Distance-regular graphs with a relatively small eigenvalue multiplicity

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Abstract

Godsil showed that if Γ is a distance-regular graph with diameter \( D \geq 3 \) and valency \( k \geq 3 \), and \( \theta \) is an eigenvalue of \( \Gamma \) with multiplicity \( m \geq 2 \), then

\[ k \leq \frac{(m+2)(m-1)}{2}. \]

In this paper we will give a refined statement of this result. We show that if \( \Gamma \) is a distance-regular graph with diameter \( D \geq 3 \), valency \( k \geq 2 \) and an eigenvalue \( \theta \) with multiplicity \( m \geq 2 \), such that \( k \) is close to \( \frac{(m+2)(m-1)}{2} \), then \( \theta \) must be a tail. We also characterize the distance-regular graphs with diameter \( D \geq 3 \), valency \( k \geq 3 \) and an eigenvalue \( \theta \) with multiplicity \( m \geq 2 \) satisfying \( k = \frac{(m+2)(m-1)}{2} \).

1 Introduction

For definitions and preliminaries, see Sections 2 and 3. In [6], Godsil showed

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Let $\theta$ be an eigenvalue of $\Gamma$ with multiplicity $m \geq 2$. Then $k \leq \frac{(m+2)(m-1)}{2}$.

In this paper we will give, in Theorem 13, a refined statement of this result. We show that if $\Gamma$ is a distance-regular graph with diameter $D \geq 3$, valency $k \geq 2$ and an eigenvalue $\theta$ with multiplicity $m \geq 2$, such that $k$ is close to $\frac{(m+2)(m-1)}{2}$, then $\theta$ must be a so-called tail. This, for example, implies that several Krein parameters vanish. Using the fact that $\theta$ is a (light) tail, we are also able to characterize in Theorem 14 the distance-regular graphs with diameter $D \geq 3$, valency $k \geq 3$ and an eigenvalue $\theta$ with multiplicity $m \geq 2$ satisfying $k = \frac{(m+2)(m-1)}{2}$.

In Section 2 we give the necessary definitions, and in Section 3 some preliminary results. In Section 4 we characterize the (non-bipartite) Taylor graphs as the non-bipartite distance-regular graphs with diameter at least three, having a light tail such that its accompanying eigenvalue equals $-1$ (Theorem 12). In Section 5 we state and prove Theorem 13 and Theorem 14.

2 Definitions

All the graphs considered in this paper are finite, undirected and simple (for unexplained terminology, examples and more details, see [4, 7]). Suppose that $\Gamma$ is a connected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, where $E(\Gamma)$ consists of unordered pairs of two adjacent vertices. The distance $d_{\Gamma}(x, y)$ between any two vertices $x$ and $y$ in a graph $\Gamma$ is the length of a shortest path connecting $x$ and $y$. If the graph $\Gamma$ is clear from the context, then we simply use $d(x, y)$. We define the diameter $D$ of $\Gamma$ as the maximum distance in $\Gamma$. For a vertex $x \in V(\Gamma)$, define $\Gamma_i(x)$ to be the set of vertices which are at distance precisely $i$ from $x$ ($0 \leq i \leq D$). In addition, define $\Gamma_{-1}(x) = \Gamma_{D+1}(x) := \emptyset$. We write $\Gamma(x)$ instead of $\Gamma_1(x)$.

A connected graph $\Gamma$ with diameter $D$ is called distance-regular if there are integers $b_i, c_i$ ($0 \leq i \leq D$) such that for any two vertices $x, y \in V(\Gamma)$ with $d(x, y) = i$, there are precisely $c_i$ neighbors of $y$ in $\Gamma_{i-1}(x)$ and $b_i$ neighbors of $y$ in $\Gamma_{i+1}(x)$, where we define $b_D = c_0 = 0$. A graph $\Gamma$ is said to be strongly regular with parameters $(v, k, \lambda, \mu)$ whenever $\Gamma$ has $v$ vertices and is regular with valency $k$, adjacent vertices of $\Gamma$ have precisely $\lambda$ common neighbors, and distinct non-adjacent vertices of $\Gamma$ have precisely $\mu$ common neighbors. Note that distance-regular graphs of diameter two are strongly regular. We define $a_i := k - b_i - c_i$ for notational convenience. Note that $a_i = |\Gamma(y) \cap \Gamma_{i}(x)|$ holds for any two vertices $x, y$ with $d(x, y) = i$ ($0 \leq i \leq D$).

For a distance-regular graph $\Gamma$ and a vertex $x \in V(\Gamma)$, we denote $k_i := |\Gamma_i(x)|$ and $p_{ij}^h := |\{w \mid w \in \Gamma_i(x) \cap \Gamma_j(y)\}|$ for any $y \in \Gamma_h(x)$. It is easy to see that $k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_i c_{i+1} \cdots c_D}$ and hence it does not depend on $x$. The numbers $a_i$, $b_i$, and $c_i$ ($1 \leq i \leq D$) are called the intersection numbers, and the array $\{b_0, b_1, \cdots, b_{D-1}; c_1, c_2, \cdots, c_D\}$ is called the intersection array of $\Gamma$.

Suppose that $\Gamma$ is a distance-regular graph with diameter $D \geq 2$ and valency $k \geq 2$, and let $A_i$ be the matrix of $\Gamma$ such that the rows and the columns of $A_i$ are indexed by the vertices in $\Gamma$.
vertices of $\Gamma$ and the $(x, y)$-entry is 1 whenever $x$ and $y$ are at distance $i$ and 0 otherwise.

We call $A_i$ the $i$th distance matrix of $\Gamma$. We abbreviate $A := A_1$ and call this the adjacency matrix of $\Gamma$. The eigenvalues of the graph $\Gamma$ are the eigenvalues of $A$.

We find that $A_0, A_1, \ldots, A_D$ form a basis for a commutative subalgebra $M$ of $\text{Mat}_X(\mathbb{C})$.

We call $M$ the Bose-Mesner algebra of $\Gamma$. It turns out that $A$ generates $M$ [1, p. 190]. By [4, p. 45], $M$ has a second basis $E_0, E_1, \ldots, E_D$ of the primitive idempotents of $\Gamma$, and $A$ can be written as $A = \sum_{i=0}^{D} \theta_i E_i$, where $\theta_i$ is the eigenvalue of $\Gamma$ associated with $E_i$ ($0 \leq i \leq D$). We denote by $m_i$ the multiplicity of $\theta_i$. For an eigenvalue $\theta = \theta_i$ we will also write $E_{\theta}$ instead of $E_i$.

For an eigenvalue $\theta$ of $\Gamma$, the sequence $(\omega_i)_{i=0,1,\ldots,D} = (\omega_i(\theta))_{i=0,1,\ldots,D}$ satisfying $\omega_0 = 0, \omega_1 = 0, \omega_2 = 0$ (resp. $= 1$) for $0 \leq t < \theta < j$ for $i < t < j$.

Let $\circ$ denote the entrywise product in $\text{Mat}_X(\mathbb{C})$. Observe that $A_i \circ A_j = \delta_{ij} A_i$ for $0 \leq i, j \leq D$, so $M$ is closed under $\circ$. Thus there exist complex scalars $q_{ij}^h$ ($0 \leq h, i, j \leq D$) such that

$$E_i \circ E_j = |V(\Gamma)|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \quad (0 \leq i, j \leq D).$$

By [2, p. 170], $q_{ij}^h$ is real and nonnegative for $0 \leq h, i, j \leq D$. The $q_{ij}^h$ are called the Krein parameters. The graph $\Gamma$ is said to be Q-polynomial (with respect to the given ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents) whenever $q_{ij}^h = 0$ (resp. $q_{ij}^h \neq 0$) whenever one of $h, i, j$ is greater than (resp. equal to) the sum of the other two ($0 \leq h, i, j \leq D$) [4, p. 59].

For each vertex $x \in V(\Gamma)$, we let $\Delta(x)$ denote the subgraph of $\Gamma$ induced on $\Gamma(x)$. We refer to $\Delta(x)$ as the local graph at vertex $x$. We observe that $\Delta(x)$ has $k$ vertices, and is regular with valency $a_1$.

A graph $\Gamma$ is called bipartite if it has no odd cycle. (A distance-regular graph $\Gamma$ with diameter $D$ is bipartite if and only if $a_1 = a_2 = \ldots = a_D = 0$.) An antipodal graph is a connected graph $\Gamma$ with diameter $D \geq 2$ for which being at distance 0 or $D$ is an equivalence relation. If, moreover, all equivalence classes have the same size $r$, then $\Gamma$ is also called an antipodal $r$-cover. A distance-regular graph $\Gamma$ with intersection array $\{k, \mu, 1; 1, \mu, k\}$ is called a Taylor graph. These are precisely the distance-regular antipodal 2-covers with diameter 3.

We define tails as follows: An eigenvalue $\theta$ of a distance-regular graph $\Gamma$ with valency $k$ is called a tail if $\theta \neq k$ and $E_\theta \circ E_\theta = \alpha J + \beta E_\theta + \gamma E_{\theta'}$ for some eigenvalue $\theta' \neq k, \theta$ and some $\alpha, \beta$ and $\gamma \neq 0$. We call $\theta'$ the accompanying eigenvalue for the tail $\theta$. We call $\theta$ a light tail if $\beta = 0$ and heavy otherwise. Note that $\theta > 0$ and $\beta > 0$. (Note that in [13], [10], they also allow $\gamma = 0$ for a tail and a light tail, respectively. Note that for diameter $D \geq 3$ this case of $\gamma = 0$ only occurs if $\Gamma$ is an antipodal distance-regular graph of diameter $D = 3$ and $\theta = -1$ [10, Theorem 4.1(b)].)
3 Preliminaries

In this section we will give some preliminary results.

The following lemma is a special case of the Absolute Bound and we state it for distance-regular graphs only.

**Lemma 2.** ([15]) Let $\Gamma$ be a distance-regular graph with diameter $D \geq 2$. Then
\[ \sum_{q_i \neq 0} m_j \leq \frac{m_i(m_i + 1)}{2} \quad (0 \leq j \leq D). \]

The next result relates the multiplicity of an eigenvalue and its number of vertices for a strongly regular graph. A graph $\Gamma$ is called coconnected if its complement is connected.

**Lemma 3.** Let $\Gamma$ be a connected and coconnected strongly regular graph with $v$ vertices and distinct eigenvalues $k > \sigma > \tau$ with corresponding multiplicities $1, f, g$. Then
(i) $v \leq \min\{\frac{f(f+3)}{2}, \frac{g(g+3)}{2}\}$.
(ii) If $v > \frac{g(g+1)}{2}$, then $\tau$ is a light tail, that is, $\mu = \frac{-2(\sigma+1)\tau(\sigma+\tau^2)}{\tau-\sigma(\sigma+2)}$.
(iii) If $v > \frac{f(f+1)}{2}$, then $\sigma$ is a light tail, that is, $\mu = \frac{-2(\tau+1)\sigma(\sigma+\tau^2)}{\sigma-\tau(\tau+2)}$.
(iv) If $v = \frac{g(g+3)}{2}$, then
\[
\begin{align*}
\mu &= \sigma^3(2\sigma + 3), \\
k &= 2\mu, \\
\lambda &= \sigma(2\sigma^3 + \sigma^2 - 3\sigma + 1), \\
v &= (2\sigma + 1)^2(2\sigma^2 + 2\sigma - 1), \\
\tau &= -\sigma^2(2\sigma + 3),
\end{align*}
\]
and $\sigma > 0$ and $\tau < -1$ are integers except for the case $\sigma = -\frac{1+\sqrt{5}}{2}$, $\tau = -\frac{1-\sqrt{5}}{2}$ and $\Gamma$ is the pentagon.
(v) If $v = \frac{f(f+3)}{2}$, then
\[
\begin{align*}
\mu &= \tau^3(2\tau + 3), \\
k &= 2\mu, \\
\lambda &= \tau(2\tau^3 + \tau^2 - 3\tau + 1), \\
v &= (2\tau + 1)^2(2\tau^2 + 2\tau - 1), \\
\sigma &= -\tau^2(2\tau + 3),
\end{align*}
\]
and $\sigma > 0$ and $\tau < -1$ are integers except for the case $\sigma = -\frac{1+\sqrt{5}}{2}$, $\tau = -\frac{1-\sqrt{5}}{2}$ and $\Gamma$ is the pentagon.

**Proof:** (i) This follows from the absolute bound, Lemma 2. See also [19, p.169].
(ii) It follows from [15, Theorem 2] and [5, Theorem 6.1].
(iii) If we take the complement of $\Gamma$ then it is a strongly regular graph satisfying (ii), and the result follows easily.
(iv) See [14] (cf. [19, p.169–170]).
(v) If we take the complement of $\Gamma$ then it is a strongly regular graph satisfying (iv), and the result follows easily.

Next, we introduce the Fundamental Bound and tight distance-regular graphs.

**Lemma 4.** ([9, Theorem 6.2]) Let $\Gamma$ be a distance-regular graph with diameter $D \geq 3$, valency $k$ and distinct eigenvalues $k = \theta_0 > \theta_1 > \ldots > \theta_D$. Then the following inequality holds.

$$
\left( \theta_1 + \frac{k}{a_1 + 1} \right) \left( \theta_D + \frac{k}{a_1 + 1} \right) \geq -\frac{ka_1 b_1}{(a_1 + 1)^2} \quad (1)
$$

We refer to (1) as the *Fundamental Bound*. A distance-regular graph $\Gamma$ is *tight* if $\Gamma$ is not bipartite and equality holds in (1).

The next lemma gives some known results on tight distance-regular graphs.

**Lemma 5.** Let $\Gamma$ be a distance-regular graph with diameter $D \geq 3$, valency $k$ and distinct eigenvalues $k = \theta_0 > \theta_1 > \ldots > \theta_D$. Then

(i) ([9, Theorem 12.6]) $\Gamma$ is tight if and only if for all $x \in V(\Gamma)$, the local graph $\Delta(x)$ is connected strongly regular with distinct eigenvalues $a_1, -1 - \frac{b_1}{\theta_D + 1}, -1 - \frac{b_1}{\theta_i + 1}$.

(ii) ([9, Theorem 11.7]) If $\Gamma$ is tight, then the intersection number $a_D$ satisfies $a_D = 0$.

(iii) ([18, Lemma 3.5], cf.[17]) If $\Gamma$ is tight, then the Krein parameter $q_{1D}^i$ satisfies $q_{1D}^i = 0$ unless $i = D - 1 \ (0 \leq i \leq D)$.

The next result is due to Terwilliger and concerns the eigenvalues of the local graph $\Delta(x)$ at a vertex $x$ of a distance-regular graph $\Gamma$.

**Proposition 6.** ([4, Theorem 4.4.4]) Let $\Gamma$ be a distance-regular graph with diameter $D \geq 3$, valency $k$ and distinct eigenvalues $k = \theta_0 > \theta_1 > \ldots > \theta_D$ with corresponding multiplicities $1 = m_0, m_1, \ldots, m_D$. If $\theta_i$ has multiplicity $m_i$ with $1 < m_i < k$, then $\theta_i \in \{\theta_1, \theta_D\}$. Putting $b = \frac{b_1}{\theta_1 + 1}$ we have that each local graph $\Delta(x)$ has eigenvalue $-1 - b$ with multiplicity at least $k - m_i$; in case $-1 - b = a_1$ its multiplicity is at least $k - m_i + 1$.

The following lemma is a consequence of Proposition 6.

**Lemma 7.** Let $\Gamma$ be a distance-regular graph with diameter $D \geq 3$, valency $k$ and distinct eigenvalues $k = \theta_0 > \theta_1 > \ldots > \theta_D$ with corresponding multiplicities $1 = m_0, m_1, \ldots, m_D$. Then $m_1 + m_D \geq k + 1$.

**Proof:** As the sum of the multiplicities of $-1 - \frac{b_1}{\theta_i + 1}$ and $-1 - \frac{b_1}{\theta_D + 1}$ as eigenvalues of the local graph at vertex $x$ is at most $k - 1$ if $-1 - \frac{b_1}{\theta_D + 1} \neq a_1$ and at most $k$ if equals $a_1$, the result follows.

In the next lemma we show that the accompanying eigenvalue of a light tail $\theta$ is the third-largest eigenvalue, if $\theta$ is the second-largest eigenvalue.
Lemma 8. Let Γ be a distance-regular graph with diameter $D \geq 3$, valency $k$ and distinct eigenvalues $k = \theta_0 > \theta_1 > \ldots > \theta_D$. If $\theta = \theta_1$ is a light tail, then the accompanying eigenvalue $\theta'$ satisfies $\theta' = \theta_2$.

Proof: Let $E_i$ be the primitive idempotent corresponding to $\theta_i$. Now $E_1 \circ E_1 = \alpha E_0 + \beta E_i$, where $\alpha$ and $\beta$ are positive numbers. As the standard sequence corresponding to $\theta_1$ is strictly decreasing, this implies that the standard sequence corresponding to $\theta_i$ has at most two sign changes ([10, Theorem 4.1(iii)]). But as $i \neq 0, 1$ it follows that $i = 2$. \hfill \qed

4 Characterizations of Taylor graphs

In this section we will give some characterizations of the Taylor graphs. We start with the following result, due to Taylor.

Lemma 9. ([4, Proposition 1.5.1, Theorem 1.5.3])

(i) If $\Gamma$ is a Taylor graph with valency $k$, then for every $x \in V(\Gamma)$, the local graph $\Delta(x)$ is strongly regular with parameters $(v', k', \lambda', \mu')$ and satisfies $\alpha_1 = k' = 2\mu'$, and $v' = k$.

(ii) If $\Delta$ is a (non-complete) connected strongly regular graph with $(v', k', \lambda', \mu')$ such that $k' = 2\mu'$, then there exists a Taylor graph $\Gamma$ and a vertex $x$ of $\Gamma$ such that the local graph $\Delta(x)$ of $\Gamma$ is isomorphic to $\Delta$.

Remark: We denote by $\text{Tay}(\Delta)$, the Taylor graph as in Lemma 9(ii), where $\Delta$ is a (non-complete) connected strongly regular graph with $(v', k', \lambda', \mu')$ satisfying $k' = 2\mu'$.

The next result gives some sufficient conditions for a distance-regular graph to be tight.

Lemma 10. Let $\Gamma$ be a distance-regular graph with diameter $D \geq 3$, valency $k$ and distinct eigenvalues $k = \theta_0 > \theta_1 > \ldots > \theta_D$ with corresponding multiplicities $1 = m_0, m_1, \ldots, m_D$. Then the following hold.

(i) If $m_1 + m_D = k + 1$, then $\Gamma$ is an antipodal 2-cover, and $\Gamma$ is tight or bipartite.

(ii) If for all vertices $x$ the local graph $\Delta(x)$ is strongly regular and $m_1, m_D < k$, then $\Gamma$ is tight.

Proof: (i) If $m_1 + m_D = k + 1$, then we need to consider two cases: $m_D = 1$ and $m_D \geq 2$.

If $m_D = 1$, then $\Gamma$ is bipartite and $\theta_D = -k$ by [4, Proposition 4.4.8(i)]. If $m_i = 1$ and $i \geq 1$, then $i = D$, $\theta_D = -k$ and $\Gamma$ is bipartite. So from now we may assume $m_1 \geq 2$ and $m_D \geq 2$. Now let $m_D \geq 2$. Then $m_1 = k + 1 - m_D < k$. If $\theta_D = -1 - \frac{b_1}{\theta_{i+1}}$, then the local graph $\Delta(x)$ at vertex $x$ has eigenvalues $a_1$ and $-1 - \frac{b_1}{\theta_{i+1}}$ with corresponding multiplicities $k - m_D + 1$ and $k - m_1$ by Proposition 6. So this means that $\Delta(x)$ is a disjoint union of cliques. Since $\theta_1 > 0$, we find that $-1 - \frac{b_1}{\theta_{D+1}} < -1$. But it is not possible. So we find that $\theta_D \neq -1 - \frac{b_1}{\theta_{i+1}}$. Then again by Proposition 6 we find that for all vertices $x$ the local graph $\Delta(x)$ has eigenvalues $a_1$, $-1 - \frac{b_1}{\theta_D+1}$, $-1 - \frac{b_1}{\theta_{i+1}}$ with corresponding multiplicities 1,
So this means that $\Delta(x)$ is strongly regular by [7, Lemma 10.1.5], and hence by Lemma 5(i), we find $\Gamma$ is tight. So we have shown that $\Gamma$ is tight or bipartite. This means that $a_D = 0$ by Lemma 5(ii). By [8] it follows that $k_D = 1$ as otherwise $-1 - \frac{b_1}{\sigma_1 + 1}$ has multiplicity at least $k + 1 - m_1$ in $\Delta(x)$ for any vertex $x$. This shows (i).

(ii) Let $x$ be a vertex of $\Gamma$ and consider the local graph $\Delta(x)$. Proposition 6 implies that $-1 - \frac{b_1}{\sigma_1 + 1}$ and $-1 - \frac{b_1}{\sigma_D + 1}$ are both eigenvalues of $\Delta(x)$. Now $-1 - \frac{b_1}{\sigma_1 + 1} \neq -1$, so that means $\Delta(x)$ is not the disjoint union of cliques, and hence is connected. But this shows that $\Gamma$ is tight in similar fashion as in (i).

Remark: (i) The bipartite distance-regular graphs with an eigenvalue having multiplicity $k$ are determined by N. Yamazaki [21] and K. Nomura [16]. They found the following:
(a) $2d$-gons,
(b) complete bipartite graphs,
(c) complements of $2 \times (k + 1)$-grids,
(d) Hadamard graphs,
(e) antipodal 2-covers with the intersection array $\{k, k - 1, k - c, c, 1; 1, c, k - c, k - 1, k\}$, where $k = \gamma(\gamma^2 + 3\gamma + 1)$, $c = \gamma(\gamma + 1)$ and $\gamma \geq 2$,
(f) hypercubes.

For the fifth case, if $\gamma = 2$, then the graph is 2-cover of Higman-Sims graph, and for $\gamma \geq 3$, no graph is known.

(ii) The Taylor graphs have $m_1 + m_3 = k + 1$. Besides them there are feasible intersection arrays known for diameter 4 with $m_1 + m_4 = k + 1$. These are
\[
\{56, 45, 12, 1; 1, 12, 45, 56\}, \\
\{115, 96, 20, 1; 1, 20, 96, 115\}, \\
\{204, 175, 30, 1; 1, 30, 175, 204\} \text{ and,} \\
\{329, 288, 42, 1; 1, 42, 288, 329\}.
\]

For the first intersection array, it is known that there are no distance-regular graphs with this intersection array ([3, 11.4.6 Theorem]). There are no feasible intersection arrays known for larger diameter.

In Theorem 12 below, we show that the (non-bipartite) Taylor graphs are the distance-regular graphs with diameter $D \geq 3$, valency $k$ and intersection number $a_1 \neq 0$ having a light tail such that its accompanying eigenvalue equals $-1$. To show this result we first need the following lemma.

Lemma 11. Let $\Gamma$ be a distance-regular graph with diameter $D \geq 3$, valency $k$, intersection number $a_1 \neq 0$ and distinct eigenvalues $k = \theta_0 > \theta_1 > \ldots > \theta_D$. Let $\theta$ be a light tail of $\Gamma$ with standard sequence $1 = \omega_0, \omega_1, \ldots, \omega_D$ and let $\theta'$ be the accompanying eigenvalue of $\theta$. For all $x \in V(\Gamma)$, let the local graph $\Delta(x)$ be a (non-complete) strongly regular graph with parameters $(v' = k, k' = a_1, \lambda', \mu')$. Then the following statements are equivalent.
(i) $\theta' = -1$.
(ii) $\theta$ is a root of $x^2 - (a_1 - b_1)x - k$.
(iii) $k' = 2\mu'$.
(iv) $\omega_2 = -\omega_1$.

Proof: The equivalence (i)$\iff$(ii) follows from [10, Theorem 4.1(a)].
The equivalence (ii)$\iff$(iii) follows from [10, Corollary 6.3].
The equivalence (ii)$\iff$(iv) is straightforward. \hfill \blacksquare

In the next result we show that any of the 4 statements in Lemma 11 is equivalent with $\Gamma$ be a Taylor graph.

**Theorem 12.** Let $\Gamma$ be a distance-regular graph with diameter $D \geq 3$, valency $k$, intersection number $a_1 \neq 0$ and distinct eigenvalues $k = \theta_0 > \theta_1 > \ldots > \theta_D$. Let $\theta \neq \pm k$ be an eigenvalue of $\Gamma$. Then the following statements are equivalent:
(i) $\theta$ is a light tail of $\Gamma$ with standard sequence $1 = \omega_0, \omega_1, \ldots, \omega_D$ such that its accompanying eigenvalue $\theta'$ equals $-1$.
(ii) $\Gamma$ is a Taylor graph and $\theta \in \{\theta_1, \theta_3\}$.

Proof: (i)$\Rightarrow$(ii) As $a_1 \neq 0$ and $\theta$ is a light tail it follows that $\theta \in \{\theta_1, \theta_D\}$ by [10, Remarks 3.3(iii)]. If $\theta = \theta_1$, then $\theta_2 = \theta' = -1$ by Lemma 8. If $D \geq 4$, then $\theta_2 \geq \min\{0, a_2, a_4\} \geq 0$. This implies that $D = 3$. By [10, Theorem 5.1] and Lemma 11 we find $c_3 = k \frac{\omega_3(1-\omega_1)}{\omega_3-\omega_2} = k \frac{\omega_3(1-\omega_1)}{\omega_3+\omega_3}$. This implies $c_3 = k$ and $\omega_3 = -1$ as $\omega_1 > 0$ and hence $\Gamma$ is an antipodal $r$-cover. By [4, p.142–143], $\omega_3 = -1/(r - 1)$ and hence $\Gamma$ is a Taylor graph. Let us assume that $\theta = \theta_D$, then we need to consider two cases: $D = 3$ and $D \geq 4$.

If $D = 3$, then let $\alpha$ be the largest root of $x^2 - (a_1 - b_1)x - k$. Let $\text{Tay}(\Delta)$ be the Taylor graph corresponding to $\Delta = \Delta(x)$ as in Lemma 9(ii). Here note that as $\theta$ is a light tail the local graph $\Delta = \Delta(x)$ is a (non-complete) strongly regular graph with parameters $(v' = k, k' = a_1, \lambda', \mu')$ and it satisfies $k' = 2\mu'$ by Lemma 11. Now $\Delta$ has the smallest eigenvalue $-1 - \frac{b_1}{\alpha + 1}$ as $\alpha$ is an eigenvalue of $\text{Tay}(\Delta)$ and $\text{Tay}(\Delta)$ is tight. This implies $\theta_1 \leq \alpha$ ([4, Theorem 4.4.3]). But then $a_1 + a_2 + a_3 = k + \theta_1 + \theta_2 + \theta_3 \leq k + \alpha + \theta_2 + \theta_3 = 2a_1$ as $\text{Tay}(\Delta)$ has eigenvalues $k, \alpha, \theta_2, \theta_3$. Hence $a_1 \geq a_2 + a_3$. But $a_2 + a_3 \geq a_1$ by [11, Proposition 4]. So $a_2 + a_3 = a_1$ and this implies $a_3 = 0$ and $b_2 = 1$ and hence $\Gamma$ is a Taylor graph. If $D \geq 4$, then by [12, Theorem 3.1(iii)], $\theta_1 \geq \frac{a_1+\sqrt{a_1^2+4k}}{2} > a_1 + 1$. But again from the proof of $D = 3$ we have $\theta_1 \leq \alpha$, where $\alpha$ is the largest root of $x^2 - (a_1 - b_1)x - k$. But if we evaluate the polynomial $x^2 - (a_1 - b_1)x - k$ in point $a_1 + 1$ we see that it is always non-negative. This means that $\alpha \leq a_1 + 1$ and $\alpha \geq \theta_1 > a_1 + 1$, a contradiction. So this case can not occur.\hfill \blacksquare

(ii)$\Rightarrow$(i) It is easily checked that if $\Gamma$ is a Taylor graph then $\theta \in \{\theta_1, \theta_3\}$ is a light tail and its accompanying eigenvalue $\theta'$ equals $-1$.\hfill \blacksquare
5 The refined bound

In this section, we will show the following refined version of Theorem 1.

**Theorem 13.** Let $\Gamma$ be a distance-regular graph with diameter $D \geq 3$, valency $k \geq 2$ and distinct eigenvalues $k = \theta_0 > \theta_1 > \ldots, \theta_D$. Let $\theta \neq \pm k$ be an eigenvalue of $\Gamma$ with multiplicity $m \geq 2$. Then $k \leq \frac{(m+2)(m-1)}{2}$. More precisely, the following hold.

(i) If $m = 2$, then $k = 2$.
(ii) If $\theta$ is not a tail and $m \geq 3$, then $k \leq \frac{(m-1)(m+4)}{4}$.
(iii) If $\theta$ is a heavy tail with $\theta' \notin \{\theta_1, \theta_D\}$ and $m \geq 3$, then $k \leq \frac{(m+1)(m-2)}{2}$.
(iv) If $\theta$ is a heavy tail with $\theta' \in \{\theta_1, \theta_D\}$ and $m \geq 3$, then $k \leq \frac{(m-2)(m+3)}{2}$.
(v) If $\theta$ is a light tail, then $k \leq \frac{(m+2)(m-1)}{2}$.

**Proof:** Let $\theta = \theta_i \neq \pm k$ be an eigenvalue of $\Gamma$ with multiplicity $m = m_i$. [4, Proposition 4.4.8(ii)] shows $k = 2$ if and only if $m = 2$. This shows (i). So from now on we may assume $m \geq 3$ and $k \geq 3$. We will first consider the case $m < k$ and later we will consider $m \geq k$.

Let us first assume $m < k$. Then $i \in \{1, D\}$ by Proposition 6, and $a_1 \neq 0$ by [10, Theorem 3.2]. If there are at least two distinct $j_1, j_2 \notin \{0, i\}$ satisfying $q_{iji}^j \neq 0 \neq q_{iji}^j$, then by Lemma 2 and Lemma 7 we have $\frac{m(m+1)}{2} \geq m_0 + m_{j_1} + m_{j_2} \geq 1 + k + k - m + 1$ and hence $k \leq \frac{(m-1)(m+4)}{4}$. If $q_{ii}^j = 0$ for all $j \notin \{0, i\}$, then by [10, Theorem 4.1(b)], $\Gamma$ is antipodal with diameter 3 and $\theta = \theta_2 = -1$. But then $m = k$. This shows (ii) if $m < k$. Now let us assume $\theta$ is a tail and $\theta'$ its accompanying eigenvalue. Let $m'$ be the multiplicity of $\theta'$. If $\theta$ is a heavy tail with $\theta' \notin \{\theta_1, \theta_D\}$, then by Lemma 2 and Proposition 6, $\frac{m(m+1)}{2} \geq 1 + m + k$ and this shows (iii) if $m < k$. If $\theta$ is a heavy tail with $\theta' \in \{\theta_1, \theta_D\}$, then by Lemma 2, Proposition 6, and Lemma 7, $\frac{m(m+1)}{2} \geq 1 + m + m' \geq 1 + m + k + 1 - m = k + 2$. But if $\frac{m(m+1)}{2} = k + 2$, then $m + m' = k + 1$ and it follows by Lemma 10 that $\Gamma$ is tight. But $q_{ijD}^j = 0$ if $j \neq D - 1$ by Lemma 5(iii), so this gives a contradiction. This shows (iv) when $m < k$. Now if $\theta$ is a light tail, then for all vertices $x$ the local graph $\Delta(x)$ is strongly regular by [10, Corollary 6.3]. If $m' < k$, then $\{\theta, \theta'\} = \{\theta_1, \theta_D\}$ and by Lemma 10(ii) $\Gamma$ is tight. But this is not possible by Lemma 5(iii). This means $m' \geq k$. Now by Lemma 2, $\frac{m(m+1)}{2} \geq m_0 + m' \geq 1 + k$. This shows (v). So we have shown the theorem if $m < k$.

As $m \leq \frac{(m-1)(m+4)}{4}$, $m \leq \frac{(m-2)(m+3)}{2}$, and $m \leq \frac{(m+2)(m-1)}{2}$ if $m \geq 3$, it follows that cases (ii), (iv), and (v) also hold if $m \geq k \geq 3$. For case (iii) and $m \geq k \geq 3$ we see that $m \leq \frac{(m+1)(m-2)}{2}$ unless $m = 3$. If $m = 3$ and $m \geq k \geq 3$, then we see that $k = 3$ and $a_1 = 0$ as $D \geq 3$. But then $\theta$ is a light tail, a contradiction with the assumption that $\theta$ is a heavy tail.

In the following theorem, we characterize the distance-regular graphs with valency at least three which attain the bound in Theorem 13.

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Theorem 14. Let $\Gamma$ be a distance-regular graph with diameter $D \geq 3$, valency $k \geq 3$ and an eigenvalue $\theta$ having multiplicity $m \geq 2$. Then the following statements are equivalent. 

(i) $k = \frac{(m+2)(m-1)}{2}$.

(ii) $\Gamma$ is a Taylor graph with intersection array $\{(2\alpha + 1)^2(2\alpha^2 + 2\alpha - 1), 2\alpha^3(2\alpha + 3), 1; 1, 2\alpha^3(2\alpha + 3), (2\alpha + 1)^2(2\alpha^2 + 2\alpha - 1)\}$ where $\alpha$ is an integer $\neq 0, -1$ or $\alpha = -\frac{1+\sqrt{5}}{2}$, (and $m = 4\alpha^2 + 4\alpha - 1$).

Proof: (i) $\Rightarrow$ (ii) The only distance-regular graphs with an eigenvalue having multiplicity 2 are the polygons. So $\theta$ has multiplicity $m \geq 3$. As $m < \frac{(m+2)(m-1)}{2}$ if $m \geq 3$, we have $m < k$ and hence $a_1 \neq 0$. By Theorem 13, the eigenvalue $\theta$ is a light tail. To complete the proof, we will show that for any vertex $x$ of $\Gamma$, the local graph $\Delta(x)$ at the vertex $x$ is a strongly regular graph with parameters $(v', k', \lambda', \mu')$ satisfying $k' = 2\mu'$. Then by Lemma 11 the accompanying eigenvalue $\theta'$ of $\theta$ is equal to $-1$, and hence by Theorem 12 the graph $\Gamma$ is a Taylor graph with the parameters as stated in the theorem.

Let $x$ be a vertex of $\Gamma$. Then the local graph $\Delta(x)$ is a strongly regular graph. If $\Delta(x)$ is not connected, then $\Delta(x)$ is the disjoint union of $\frac{k}{a_1+1}$ complete graphs with $a_1 + 1$ vertices. Then by [10, Corollary 6.3], we have

$$\theta = \theta_D = -1 - \frac{b_1}{a_1 + 1} = -\frac{k}{a_1 + 1}$$

and also by [10, Theorem 3.2] we have

$$m = k - \frac{b_1}{a_1 + 1} \geq \frac{k}{2} + 1.$$ 

As $k = \frac{(m+2)(m-1)}{2} \geq 2m - 1$ if $m \geq 3$, we find that $\Delta(x)$ must be connected. By [10, Corollary 6.3] we find that $\Delta(x)$ has an eigenvalue $\frac{a_1\theta}{\theta+k}$ with multiplicity $m - 1$. Now by parts (iv) and (v) in Lemma 3, we find that the local graph $\Delta(x)$ at the vertex $x$ is strongly regular with parameters $(v', k', \lambda', \mu') = ((2\alpha + 1)^2(2\alpha^2 + 2\alpha - 1), 2\alpha^3(2\alpha + 3), \alpha(2\alpha^3 + \alpha^2 - 3\alpha + 1), \alpha^3(2\alpha + 3))$ satisfying $k' = 2\mu'$, where $\alpha$ is an integer $\neq 0, -1$ or $\alpha = -\frac{1+\sqrt{5}}{2}$. This shows (i).

(ii) $\Rightarrow$ (i) Trivial.

This finishes the proof.

Remark: Note that the distance-2 graph of a graph $\Gamma = (V(\Gamma), E(\Gamma))$ has as vertex set $V(\Gamma)$ and two vertices are adjacent if they have distance 2 in $\Gamma$. Then the distance-2 graph of a Taylor graph with intersection array

$$\{(2\alpha + 1)^2(2\alpha^2 + 2\alpha - 1), 2\alpha^3(2\alpha + 3), 1; 1, 2\alpha^3(2\alpha + 3), (2\alpha + 1)^2(2\alpha^2 + 2\alpha - 1)\},$$

where $\alpha$ is an integer $\neq 0, 1$ or $\alpha = -\frac{1+\sqrt{5}}{2}$, is again a Taylor graph with intersection array

$$\{(2\beta + 1)^2(2\beta^2 + 2\beta - 1), 2\beta^3(2\beta + 3), 1; 1, 2\beta^3(2\beta + 3), (2\beta + 1)^2(2\beta^2 + 2\beta - 1)\},$$

where $\beta = -\alpha - 1$. 

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Also, the following hold:
(i) $\Gamma$ is the Icosahedron if $\alpha = \frac{-1 \pm \sqrt{5}}{2}$,
(ii) $\Gamma$ is the Gosset graph if $\alpha = 1$,
(iii) $\Gamma$ is the distance-2 graph of Gosset graph if $\alpha = -2$,
(iv) $\Gamma$ is the Tay(McLaughlin graph) (see [20]) if $\alpha = -3$,
(v) $\Gamma$ is the distance-2 graph of Tay(McLaughlin graph) if $\alpha = 2$,
(vi) For the other $\alpha$ nothing is known.

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References


