19 then provides the important connection: subtract one from each entry of the indegree sequence relative to \mathcal{O} to recapture c .

A careful description of our algorithm follows. The input is a quasi-superstable divisor on G and a vertex ordering, and the output is a G -semiorder P and the G -semiorientation \mathcal{O}_P compatible with P . Letting \mathfrak{S} be the symmetric group on $\{0, \dots, n\}$, identify $\sigma \in \mathfrak{S}$ with the vertex ordering $(v_{\sigma(0)}, \dots, v_{\sigma(n)})$. Then our algorithm defines two mappings $\phi : \mathcal{S} \times \mathfrak{S} \rightarrow \mathbb{I}$ and $\eta : \mathcal{S} \times \mathfrak{S} \rightarrow \mathbb{O}$. Theorem 19, to follow, validates the algorithm and shows that it produces a commutative diagram

$$\begin{array}{ccc} \mathcal{S} \times \mathfrak{S} & \xrightarrow{\phi} & \mathbb{I} \\ & \searrow \eta & \downarrow \nu \\ & & \mathbb{O}, \end{array} \quad (2)$$

where ν is the mapping defined in Section 3.1.

Given a $c \in \mathcal{S}$ and a vertex ordering $\sigma \in \mathfrak{S}$, the algorithm proceeds as follows:

INITIALIZATION

Let c_{\max} be the maximal stable configuration of $K(G)$, and let

$$b = c_{\max} - c.$$

Since $c \in \mathcal{S}$, we can write $c = \tilde{c} - 1_V$ for some superstable \tilde{c} on $K(G)$. Therefore, $b = (c_{\max} - \tilde{c}) + 1_V$, i.e., b is the configuration on $K(G)$ obtained from $c_{\max} - \tilde{c}$ by firing the sink \tilde{q} of $K(G)$. By Proposition 7, $c_{\max} - \tilde{c}$ is a recurrent configuration on $K(G)$; so by Proposition 8, every element of V will fire exactly once while stabilizing b .

Let u_1, \dots, u_k be the vertices that are unstable in b . Take this list of vertices to be ordered according to σ , that is, if $u_i = v_{\sigma(\ell)}$ and $u_j = v_{\sigma(m)}$ with $\ell < m$, then $i < j$. For each u_i , associate an interval,

$$J(u_i) = [i/(k+1), 1 + i/(k+1)].$$

Thus, all the $J(u_i)$ overlap. Form a queue,

$$Q = (u_1, \dots, u_k).$$

Initialize the partial orientation of G as $\mathcal{O} = \emptyset$.

LOOP

Repeat the following until the queue is empty:

Say the queue is $Q = (w_0, \dots, w_\ell)$, and

$$J(w_0) = [\alpha, \alpha + 1].$$

If $\ell > 0$, define ε using the interval for w_1 :

$$J(w_1) = [\alpha + \varepsilon, \alpha + \varepsilon + 1].$$

Otherwise, take $\varepsilon = 1$.

Fire w_0 and replace b by the resulting configuration. Remove w_0 from the queue and mark it so that it will never again appear in the queue.

Let z_1, \dots, z_t be the vertices that just became unstable with the firing of w_0 and that have not yet been fired by the algorithm, listed in order according to σ . Define

$$J(z_i) = [\alpha + 1 + i\varepsilon/(t + 1), \alpha + 2 + i\varepsilon/(t + 1)].$$

Add the z_i to Q , in order:

$$Q = (w_1, \dots, w_\ell, z_1, \dots, z_t).$$

For each edge $e = \{w_0, v\}$ that is not oriented or marked as blank,

1. if v was already unstable before the firing of w_0 , mark e as blank, so e will not subsequently be added to \mathcal{O} ;
2. otherwise orient the edge out from v , i.e., add (w_0, v) to \mathcal{O} .

OUTPUT

The intervals $\{J(v) : v \in V\}$ determine a semiorder. Identifying v with $J(v)$ gives a semiorder, P , on V . Define $\phi(c) = P$ and $\eta(c) = \mathcal{O}$. □

Example 18. Let G be the house graph of Figure 7, and fix the vertex ordering, v_1, \dots, v_5 . Then $c = (-1, 0, 0, 0, 0)$ is a quasi-superstable divisor on G . Figure 11 illustrates the application of the superstables algorithm to c and the given vertex ordering, producing a G -semiorientation, \mathcal{O} . The dotted lines emanating from each vertex to the exterior of the house graph represent the edges to the sink of $K(G)$.

We have $c_{\max} = (2, 2, 3, 2, 3)$ for $K(G)$. The algorithm starts with the configuration $b = c_{\max} - c$ in the top left corner of Figure 11. The vertex v_1 is unstable in b , and firing produces two oriented edges and two unstable vertices, v_2 and v_5 . Since v_2 comes first in the vertex ordering, it is fired next. The subscripts keep track of vertex-firing precedence. Note that when v_5 is fired, the edge $\{v_3, v_5\}$ is marked blank since, by that time, v_3 is unstable. Proceeding clockwise around the diagram, the algorithm terminates at the bottom left corner.

Let $\tilde{c} := c + 1_V$, a superstable configuration on $K(G)$. When the algorithm terminates, the resulting configuration is $c_{\max} - \tilde{c}$, which is recurrent by Proposition 7. Adding 1_V reproduces the starting configuration c . However, adding 1_V is equivalent to firing the sink of $K(G)$, which explains (via Dhar's burning algorithm) why every vertex was guaranteed to fire.

Theorem 19. *Given $P \in \mathbb{I}$, for each vertex v let $n_P(v)$ denote the cardinality of the set*

$$\{u \in V : u < v \text{ and } \{u, v\} \in E\}.$$

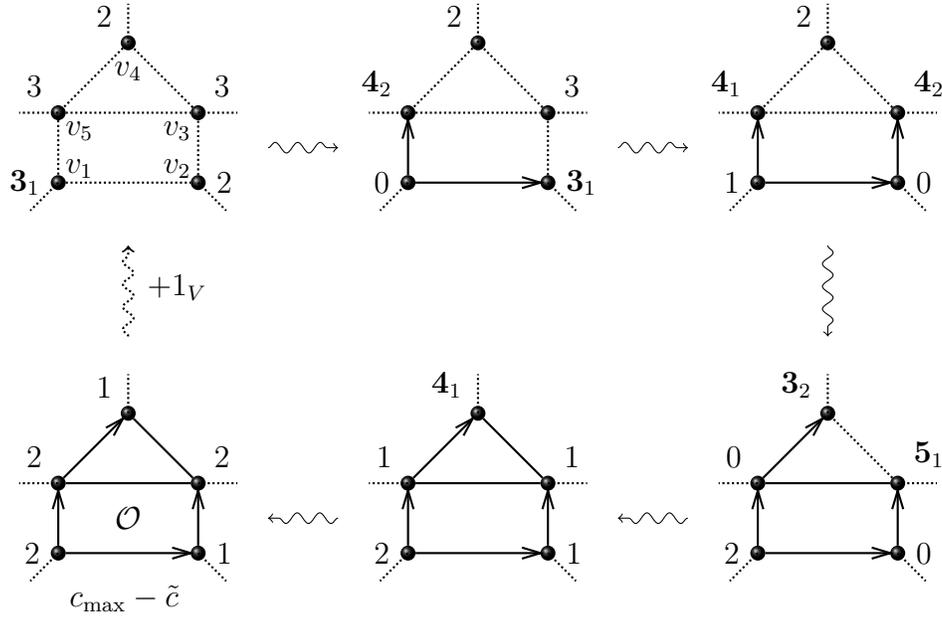


Figure 11: Superstables algorithm (cf. Example 18). Start in the upper-left corner. Bold numbers connote unstable vertices. Subscripts indicate firing precedence.

There are mappings

$$\begin{aligned} \theta : \mathbb{I} &\rightarrow \mathcal{S} \\ P &\mapsto \sum_{v \in V} (n_P(v) - 1) v, \end{aligned}$$

and

$$\begin{aligned} \psi : \mathbb{O} &\rightarrow \mathcal{S} \\ \mathcal{O} &\mapsto \sum_{v \in V} (\text{indeg}_{\mathcal{O}}(v) - 1) v. \end{aligned}$$

Let $\pi : \mathcal{S} \times \mathfrak{S} \rightarrow \mathcal{S}$ be the first projection mapping, and let ϕ and η be the mappings defined by the superstables algorithm.

The following diagram commutes:

$$\begin{array}{ccc} & \mathcal{S} \times \mathfrak{S} & \\ & \downarrow \pi & \\ \mathbb{I} & \mathcal{S} & \mathbb{O} \\ \uparrow \phi & \leftarrow \eta & \\ \theta & & \psi \\ \downarrow & & \downarrow \\ \mathbb{I} & \xrightarrow{\nu} & \mathbb{O} \end{array} \quad (3)$$

All mappings in the diagram are surjective except possibly for ϕ and η .

Proof. The image of ϕ is in \mathbb{I} . In stabilizing b , each vertex eventually becomes unstable and is assigned an interval. Thus, the semiorder produced by the algorithm is a semiorder on all the vertices of G .

The image of θ is in \mathcal{S} . Let $P \in \mathbb{I}$. Then P is isomorphic to the semiorder on a collection of intervals, $\{J(v) : v \in V\}$. Letting $c = \theta(P)$, we must show that $c + 1_V$ is a superstable configuration of $K(G)$. Let c_{\max} be the maximal stable configuration on $K(G)$, and define $b = c_{\max} - (c + 1_V)$. Note that $b \geq 0$. Starting with b and firing the sink of $K(G)$ gives $\tilde{b} = c_{\max} - c$. By Propositions 7 and 8, we must show that there is an ordering of V forming a legal firing sequence for \tilde{b} .

Choose any $\sigma \in \mathfrak{S}$ so that $\sigma(i) < \sigma(j)$ if $\min J_i < \min J_j$. In particular, this means that if $v_i < v_j$ in P , then $\sigma(i) < \sigma(j)$. The v -th component of \tilde{b} is

$$\tilde{b}_v = \deg_{K(G)}(v) - n_P(v).$$

We can legally fire the vertices in the order given by σ since when it becomes v 's turn to fire, it will have received $n_P(v)$ grains of sand from the firings of those vertices u neighboring v such that $u < v$ and will thus be unstable.

We have $\theta \circ \phi = \pi$. Let $P = \phi(c, \sigma)$ and $b = c_{\max} - c$ where c_{\max} is the maximal stable configuration on $K(G)$. Let $v \in V$. If v is unstable in b , then $c_v = -1$ and there are no vertices smaller than v in P . Hence, $n_P(v) - 1 = c_v = -1$, as required. Otherwise, in the course of the algorithm, say u is the vertex whose firing causes v to become unstable. Then the vertices that are less than v in P are exactly the vertices w such that $w \leq u$. Among these w , only those that are attached to v by an edge contribute to making v unstable. Thus, exactly $n_P(v)$ grains of sand are added to v in b to make v unstable. Since $b_v = \deg_{K(G)} v - 1 - c_v$, we have that $n_P(v) = 1 + c_v$, as required.

The mapping ν is surjective. The surjectivity of ν follows from Theorem 16.

We have $\psi \circ \nu = \theta$ and the image of ψ is in \mathcal{S} . Let $P \in \mathbb{I}$ and $\mathcal{O} = \nu(P)$. Since \mathcal{O} is compatible with P , we have that $n_P(v) = \text{indeg}_{\mathcal{O}}(v)$ for each $v \in V$. Hence, $\psi(\nu(P)) = \theta(P)$ as mappings of configurations. Since ν is surjective, the image of ψ is contained in the image of θ , hence in \mathcal{S} .

We have $\nu \circ \phi = \eta$. Given $(c, \sigma) \in \mathcal{S} \times \mathfrak{S}$, let $\mathcal{O} = \eta(c, \sigma)$ and $P = \phi(c, \sigma)$. We must show that \mathcal{O} is compatible with P . Let b be as in the algorithm. Run the algorithm up until it is a vertex u 's turn to fire. Suppose $\{u, v\} \in E$. Then v being stable at this point is equivalent to $u < v$ in P and equivalent to $(u, v) \in \mathcal{O}$. \square

Corollary 20. *Let $\sigma \in \mathfrak{S}$. Then the mappings*

$$\phi_\sigma: \mathcal{S} \rightarrow \mathbb{I}, \quad \eta_\sigma: \mathcal{S} \rightarrow \mathbb{O},$$

defined by $\phi_\sigma(c) = \phi(c, \sigma)$ and $\eta_\sigma(c) = \eta(c, \sigma)$, are injective with left inverses θ and ψ , respectively.

Let \mathcal{S}_{\max} denote the maximal quasi-superstables under the relation " $<$ " defined in Section 2.4, and let \mathbb{O}_{\max} denote the acyclic orientations of G (elements of \mathbb{O} in which

each edge is oriented). For each $c \in \mathcal{S}$, define the degree of c to be $\deg(c) := \sum_{v \in V} c_v$. Then,

1. the restriction of η_σ to \mathcal{S}_{\max} gives a bijection

$$\eta_\sigma : \mathcal{S}_{\max} \rightarrow \mathbb{O}_{\max},$$

2. for $c \in \mathcal{S}$, we have $\deg(c) \leq g - 1$, where $g := |E| - |V| + 1$ is the genus of G , with equality if and only if $c \in \mathcal{S}_{\max}$.

Proof. Both ϕ_σ and η_σ are injective on \mathcal{S} with the claimed left inverses since the restriction of π in diagram 3 of Theorem 19 to $\mathcal{S} \times \{\sigma\}$ is bijective.

To show $\eta_\sigma(\mathcal{S}_{\max}) \subseteq \mathbb{O}_{\max}$, let $c \in \mathcal{S}$ and suppose that $\mathcal{O} := \eta_\sigma(c) \notin \mathbb{O}_{\max}$. According to Theorem 16 there is a point $t = (t_0, \dots, t_n) \in \mathbb{R}^{n+1}$ and a corresponding G -semiorder, P_t , such that $\nu(P_t) = \mathcal{O}$. For ease of notation, we may assume that $t_0 \leq \dots \leq t_n$. Choose $t' \in \mathbb{R}^{n+1}$ such that $t'_i \geq t_i$ for all i and $t'_{i+1} > t'_i + 1$ for $0 \leq i \leq n - 1$. Let $\mathcal{O}' := \nu(P_{t'})$ and $c' := \psi(\mathcal{O}')$. Then the set of oriented edges \mathcal{O} is a proper subset of \mathcal{O}' . Hence, $c < c'$, showing $c \notin \mathcal{S}_{\max}$, as desired.

Now let $c \in \mathcal{S}$, and choose any $\mathcal{O} \in \mathbb{O}$ such that $c = \psi(\mathcal{O})$. From the definition of ψ , we have $\deg(c) = |\mathcal{O}| - |V| \leq |E| - |V| = g - 1$, with equality exactly when $\mathcal{O} \in \mathbb{O}_{\max}$. If $c < c'$ for some quasi-superstable c' , then since $\deg(c) < \deg(c')$, it follows that $\mathcal{O} \notin \mathbb{O}_{\max}$. This shows, for instance, that $\psi(\mathbb{O}_{\max}) \subseteq \mathcal{S}_{\max}$.

It remains to be shown that ψ restricted to \mathbb{O}_{\max} is injective. Let $c \in \mathcal{S}_{\max}$ and choose any \mathcal{O} such that $c = \psi(\mathcal{O})$. We have seen that $\mathcal{O} \in \mathbb{O}_{\max}$. From the definition of ψ , we see that $\text{indeg}_{\mathcal{O}}(v)$ is determined by c for each $v \in V$. Then $\text{outdeg}_{\mathcal{O}}(v) = \deg_G(v) - \text{indeg}_{\mathcal{O}}(v)$ since \mathcal{O} orients every edge of G , and as noted in [3], these outdegrees determine \mathcal{O} . To see this, consider the graph G oriented by \mathcal{O} . Since \mathcal{O} is acyclic, this oriented graph must have sinks, i.e., vertices with outdegree 0. Thus, c determines the orientation of all edges incident on these sinks. Now remove these sinks and their incoming edges. The orientation of the resulting graph is still acyclic, and its sinks may also be determined from c . Iterate to see that \mathcal{O} is determined by c , and hence, ψ is injective when restricted to \mathbb{O}_{\max} . \square

Theorem 21. *The mapping ν is bijective if and only if G is a complete graph.*

Proof. The surjectivity of ν is part of Theorem 19. Suppose that G is a complete graph and that $\nu(P_1) = \nu(P_2)$ for some $P_1, P_2 \in \mathbb{I}$. Then $v_i < v_j$ in P_1 implies $(v_i, v_j) \in \nu(P_1)$, which in turn implies $v_i < v_j$ in P_2 . Similarly, $v_i < v_j$ in P_2 implies $v_i < v_j$ in P_1 . Thus, $P_1 = P_2$, so ν is injective.

Now suppose G is not complete, i.e., there exist $v_i, v_j \in V$ such that $\{v_i, v_j\} \notin E$. Let P_1 be the G -semiorder where no vertices are comparable, and let P_2 be the G -semiorder where $v_i < v_j$ and all other pairs of vertices are incomparable. We have $\nu(P_1) = \emptyset = \nu(P_2)$. So ν is not injective. \square

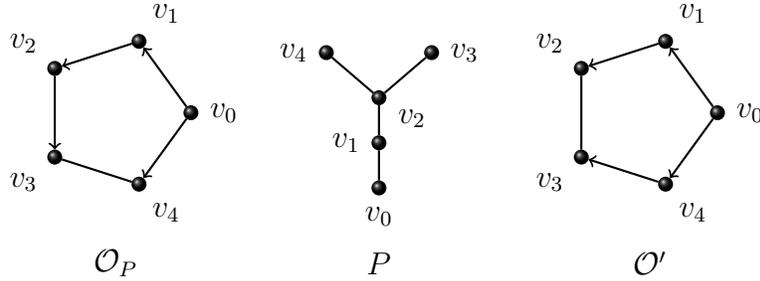


Figure 12: Figure for Example 22.

Example 22. The mappings ϕ and η are not necessarily surjective. Consider the case where G is a cycle graph on five vertices. Figure 12 displays the Hasse diagram of a G -semiorder, P , and its compatible G -semiorientation $\nu(P) = \mathcal{O}_P$. For any ordering σ of the vertices, $\eta(\psi(\mathcal{O}_P), \sigma) = \mathcal{O}'$ where \mathcal{O}' is as pictured in Figure 12.

3.4 Labeling regions with superstables

We can inductively label the regions of the G -semiorder arrangement, \mathcal{S} , with quasi-superstables. We call this labeling the (generalized) *Pak-Stanley* labeling after [16]. Start with the central region of \mathcal{S} , the region defined by $|x_i - x_j| < 1$ for all i, j such that $\{v_i, v_j\}$ is an edge of G . Label this region with the divisor -1_V . Put the central region in a queue, Q . Then, for as long as Q is not empty:

1. Remove the first region r from Q .
2. For each unlabeled region r' bordering r :
 - (a) Determine the unique indices $i \neq j$ such that $|x_i - x_j| < 1$ in r but $x_j > x_i + 1$ in r' .
 - (b) If r is labeled by $c = \sum_{k=0}^n c_k v_k$, then label r' by $c' = c + v_j$.
 - (c) Add r' to the end of Q . □

For each $r \in \mathcal{R}$, define $\lambda(r)$ to be the label assigned to r by the above algorithm. Theorem 23 guarantees that when the algorithm terminates, the regions are labeled with quasi-superstables and the labels are, in fact, independent of the order in which regions are removed from the queue.

Recall the bijection $\rho: \mathbb{O} \rightarrow \mathcal{R}$, and let $\tau = \rho^{-1}$. Also recall the mapping ψ from Theorem 19 and the mapping η_σ from Corollary 20.

Theorem 23. *We have $\lambda = \psi \circ \tau$. So there is a surjective mapping*

$$\lambda: \mathcal{R} \twoheadrightarrow \mathcal{S}$$

and, for each $\sigma \in \mathfrak{S}$, a commutative diagram

$$\begin{array}{ccc}
 \mathcal{S} & \xrightarrow{\eta_\sigma} & \mathbb{O} \\
 & \swarrow \lambda & \downarrow \rho \\
 & & \mathcal{R}.
 \end{array} \tag{4}$$

Proof. All the statements in the theorem follow directly from the equality $\lambda = \psi \circ \tau$, which we now prove by induction. If r is the central region then $\tau(r)$ is the partial orientation in which all edges are marked blank. Therefore, in this case, $\lambda(r) = \psi(\tau(r))$.

Let the labeling algorithm run, and suppose that the region r has just been removed from the queue. By induction, suppose that $\lambda = \psi \circ \tau$ when restricted to those regions that have been labeled so far. Let r' be an unlabeled region bordering r . Say that the inequalities that define r are the same as those that define r' except that $|x_i - x_j| < 1$ in r and $x_j > x_i + 1$ in r' . It follows that if $\mathcal{O} = \tau(r)$ then $\tau(r') = \mathcal{O} \cup \{(v_i, v_j)\}$. Therefore, going from $\tau(r)$ to $\tau(r')$, only the indegree of v_j has increased by one, so

$$\begin{aligned}
 \lambda(r') &= \lambda(r) + v_j && \text{(algorithm)} \\
 &= \psi(\tau(r)) + v_j && \text{(induction)} \\
 &= \psi(\tau(r')).
 \end{aligned}$$

The result follows by induction. □

4 Fixing a sink vertex

In this section, we finally fully explain Figure 2. For that figure, we started with a graph with a given sink vertex and constructed a hyperplane arrangement with regions labeled by partial orientations of the graph and sandpile configurations. The nonnegative configurations arising were exactly the superstable for the graph.

Let G be a graph with vertices $V = \{v_0, \dots, v_n\}$, and designate vertex v_0 as the sink.

Definition 24. The (G, v_0) -semiorientations, denoted \mathbb{O}_0 , are the G -semiorientations satisfying the additional requirement that v_0 is a source:

$$\mathbb{O}_0 = \{\mathcal{O} \in \mathbb{O} : \text{outdeg}_{\mathcal{O}}(v_0) = \deg(v_0)\}.$$

The set of *admissible* (G, v_0) -semiorientations is

$$\tilde{\mathbb{O}}_0 = \{\mathcal{O} \in \mathbb{O}_0 : \text{indeg}_{\mathcal{O}}(v_i) \geq 1 \text{ for all } i \neq 0\}.$$

Thus, while v_0 is the sink for the sandpile model on G —i.e., the sink for the sake of defining the sandpile group, superstable configurations, and G -parking functions—it is a source for any $\mathcal{O} \in \mathbb{O}_0$.

Example 25. Figure 2 shows the admissible (G, q) -semiorientations for the graph in Figure 3. Due to lack of space, the requisite edges (q, v_1) and (q, v_2) are understood in Figure 2, but not drawn.

Our next goal is to describe the image of \mathbb{O}_0 under the mapping $\psi: \mathbb{O} \rightarrow \mathcal{S}$ from Theorem 19, thus accounting for the configurations labeling the regions in Figure 2. We show below that the image of \mathbb{O}_0 is the set of quasi-superstables assigning the value -1 to v_0 and whose only other negative values must occur at vertices not connected to v_0 by an edge. Further, the image of $\tilde{\mathbb{O}}_0 \subseteq A_0$ is the set of G -parking functions of G .

Let $\tilde{V} = V \setminus \{v_0\}$, and let

$$X = \{v \in \tilde{V} : \{v, v_0\} \notin E\}.$$

Define $1_X = \sum_{v \in X} v$, a configuration on G (having chosen v_0 as the sink). Let $K(G)_0$ denote the graph G but with an edge $\{v, v_0\}$ added for each $v \in X$, and fix v_0 as its sink. Thus, configurations on G and on $K(G)_0$ are elements of $\mathbb{Z}\tilde{V}$, the free abelian group on \tilde{V} , a subgroup of $\mathbb{Z}V$, the configurations on $K(G)$. Recall that $K(G)$ is the graph used to define quasi-superstables.

Theorem 26.

1. Define

$$\mathcal{S}_0 = \{c \in \mathcal{S} : c_{v_0} = -1 \text{ and } c + v_0 \geq -1_X\}.$$

Then

$$\begin{aligned} \psi(\mathbb{O}_0) &= \{\tilde{c} - v_0 : \tilde{c} + 1_X \text{ a superstable on } K(G)_0\} \\ &= \mathcal{S}_0. \end{aligned}$$

2. Define

$$\tilde{\mathcal{S}}_0 = \{c \in \mathcal{S} : c_{v_0} = -1 \text{ and } c + v_0 \geq 0\} \subseteq \mathcal{S}_0.$$

Then

$$\begin{aligned} \psi(\tilde{\mathbb{O}}_0) &= \{\tilde{c} - v_0 : \tilde{c} \text{ a superstable on } G\} \\ &= \{\tilde{c} - v_0 : \tilde{c} + 1_X \text{ a superstable on } K(G)_0 \text{ and } \tilde{c} \geq 0\} \\ &= \tilde{\mathcal{S}}_0. \end{aligned}$$

Thus, $\psi(\tilde{\mathbb{O}}_0) = \tilde{\mathcal{S}}_0$ is the set of G -parking functions with respect to v_0 .

Proof. We first prove

Claim A: If $c \in \mathbb{Z}\tilde{V}$, then $c + 1_X$ is superstable on $K(G)_0$ if and only if $c - v_0 + 1_V$ is superstable on $K(G)$ and $c \geq -1_X$.

For any graph H , let E_H denote its edges. Recall that \tilde{q} is the sink vertex for $K(G)$. Let $c \in \mathbb{Z}\tilde{V}$ with $c \geq -1_X$, and consider $c - v_0 + 1_V$ as a configuration on $K(G)$. Let $U \subseteq V$. If $v_0 \in U$, then we cannot legally fire U from $c - v_0 + 1_V$: since $(c - v_0 + 1_V)_{v_0} = 0$ and $\{v_0, \tilde{q}\} \in E_{K(G)}$, firing U would result in a configuration with a negative v_0 -component.

Thus, $c - v_0 + 1_V$ is superstable if and only if there are no nonempty subsets $U \subseteq \tilde{V}$ that can be legally fired.

For $v \in U \subseteq \tilde{V}$, consider the edges incident with v that lead out of U :

$$\begin{aligned} M(v, U) &= \{w \in (V \cup \{\tilde{q}\}) \setminus U : \{v, w\} \in E_{K(G)}\} \\ M(v, U)_0 &= \{w \in V \setminus U : \{v, w\} \in E_{K(G)_0}\}. \end{aligned}$$

Then,

$$M(v, U) \setminus M(v, U)_0 = \{\tilde{q}\} \quad \text{and} \quad M(v, U)_0 \setminus M(v, U) = \begin{cases} \emptyset & \text{if } v \notin X, \\ \{v_0\} & \text{if } v \in X. \end{cases}$$

So the cardinality of $M(v, U)$ is

$$|M(v, U)| = \begin{cases} |M(v, U)_0| + 1 & \text{if } v \notin X, \\ |M(v, U)_0| & \text{if } v \in X. \end{cases} \quad (5)$$

Now $c - v_0 + 1_V$ is superstable on $K(G)$ if and only if $(c - v_0 + 1_V)_v < |M(v, U)|$ for all $v \in U$ for all nonempty $U \subseteq \tilde{V}$, and $c + 1_X$ is superstable on $K(G)_0$ if and only if $(c + 1_X)_v < |M(v, U)_0|$ for all $v \in U$ for all nonempty $U \subseteq \tilde{V}$. For $v \in \tilde{V}$ we have

$$(c - v_0 + 1_V)_v = c_v + 1 = \begin{cases} (c + 1_X)_v + 1 & \text{if } v \notin X, \\ (c + 1_X)_v & \text{if } v \in X. \end{cases}$$

So Claim A follows from (5). The condition $c \geq -1_X$ is required in the statement of the claim since superstables must be nonnegative.

Now let

$$P = \{\tilde{c} - v_0 : \tilde{c} + 1_X \text{ superstable on } K(G)_0\}.$$

The fact that $P = \mathcal{S}_0$ follows directly from Claim A. We now show that $\psi(\mathcal{O}_0) = \mathcal{S}_0$ to finish the proof of part (1). Let $\mathcal{O} \in \mathcal{O}_0$. Then $\psi(\mathcal{O}) \in \mathcal{S}$ by Theorem 19. Since v_0 is a source for \mathcal{O} , we have $\psi(\mathcal{O})_{v_0} = -1$, and if $v \in \tilde{V} \setminus X$, then $\psi(\mathcal{O})_v \geq 0$. Thus, $\psi(\mathcal{O}) \in \mathcal{S}_0$. Conversely, given $c \in \mathcal{S}_0$, run the superstables algorithm from Section 3.3 with any vertex ordering of V in which v_0 appears first. Using the notation from the initialization stage of the algorithm, let $b = c_{\max} - c$. Since $c_{v_0} = -1$, the vertex v_0 is unstable in b and will fire first. Since $c + v_0 \geq -1_X$, no vertex $v \in \tilde{V} \setminus X$ is unstable in b . So when v_0 fires, the algorithm will orient each edge incident on v_0 out from v_0 . If \mathcal{O} is the semiorientation produced by the algorithm, it follows that $\mathcal{O} \in \mathcal{O}_0$ and, by Theorem 19, we have $\psi(\mathcal{O}) = c$. Thus, $\psi(\mathcal{O}_0) = \mathcal{S}_0$.

To prove part (2), let

$$\begin{aligned} N &= \{\tilde{c} - v_0 : \tilde{c} \text{ a superstable on } G\}, \\ \tilde{P} &= \{\tilde{c} - v_0 : \tilde{c} + 1_X \text{ a superstable on } K(G)_0 \text{ and } \tilde{c} \geq 0\} \subseteq P. \end{aligned}$$

From part (1), it follows directly that $\psi(\tilde{\mathcal{O}}_0) = \tilde{P} = \tilde{\mathcal{S}}_0$. To show $N = \tilde{\mathcal{S}}_0$ and finish, proceed exactly as in the proof of Claim A. Let $\tilde{c} \in \mathbb{Z}\tilde{V}$ with $\tilde{c} \geq 0$. We must show that \tilde{c} is superstable on G if and only if $\tilde{c} - v_0 + 1_V$ is superstable on $K(G)$. Given $v \in U \subset \tilde{V}$, this time consider the set

$$M(v, U)_G = \{w \in V \setminus U : \{v, w\} \in E_G\},$$

and note that $|M(v, U)| = |M(v, U)_G| + 1$, with $M(v, U)$ defined as before, from which the result follows. \square

Part 2 of the following corollary recaptures a well-known result from sandpile theory. For instance, it occurs as Lemma 5 in [4] as a statement about recurrent configurations, equivalent to our statement in light of Proposition 7.

Corollary 27. *Order the superstable configurations on G by the relation “ $<$ ” defined in Section 2.4. Let \mathcal{S}_{\max} denote the maximal quasi-superstables on G , and let $g := |E| - |V| - 1$ be the genus of G , as in Corollary 20. Let \tilde{c} be a superstable configuration on G . Then,*

1. \tilde{c} is maximal if and only if $\tilde{c} - v_0 \in \tilde{\mathcal{S}}_0 \cap \mathcal{S}_{\max}$,
2. $\deg(\tilde{c}) := \sum_{v \in \tilde{V}} \tilde{c}_v \leq g$ with equality if and only if \tilde{c} is maximal.

Proof. By Theorem 26 (2), $\tilde{c} - v_0 \in \tilde{\mathcal{S}}_0$. Hence, part (2) follows immediately from part (1) by Corollary 20 (2). We now prove part (1).

(\Rightarrow) Suppose $\tilde{c} - v_0 \notin \mathcal{S}_{\max}$. By Theorem 26 (2), there exists $\mathcal{O} \in \tilde{\mathcal{O}}_0$ such that $\psi(\mathcal{O}) = \tilde{c} - v_0$. By Corollary 20, we have $\mathcal{O} \notin \mathcal{O}_{\max}$, i.e., \mathcal{O} is not an acyclic orientation of G , but by the beginning of the proof to Corollary 20, there exists $\mathcal{O}' \in \mathcal{O}_{\max}$ such that \mathcal{O} is a proper subset of \mathcal{O}' . It follows that $\psi(\mathcal{O}) = \tilde{c} - v_0 < \psi(\mathcal{O}') =: c'$, where $c' \in \mathcal{S}$. Since $\mathcal{O} \subset \mathcal{O}'$ and \mathcal{O}' is acyclic, we have $c'_{v_0} = -1$ and $c' + v_0 \geq 0$, i.e., $c' \in \tilde{\mathcal{S}}_0$. Define \tilde{c}' by $c' = \tilde{c}' - v_0$. Then \tilde{c}' is a superstable configuration on G by Theorem 26, and $\tilde{c} < \tilde{c}'$. So \tilde{c} is not a maximal superstable.

(\Leftarrow) Conversely, suppose $\tilde{c} - v_0 \in \mathcal{S}_{\max}$. Take any superstable configuration \tilde{c}' such that $\tilde{c} \leq \tilde{c}'$. Then $\tilde{c}' - v_0 \in \mathcal{S}$ by Theorem 26 (2), and $\tilde{c} - v_0 \leq \tilde{c}' - v_0$. By maximality of $\tilde{c} - v_0$, it follows that $\tilde{c} = \tilde{c}'$. Hence, \tilde{c} is a maximal superstable. \square

Definition 28. The (G, v_0) -semiorder arrangement, denoted \mathcal{S}_0 , is the set of hyperplanes in \mathbb{R}^n given by

$$x_i - x_j = 1,$$

for all i, j not equal to 0 such that $\{v_i, v_j\} \in E$.

Definition 29. The regions of \mathcal{S}_0 , denoted \mathcal{R}_0 , are the connected components of $\mathbb{R}^n \setminus \mathcal{S}_0$.

Define the subset

$$T_0 = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_i > x_0 + 1 \text{ whenever } \{v_i, v_0\} \in E_G\}$$

and let

$$\mathcal{R}'_0 = \{r \in \mathcal{R} : r \subseteq T_0\}.$$

The elements of \mathcal{R}'_0 are exactly those regions with corresponding semiorientations (under ρ) having v_0 as a source. Hence, the bijection $\rho: \mathbb{O} \rightarrow \mathcal{R}$ restricts to a bijection $\mathbb{O}_0 \rightarrow \mathcal{R}'_0$. The projection mapping $(x_0, \dots, x_n) \rightarrow (x_1, \dots, x_n)$, omitting the 0-th coordinate, induces a bijection

$$\pi_0: \mathcal{R}'_0 \rightarrow \mathcal{R}_0.$$

Therefore, we have the following theorem.

Theorem 30. *The mapping*

$$\rho_0 := \pi_0 \circ \rho: \mathbb{O}_0 \rightarrow \mathcal{R}_0$$

is a bijection.

Example 31. The (G, v_0) -semiorder arrangement for graph G of Figure 3 with $v_0 = q$ is drawn in Figure 2. Its regions, \mathcal{R}_0 , are the projections of the regions \mathcal{R}'_0 displayed in Figure 6.

The central region of \mathcal{S}_0 is the region defined by $|x_i - x_j| < 1$ for all distinct i, j not equal to 0 such that $\{v_i, v_j\}$ is an edge of G . Inductively label the regions of \mathcal{S}_0 as in Section (3.4), but starting with the central region labeled with the configuration that assigns 0 to all v_i such that $\{v_i, v_0\} \in E$ and -1 to all other vertices, including v_0 . For each $r \in \mathcal{R}_0$, define $\lambda_0(r)$ to be the label assigned to r in this fashion.

Define $\tau_0 = \rho_0^{-1}$. There is a version of Theorem 3.4 in this context (proved similarly):

Theorem 32. *We have $\lambda_0 = \psi \circ \tau_0$. So there is a surjective mapping*

$$\lambda_0: \mathcal{R}_0 \twoheadrightarrow \mathcal{S}_0$$

and, for each $\sigma \in \mathfrak{S}_0$, a commutative diagram

$$\begin{array}{ccc} \mathcal{S}_0 & \xrightarrow{\eta_{\sigma,0}} & \mathbb{O}_0 \\ & \swarrow \lambda_0 & \downarrow \rho_0 \\ & & \mathcal{R}_0, \end{array} \tag{6}$$

where $\eta_{\sigma,0}$ is the restriction of η_σ to \mathcal{S}_0 .

For $i = 1, \dots, n$, if $\{v_i, v_0\} \in E_G$, let

$$\mathbb{R}_{(i,0)}^n = \mathbb{R}^n,$$

otherwise, if $\{v_i, v_0\} \notin E_G$, let

$$\mathbb{R}_{(i,0)}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i > x_j + 1 \text{ for some } j \text{ with } \{v_i, v_j\} \in E\}.$$

Define $\tilde{\mathbb{R}}^n = \bigcap_{i=1}^n \mathbb{R}_{(i,0)}^n$.

Definition 33. The *admissible regions* of \mathcal{S}_0 , denoted $\tilde{\mathcal{R}}_0$, are the connected components of $\mathbb{R}^n \setminus \mathcal{S}_0$.

Note that $\tilde{\mathcal{R}}_0 = \mathcal{R}_0$ exactly when each nonsink vertex is connected by an edge to the sink.

The admissible regions of \mathcal{S}_0 are exactly those regions whose corresponding G -semi-orientations satisfy $\text{indeg}_{\mathcal{O}}(v_i) \geq 1$ for all $i \neq 0$. So combining Theorem 26 and Theorem 32 gives

Theorem 34.

$$\lambda_0(\tilde{\mathcal{R}}_0) = \tilde{\mathcal{S}}_0 = \{c - v_0 : c \text{ a superstable on } G\}.$$

Example 35. The 9 admissible regions in Figure 2 are labeled by the 8 distinct superstables (or G -parking functions if one remembers that the sink is labeled by -1). The zero-configuration appears twice.

5 Conclusion.

Let A be an $(n + 1) \times (n + 1)$ matrix. Define \mathcal{H}_A to be the set of hyperplanes

$$x_i - x_j = A_{ij},$$

for all $i \neq j$ such that $\{v_i, v_j\} \in E$. For example, $\mathcal{H}_A = \mathcal{S}$ if A has all 1s as its entries.

Define the *regions* of \mathcal{H}_A , denoted \mathcal{R}_A , to be the connected components of $\mathbb{R}^{n+1} \setminus \mathcal{H}_A$. The set of inequalities $x_i - x_j < A_{ij}$ for all i and j defines the *central region* of \mathcal{H}_A . We say that \mathcal{H}_A has a *central region* if this central region is nonempty.

Conjecture 36. Suppose that \mathcal{H}_A has a central region. Labeling the regions of \mathcal{H}_A as in Section 3.4 defines a surjection

$$\lambda_A : \mathcal{R}_A \twoheadrightarrow \mathcal{S}.$$

A similar conjecture holds if one first chooses v_0 as a sink: replace A with an $n \times n$ matrix, and label regions as in Section 4. So the central region would be labeled with the configuration that assigns 0 to vertices connected to v_0 and -1 to the other vertices (including v_0). We conjecture that the nonnegative configurations that arise as labels are exactly the G -parking functions. The G -Shi conjecture of Duval, Klivans, and Martin is a special case.

In the spirit of [1] and [16], it would be interesting to extend our results to the case of multigraphs: graphs in which multiple edges are allowed between vertices. If there are k edges between v_i and v_j , one might replace the two hyperplanes $x_i - x_j = \pm 1$ with the $2k$ hyperplanes $x_i - x_j = \pm 1, \dots, \pm k$.

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