The Ramsey number of loose paths in 3-uniform hypergraphs

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Abstract

Recently, asymptotic values of 2-color Ramsey numbers for loose cycles and also loose paths were determined. Here we determine the 2-color Ramsey number of 3-uniform loose paths when one of the paths is significantly larger than the other: for every $n \geqslant \left\lfloor \frac{5m}{4} \right\rfloor$, we show that

$$R(\mathcal{P}^3_n, \mathcal{P}^3_m) = 2n + \left\lfloor \frac{m+1}{2} \right\rfloor.$$

Keywords: Ramsey Number, Loose Path, Loose Cycle.

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1 Introduction

A *hypergraph* $\mathcal{H}$ is a pair $\mathcal{H} = (V, E)$, where V is a finite nonempty set (the set of vertices) and E is a collection of distinct nonempty subsets of V (the set of edges). A *k-uniform hypergraph* is a hypergraph such that all its edges have size k. For two k-uniform hypergraphs $\mathcal{H}$ and $\mathcal{G}$, the *Ramsey number* $R(\mathcal{H}, \mathcal{G})$ is the smallest number N such that, in any red-blue coloring of the edges of the complete k-uniform hypergraph K_N^k on N vertices there is either a red copy of $\mathcal{H}$ or a blue copy of $\mathcal{G}$. There are several natural definitions for a cycle and a path in a uniform hypergraph. Here we consider the one called loose. A k-uniform *loose cycle* $\mathcal{C}_n^k$ (shortly, a *cycle of length* n), is a hypergraph with vertex set $\{v_1, v_2, \ldots, v_{n(k-1)}\}$ and with the set of n edges $e_i = \{v_1, v_2, \ldots, v_k\} + i(k-1)$, $i = 0, 1, \ldots, n-1$, where we use mod $n(k-1)$ arithmetic and adding a number t to a set $H = \{v_1, v_2, \ldots, v_k\}$ means a shift, i.e. the set obtained by adding t to subscripts of each element of H. Similarly, a k-uniform *loose path* $\mathcal{P}_n^k$ (simply, a *path of length* n), is a hypergraph with vertex set $\{v_1, v_2, \ldots, v_{n(k-1)+1}\}$ and with the set of n edges $e_i = \{v_1, v_2, \ldots, v_k\} + i(k-1)$, $i = 0, 1, \ldots, n-1$ and we denote this path by $e_0 e_1 \cdots e_{n-1}$. For $k = 2$ we get the usual definitions of a cycle and a path. In this case, a classical result in graph theory (see [1]) states that $R(P_n, P_m) = n + \lfloor \frac{m+1}{2} \rfloor$, where $n \geqslant m \geqslant 1$. Moreover, the exact values of $R(P_n, C_m)$ and $R(C_n, C_m)$ for positive integers n and m are determined [5]. For $k = 3$ it was proved in [4] that $R(\mathcal{C}_n^3, \mathcal{C}_n^3)$, and consequently $R(\mathcal{P}_n^3, \mathcal{P}_n^3)$ and $R(\mathcal{P}_n^3, \mathcal{C}_n^3)$, are asymptotically equal to $\frac{5n}{2}$. Subsequently, Gyárfás et. al. in [3] extended this result to the k-uniform loose cycles and proved that $R(\mathcal{C}_n^k, \mathcal{C}_n^k)$, and consequently $R(\mathcal{P}_n^k, \mathcal{P}_n^k)$ and $R(\mathcal{P}_n^k, \mathcal{C}_n^k)$, are asymptotically equal to $\frac{1}{2}(2k-1)n$. For small cases, Gyárfás et. al. (see [2]) proved that $R(\mathcal{P}_3^k, \mathcal{P}_3^k) = R(\mathcal{P}_3^k, \mathcal{C}_3^k) = R(\mathcal{C}_3^k, \mathcal{C}_3^k) + 1 = 3k-1$ and $R(\mathcal{P}_4^k, \mathcal{P}_4^k) = R(\mathcal{P}_4^k, \mathcal{C}_4^k) = R(\mathcal{C}_4^k, \mathcal{C}_4^k) + 1 = 4k-2$. To see a survey on Ramsey numbers involving cycles see [6].

It is easy to see that $N = (k-1)n + \lfloor \frac{m+1}{2} \rfloor$ is a lower bound for the Ramsey number $R(\mathcal{P}_n^k, \mathcal{P}_m^k)$. To show this, partition the vertex set of $\mathcal{K}_{N-1}^k$ into parts A and B, where $|A| = (k-1)n$ and $|B| = \lfloor \frac{m+1}{2} \rfloor - 1$, color all edges that contain a vertex of B blue, and the rest red. Now, this coloring can not contain a red copy of $\mathcal{P}_n^k$, since such a copy has $(k-1)n+1$ vertices. Clearly the longest blue path has length at most $m-1$, which proves our claim. Using the same argument we can see that N and $N-1$ are the lower bounds for $R(\mathcal{P}_n^k, \mathcal{C}_m^k)$ and $R(\mathcal{C}_n^k, \mathcal{C}_m^k)$, respectively. In [2], motivated by the above facts and some other results, the authors conjectured that these lower bounds give the exact values of the mentioned Ramsey numbers for $k = 3$. In this paper, we consider this problem and we prove that $R(\mathcal{P}_n^3, \mathcal{P}_m^3) = 2n + \lfloor \frac{m+1}{2} \rfloor$ for every $n \geqslant \lfloor \frac{5m}{4} \rfloor$. Throughout the paper, for a 2-edge coloring of a uniform hypergraph $\mathcal{H}$, say red and blue, we denote by $\mathcal{F}_{red}$ and $\mathcal{F}_{blue}$ the induced hypergraph on edges of colors red and blue, respectively.

2 Preliminaries

In this section, we present some lemmas which are essential in the proof of the main results.

Lemma 1. *Let $n \geqslant m \geqslant 3$ and $\mathcal{K}^k_{(k-1)n+\lfloor \frac{m+1}{2} \rfloor}$ be 2-edge colored red and blue. If $\mathcal{C}^k_n \subseteq \mathcal{F}_{red}$, then either $\mathcal{P}^k_n \subseteq \mathcal{F}_{red}$ or $\mathcal{P}^k_m \subseteq \mathcal{F}_{blue}$.*

Proof. Let $e_i = \{v_1, v_2, \ldots, v_k\} + i(k-1) \pmod{n(k-1)}$, $i = 0, 1, \ldots, n-1$, be the edges of $\mathcal{C}^k_n \subseteq \mathcal{F}_{red}$ and $W = \{x_1, x_2, \ldots, x_{\lfloor \frac{m+1}{2} \rfloor}\}$ be the set of the remaining vertices. Set $e'_0 = (e_0 \setminus \{v_1\}) \cup \{x_1\}$ and for $1 \leqslant i \leqslant m-1$ let

$$e'_i = \begin{cases} (e_i \setminus \{v_{i(k-1)+1}\}) \cup \{x_{\frac{i+1}{2}}\} & \text{if } i \text{ is odd,} \\ \\ (e_i \setminus \{v_{(i+1)(k-1)+1}\}) \cup \{x_{\frac{i+2}{2}}\} & \text{if } i \text{ is even.} \end{cases}$$

If one of e'_i is red, we have a monochromatic $\mathcal{P}^k_n \subseteq \mathcal{F}_{red}$, otherwise $e'_0 e'_1 \ldots e'_{m-1}$ form a blue $\mathcal{P}^k_m$, which completes the proof. ∎

Let $\mathcal{P}$ be a loose path and x, y be vertices which are not in $\mathcal{P}$. By a $\varpi_{\{v_i,v_j,v_k\}}$-*configuration*, we mean a copy of $\mathcal{P}^3_2$ with edges $\{x, v_i, v_j\}$ and $\{v_j, v_k, y\}$ so that v_l's, $l \in \{i, j, k\}$, belong to two consecutive edges of $\mathcal{P}$. The vertices x and y are called the end vertices of this configuration. Using this notation, we have the following lemmas.

Lemma 2. *Let $n \geqslant 10$, $\mathcal{K}^3_n$ be 2-edge colored red and blue and $\mathcal{P}$, say in $\mathcal{F}_{red}$, be a maximum path. Let A be the set of five consecutive vertices of $\mathcal{P}$. If $W = \{x_1, x_2, x_3\}$ is disjoint from $\mathcal{P}$, then we have a ϖ_S-configuration in $\mathcal{F}_{blue}$ with two end vertices in W and $S \subseteq A$.*

Proof. First let $A = e \cup e'$ for two edges $e = \{v_1, v_2, v_3\}$ and $e' = \{v_3, v_4, v_5\}$. Since $\mathcal{P} \subseteq \mathcal{F}_{red}$ is maximal, at least one of the edges $e_1 = \{x_1, v_1, v_2\}$ and $e_2 = \{v_2, v_3, x_2\}$ must be blue. If both are blue, then $e_1 e_2$ is such a configuration. So first let e_1 be blue and e_2 be red. Maximality of $\mathcal{P}$ implies that at least one of the edges $e_3 = \{x_2, v_1, v_4\}$ or $e_4 = \{x_3, v_2, v_5\}$ is blue (otherwise, replacing ee' by $e_3 e_2 e_4$ in $\mathcal{P}$ yields a red path greater than $\mathcal{P}$, a contradiction), and clearly in each case we have a ϖ_S-configuration. Now, let e_1 be red and e_2 be blue. Clearly $e_5 = \{v_2, v_4, x_3\}$ is blue and $e_2 e_5$ form a ϖ_S-configuration. Now let $A = \{v_1, v_2, \ldots, v_5\}$ where $e_1 = \{x, v_1, v_2\}$, $e_2 = \{v_2, v_3, v_4\}$ and $e_3 = \{v_4, v_5, y\}$ are three consecutive edges of $\mathcal{P}$. If $\{x_i, v_2, v_3\}$ is a red edge for some $i \in \{1, 2, 3\}$, then $\{v_3, v_4, x_j\}$ and $\{v_3, v_5, x_j\}$ are blue for $j \neq i$ and so we are done. By the same argument the theorem is true if $\{x_i, v_3, v_4\}$ is red. Now we may assume $\{v_2, v_3, x_i\}$ and $\{v_3, v_4, x_i\}$ are blue for each $i \in \{1, 2, 3\}$ and so there is nothing to prove. ∎

Lemma 3. *Assume that $n \geqslant \left\lfloor \frac{5m}{4} \right\rfloor$ and $\mathcal{K}^3_{2n+\lfloor \frac{m+1}{2} \rfloor}$ is 2-edge colored red and blue. If $\mathcal{P} \subseteq \mathcal{F}_{blue}$ is a maximum path and W, $|W| \geqslant 5$, is a set of the vertices which are not covered by $\mathcal{P}$, then for every 4 consecutive edges e_1, e_2, e_3, e_4 of $\mathcal{P}$ either there is a $\mathcal{P}^3_5 \subseteq \mathcal{F}_{red}$, say Q, between $\{e_1, e_2, e_3, e_4\}$ and W with end vertices in W and with no the last vertex of e_4 as a vertex such that $|W \cap V(Q)| \leqslant 5$ or there is a $\mathcal{P}^3_4 \subseteq \mathcal{F}_{red}$, say Q, between $\{e_1, e_2, e_3\}$ and W with end vertices in W and with no the last vertex of e_3 as a vertex such that $|W \cap V(Q)| \leqslant 4$. In each of the above cases, each vertex of W except one vertex can be considered as the end vertex of Q.*

Proof. Suppose that e_1,e_2,e_3,e_4 be four consecutive edges in $\mathcal{P}$. Let $e_i = \{v_{2i-1}, v_{2i}, v_{2i+1}\}$, $1 \leqslant i \leqslant 4$, and $W = \{x_1, \ldots, x_t\}$ and $T = \{1, 2, \cdots, t\}$.

Case 1. For every $1 \leqslant i, j \leqslant t$, $\{v_1, v_2, x_i\}$ and $\{v_2, v_3, x_j\}$ are red.

Subcase 1. For every $1 \leqslant k, l \leqslant t$, the edges $\{v_3, v_4, x_k\}$ and $\{v_4, v_5, x_l\}$ are red.

For each $\{i_1, i_2, i_3, i_4\} \in P_4(T)$, edges, $\{x_{i_1}, v_1, v_2\}$,$\{v_2, x_{i_2}, v_3\}$,$\{v_3, x_{i_3}, v_4\}$,$\{v_4, v_5, x_{i_4}\}$ make a red $\mathcal{P}_4^3$ with end vertices x_{i_1} and x_{i_4}.

Subcase 2. There exists $1 \leqslant k \leqslant t$, such that the edge $\{v_3, v_4, x_k\}$ is blue.

So for each $\{i_1, i_2, i_3,\} \in P_3(T)$ with $k \neq i_2, i_3$, $\{x_{i_1}, v_1, v_2\}$,$\{v_2, v_3, x_{i_2}\}$,$\{x_{i_2}, v_5, v_4\}$, $\{v_4, v_6, x_{i_3}\}$ are the edges of a red desired $\mathcal{P}_4^3$ with end vertices x_{i_1} and x_{i_3}.

Subcase 3. There exists $1 \leqslant k \leqslant t$, such that the edge $\{v_4, v_5, x_k\}$ is blue.

If for every $1 \leqslant i, j \leqslant t$, the edges $\{v_5, v_6, x_i\}$ and $\{v_6, v_7, x_j\}$ are red, then for every $\{i_1, i_2, i_3, i_4\} \in P_4(T)$ with $i_3 \neq k$, we can find a red copy of $\mathcal{P}_5^3$ with edges $\{x_{i_1}, v_1, v_2\}$,$\{v_2, x_{i_2}, v_3\}$,$\{v_3, v_4, x_{i_3}\}$,$\{x_{i_3}, v_5, v_6\}$,$\{v_6, v_7, x_{i_4}\}$ and end vertices x_{i_1} and x_{i_4}. Otherwise there exists $1 \leqslant l \leqslant t$, such that either $\{v_{5,} v_6, x_l\}$ or $\{v_6, v_7, x_l\}$ is blue. For the first one, for every $\{i_1, i_2, i_3, i_4\} \in P_4(T)$ with $i_3 \neq k, l$ and $i_4 \neq l$, $\{x_{i_1}, v_1, v_2\}$,$\{v_2, x_{i_2}, v_3\}$,$\{v_3, v_4, x_{i_3}\}$,$\{x_{i_3}, v_7, v_6\}$,$\{v_6, v_8, x_{i_4}\}$ make a red copy of $\mathcal{P}_5^3$ with end vertices x_{i_1} and x_{i_4} and for the second one, for every $\{i_1, i_2, i_3\} \in P_3(T)$ with $l \neq i_2, i_3$ the edges, $\{x_{i_1}, v_1, v_2,\}$, $\{v_2, v_3, x_{i_2}\}$, $\{x_{i_2}, v_6, v_5\}$, $\{v_5, x_{i_3}, x_l\}$ make a red $\mathcal{P}_4^3$ with end vertices x_{i_1} and y where $y \in \{x_{i_3}, x_l\}$.

Case 2. For some $1 \leqslant i \leqslant t$, $\{v_1, v_2, x_i\}$ is blue.

Subcase 1. For every $1 \leqslant k, l \leqslant t$, the edges $\{v_5, v_6, x_k\}$ and $\{v_6, v_7, x_l\}$ are red.

For each $\{i_1, i_2, i_3, i_4\} \in P_4(T)$ with $i_j \neq i$, $1 \leqslant j \leqslant 4$, the edges, $\{x_{i_1}, x_i, v_3\}$, $\{v_3, x_{i_2}, v_2\}$, $\{v_2, v_4, x_{i_3}\}$, $\{x_{i_3}, v_5, v_6\}$, $\{v_6, v_7, x_{i_4}\}$ make a red $\mathcal{P}_5^3$ with end vertices y, $y \in \{x_{i_1}, x_i\}$, and x_{i_4}.

Subcase 2. For some $1 \leqslant k \leqslant t$, $\{v_5, v_6, x_k\}$ is blue.

In this case, for each $\{i_1, i_2, i_3, i_4\} \in P_4(T)$ with $i_j \neq i$, $1 \leqslant j \leqslant 4$, and $i_3, i_4 \neq k$, the edges $\{x_{i_1}, x_i, v_3\}$,$\{v_3, x_{i_2}, v_2\}$,$\{v_2, v_4, x_{i_3}\}$,$\{x_{i_3}, v_7, v_6\}$, $\{v_6, v_8, x_{i_4}\}$ make a red $\mathcal{P}_5^3$ with end vertices y, $y \in \{x_{i_1}, x_i\}$, and x_{i_4}.

Subcase 3. For some $1 \leqslant k \leqslant t$, $\{v_6, v_7, x_k\}$ is blue.

In this case, for each $\{i_1, i_2, i_3\} \in P_3(T)$ with $i_j \neq i$, $1 \leqslant j \leqslant 3$, and $i_2, i_3 \neq k$, the edges $\{x_{i_1}, x_i, v_3\}$,$\{v_3, v_2, x_{i_2}\}$,$\{x_{i_2}, v_4, v_6\}$,$\{v_6, v_5, x_{i_3}\}$ make a red $\mathcal{P}_4^3$ with end vertices y, $y \in \{x_{i_1}, x_i\}$, and x_{i_3}.

Case 3. For some $1 \leqslant i \leqslant t$, $\{v_2, v_{3,} x_i\}$ is blue.

Subcase 1. For every $1 \leqslant k, l \leqslant t$, the edges $\{v_3, v_4, x_k\}$ and $\{v_4, v_5, x_l\}$ are red.

For each $\{i_1, i_2, i_3\} \in P_3(T)$ with $i_j \neq i$, $1 \leqslant j \leqslant 3$, $\{x_{i_1}, x_i, v_1\}$,$\{v_1, v_2, x_{i_2}\}$, $\{x_{i_2}, v_3, v_4\}$,$\{v_4, v_5, x_{i_3}\}$ are the edges of a red $\mathcal{P}_4^3$ with end vertices y, $y \in \{x_{i_1}, x_i\}$, and x_{i_3}.

Subcase 2. For some $1 \leqslant k \leqslant t$, $\{v_3, v_4, x_k\}$ is blue.

In this case, for each $\{i_1, i_2, i_3\} \in P_3(T)$ with $i_j \neq i$, $1 \leqslant j \leqslant 3$, and $i_2, i_3 \neq k$, the edges, $\{x_{i_1}, x_i, v_1\}$,$\{v_1, v_2, x_{i_2}\}$,$\{x_{i_2}, v_5, v_4\}$,$\{v_4, v_6, x_{i_3}\}$ make a red copy of $\mathcal{P}_4^3$ with end vertices y, $y \in \{x_{i_1}, x_i\}$, and x_{i_3}.

Subcase 3. For some $1 \leqslant k \leqslant t$, $\{v_4, v_5, x_k\}$ is blue.

If for every $1 \leqslant l, h \leqslant t$, the edges $\{v_5, v_6, x_l\}$ and $\{v_6, v_7, x_h\}$ are red, then for each $\{i_1, i_2, i_3, i_4\} \in P_4(T)$ with $i_j \neq i$, $1 \leqslant j \leqslant 4$, and $i_3 \neq k$, the edges, $\{x_{i_1}, x_i, v_1\}$,$\{v_1, x_{i_2}, v_2\}$,$\{v_2, v_4, x_{i_3}\}$,$\{x_{i_3}, v_5, v_6\}$, $\{v_6, v_7, x_{i_4}\}$ make a red $\mathcal{P}_5^3$ with end vertices y, $y \in \{x_{i_1}, x_i\}$, and x_{i_4}. Otherwise there exists $1 \leqslant l \leqslant t$, such that either $\{v_{5,}v_6, x_l\}$ or $\{v_6, v_7, x_l\}$ is blue. For the first one, for each $\{i_1, i_2, i_3, i_4\} \in P_4(T)$ with $i_j \neq i$, $1 \leqslant j \leqslant 4$, $i_3 \neq k, l$ and $i_4 \neq l$, the edges $\{x_{i_1}, x_i, v_1\}$,$\{v_1, x_{i_2}, v_2\}$,$\{v_2, v_4, x_{i_3}\}$, $\{x_{i_3}, v_7, v_6\}$,$\{v_6, v_8, x_{i_4}\}$ make a red copy of $\mathcal{P}_5^3$ with end vertices y, $y \in \{x_{i_1}, x_i\}$ and x_{i_4}. For the second one, for every $\{i_1, i_2, i_3\} \in P_3(T)$ with $i_j \neq i$, $1 \leqslant j \leqslant 3$, and $i_2, i_3 \neq l$, $\{\{x_{i_1}, x_i, v_1\}, \{v_1, v_2, x_{i_2}\}, \{x_{i_2}, v_4, v_6\}, \{v_6, v_5, x_{i_3}\}\}$ is the set of the edges of a red $\mathcal{P}_4^3$ with end vertices y, $y \in \{x_{i_1}, x_i\}$, and x_{i_3}. These observations complete the proof. ■

3 Main Results

In this section, we prove that $R(\mathcal{P}_n^3, \mathcal{P}_m^3) = 2n + \lfloor \frac{m+1}{2} \rfloor$ for every $n \geqslant \lfloor \frac{5m}{4} \rfloor$. First we present several lemmas which will be our main tools in establishing the main theorem.

Lemma 4. *Assume that $n = \lfloor \frac{5m}{4} \rfloor$ and $\mathcal{K}^3_{2n+\lfloor \frac{m+1}{2} \rfloor}$ is 2-edge colored red and blue. If $\mathcal{P} = \mathcal{P}_{m-1}^3$ is a maximum blue path, then $\mathcal{P}_{n-1}^3 \subseteq \mathcal{F}_{red}$.*

Proof. Let $t = 2n + \lfloor \frac{m+1}{2} \rfloor$ and $\mathcal{P} = e_1 e_2 \ldots e_{m-1}$ be a copy of $\mathcal{P}_{m-1}^3 \subseteq \mathcal{F}_{blue}$ with edges $e_i = \{v_1, v_2, v_3\} + 2(i-1)$, $i = 1, \ldots, m-1$. Set $W = V(\mathcal{K}_t^3) \setminus V(\mathcal{P})$. Using Lemma 3 there is a red path Q_1 with end vertices x_1 and y_1 in $W_1 = W$ between E_1' and W_1 where $E_1 = \{e_i : i_1 = 1 \leqslant i \leqslant 4\}$, $\bar{E}_1 = E_1 \setminus \{e_4\}$ and $E_1' \in \{E_1, \bar{E}_1\}$. Set $i_2 = \min\{j : j \in \{i_1+3, i_1+4\}, e_j \notin E_1'\}$, $E_2 = \{e_i : i_2 \leqslant i \leqslant i_2+3\}$ and $\bar{E}_2 = E_2 \setminus \{e_{i_2+3}\}$ and $W_2 = (W \setminus V(Q)) \cup \{x_1, y_1\}$. Again using Lemma 3 there is a red path Q_2 between E_2' and W_2 such that $Q_1 \cup Q_2$ is a red path with end vertices x_2, y_2 in W_2 where $E_2' \in \{E_2, \bar{E}_2\}$ and again set $i_3 = \min\{j : j \in \{i_2+3, i_2+4\}, e_j \notin E_2'\}$, $E_3 = \{e_i : i_3 \leqslant i \leqslant i_3+3\}$, $\bar{E}_3 = E_3 \setminus \{e_{i_3+3}\}$ and $W_3 = (W \setminus V(Q_1 \cup Q_2)) \cup \{x_2, y_2\}$. Since $|W| \geqslant m$, using Lemma 3 by continuing the above process we can partition $E(\mathcal{P}) \setminus \{e_{m-1}\}$ into classes E_i'th, $|E_i'| \in \{3, 4\}$ and at most one class of size $r \leqslant 3$ of the last edges such that for each i, there is a red $Q_i = \mathcal{P}_5^3$ (resp. $Q_i = \mathcal{P}_4^3$) between E_i' and W with the properties in Lemma 3 if $|E_i'| = 4$ (resp. $|E_i'| = 3$) and $\mathcal{P}' = \cup Q_i$ is a red path with end vertices x, y in W. Let

$l_1 = |\{i : | E'_i |= 4\}|$ and $l_2 = |\{i : | E'_i |= 3\}|$. So $m - 2 = 4l_1 + 3l_2 + r$, $0 \leqslant r \leqslant 3$ and $\mathcal{P}'$ has $5l_1 + 4l_2$ edges. One can easily check that $5l_1 + 4l_2 \geqslant \frac{5}{4}(m - 2 - r)$. Also we have

$$|W \cap V(\mathcal{P}')| \leqslant 4l_1 + 3l_2 + 1 = m - 1 - r.$$

Let $T = V(\mathcal{K}_t^3) \setminus (V(\mathcal{P}) \cup V(\mathcal{P}'))$ and suppose that $m = 4k + p$ for some p, $0 \leqslant p \leqslant 4$. Therefore $|T| \geqslant r + 2$ if $p = 0, 1$ and $|T| \geqslant r + 1$ if $p = 2, 3$. Now we consider the following cases.

Case 1. $r = 0$.

Clearly $|T| \geqslant 1$ and it is easy to see that $\mathcal{P}'$ contains at least $n-2$ edges. Let $\{u\} \subseteq T$. The maximality of $\mathcal{P}$ implies that the edge $e = \{v_{2m-1}, x, u\}$ is red and hence $\mathcal{P}' \cup \{e\}$ is a red copy of $\mathcal{P}_{n-1}^3$.

Case 2. $r = 1$.

In this case, $|T| \geqslant 2$ and it is easy to see that $\mathcal{P}'$ contains at least $n - 3$ edges. Let $\{u, v\} \subseteq T$. Clearly $\mathcal{P}' \cup \{\{v_{2m-2}, x, u\}, \{v_{2m-1}, u, v\}\}$ is a red copy of $\mathcal{P}_{n-1}^3$.

Case 3. $r = 2$.

It is easily seen that $|T| \geqslant 3$ and $\mathcal{P}'$ contains at least $n-5$ edges. Let $T' = \{u, v, w\} \subseteq T$. Since $V(\mathcal{P}') \cap V(e_{m-3} \cup e_{m-2}) = \emptyset$ by Lemma 2 there is a red ϖ_S-configuration with $S \subset e_{m-3} \cup e_{m-2}$ and its end vertices in T', say u and v. The maximality of $\mathcal{P}$ implies that the edges $\{v_{2m-2}, x, u\}$ and $\{v_{2m-1}, v, w\}$ are red and clearly we have a red $\mathcal{P}_{n-1}^3$.

Case 4. $r = 3$.

In this case, for $p \in \{2, 3\}$ we have $|T| \geqslant 4$ and $\mathcal{P}'$ contains at least $n - 5$ edges. Using an argument similar to case 3 we can complete the proof. Now let $p \in \{0, 1\}$. Then $|T| \geqslant 5$ and $\mathcal{P}'$ contains at least $n - 6$ edges. Set $T' = \{u, v, w, z, t\} \subseteq T$. By Lemma 2, there is a ϖ_S-configuration C with $S \subseteq V(e_{m-3} \cup e_{m-2})$ and end vertices in T', say u and v. Clearly $\mathcal{P}' \cup \{\{y, w, v_{2m-2}\}, \{v_{2m-2}, z, t\}, \{v_{2m-1}, t, u\}\} \cup C$ is a red $\mathcal{P}_{n-1}^3$. These observations complete the proof. ∎

Lemma 5. *Let $n \geqslant \left\lfloor \frac{5m}{4} \right\rfloor$ and $\mathcal{K}^3_{2n+\lfloor \frac{m+1}{2} \rfloor}$ be 2-edge colored red and blue. If $\mathcal{P}_{n-1}^3 \subseteq \mathcal{F}_{red}$ be a maximum path, then $\mathcal{P}_m^3 \subseteq \mathcal{F}_{blue}$.*

Proof. Let $t = 2n + \lfloor \frac{m+1}{2} \rfloor$ and $\mathcal{P} = e_1 e_2 \dots e_{n-1}$ be a copy of $\mathcal{P}_{n-1}^3 \subseteq \mathcal{F}_{red}$ with end edges $e_1 = \{v_1, v_2, v_3\}$ and $e_{n-1} = \{v_{2n-3}, v_{2n-2}, v_{2n-1}\}$. By Lemma 1, we may assume that the subhypergraph induced by $V(\mathcal{P})$ does not have a red copy of $\mathcal{C}_n^3$. Let $W = V(\mathcal{K}_t^3) \setminus V(\mathcal{P})$ and let $2n - 2 = 5q + h$ where $0 \leqslant h < 5$. Partition the set $V(\mathcal{P}) \setminus \{v_1\}$ into q classes $A_1, A_2, \dots, A_q$ of size five and one class $A_{q+1} = \{v_{2n-h}, \dots, v_{2n-2}, v_{2n-1}\}$ of size h if $h > 0$, so that each class contains consecutive vertices of $\mathcal{P}$. Using Lemma 2, there is a blue ϖ_{S_1}-configuration, $\bar{c}_1$, with the set of end vertices $E_1 \subseteq W$ and $S_1 \subseteq A_1$. Let $x_1 \in E_1$ and B_1 be a 2-subset of $W \setminus E_1$. Again by Lemma 2, there is a blue ϖ_{S_2}-configuration, $\bar{c}_2$, with the set of end vertices $E_2 \subseteq (B_1 \cup \{x_1\})$ and $S_2 \subseteq A_2$. If $x_1 \notin E_2$, then let $\bar{c}_3$ be a blue ϖ_{S_3}-configuration with the set of end vertices $E_3 \subseteq \{x_1, y, z\}$ and $S_3 \subseteq A_3$ where $y \in B_1$ and $z \in W \setminus (E_1 \cup E_2)$. If $x_1 \in E_2$, then let $\bar{c}_3$ be a blue ϖ_{S_3}-configuration with the set of end vertices $E_3 \subseteq \{x_2, y, z\}$ and $S_3 \subseteq A_3$ where $x_2 \in E_2 \setminus \{x_1\}$ and $\{y, z\} \subseteq W \setminus (E_1 \cup E_2)$. We continue this process to find the set of $\{\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_{q'}\}$ of configurations. When this process terminate, we have the paths $\mathcal{P}_{l''}$ and $\mathcal{P}_{l'}$ where $l'' \geqslant l' \geqslant 0$ and $l'' + l' = 2q'$. Let x'', y'' (resp. x', y' if $l' > 0$) be the end vertices of $\mathcal{P}_{l''}$ (resp. $\mathcal{P}_{l'}$) in W. Let $T = V(\mathcal{K}_t^3) \setminus (V(\mathcal{P}) \cup V(\mathcal{P}_{l''}) \cup V(\mathcal{P}_{l'}))$. Clearly $|T| = \lfloor \frac{m+1}{2} \rfloor + 1 - (q' + i)$ where $i = 1$ if $l' = 0$ and $i = 2$ if $l' > 0$. Assume $m = 4k + r$ for some r, $0 \leqslant r \leqslant 3$. We have the following cases.

Case 1. $r = 0$.

Since $q \geqslant 2k - 1$, we have $2q' \geqslant m - 2$. On the other hand, $|W| = \lfloor \frac{m+1}{2} \rfloor + 1$ and so $2q' \leqslant m$. If $2q' = m$, then $l' = 0$ and so $\mathcal{P}_{l''=m}$ is a blue path. Now we may assume that $2q' = m - 2$, and one can easily check that the vertices $\{v_{2n-3}, v_{2n-2}, v_{2n-1}\}$ are not used in $\mathcal{P}_{l''} \cup \mathcal{P}_{l'}$. First let $l' = 0$. Then $|T| = 1$ and we may assume $T = \{u\}$. Now using the maximality of $\mathcal{P}$ and the fact that $\mathcal{C}_n^3 \not\subseteq \mathcal{F}_{red}$, $\mathcal{P}_{l''} \cup \{\{v_{2n-2}, y'', u\}, \{v_{2n-1}, u, v_1\}\}$ is a blue $\mathcal{P}_m^3$. For $l' > 0$, $\mathcal{P}_{l''} \cup \{\{v_{2n-2}, y'', x'\}\} \cup \mathcal{P}_{l'} \cup \{\{v_{2n-1}, y', v_1\}\}$ is a blue $\mathcal{P}_m^3$.

Case 2. $r = 1$.

Since $\mid W \mid = \lfloor \frac{m+1}{2} \rfloor + 1$, $2q' \leqslant m + 1$ and if the equality holds, then $l' = 0$. On the other hand, $q \geqslant 2k$ and so $2q' \geqslant m - 1$. Hence $2q' \in \{m + 1, m - 1\}$. If $2q' = m + 1$, then $l' = 0$ and there is a blue $\mathcal{P}_{m+1}^3$. Now let $2q' = m - 1$. If $l' = 0$, then $|T| = 1$, so $T = \{u\}$ and hence $\mathcal{P}_{l''} \cup \{\{v_1, u, y''\}\}$ is a blue $\mathcal{P}_m^3$. If $l' > 0$, then $\mathcal{P}_{l''} \cup \{\{v_1, y'', x'\}\} \cup \mathcal{P}_{l'}$ is a blue $\mathcal{P}_m^3$.

Case 3. $r = 2$.

Using an argument similar to the case 1, we have $2q' \in \{m, m - 2\}$ and if $2q' = m$, then $l' = 0$ and we have a blue $\mathcal{P}_{l''=m}$. Again by an argument similar to the case 1 we have a blue $\mathcal{P}_m^3$.

Case 4. $r = 3$.

In this case, partition $V(\mathcal{P}) \setminus \{v_1, v_2\}$ into $\lfloor \frac{2n-3}{5} \rfloor$ classes of size five and possibly one class of size at most four. Then we repeat the mentioned process in the first of the proof to find blue paths $\mathcal{P}_{l''}$ and $\mathcal{P}_{l'}$ with $l'' \geqslant l' \geqslant 0$ and $l'' + l' = 2q'$. Again using a similar argument in case 1, we have $2q' \in \{m + 1, m - 1, m - 3\}$. If $2q' = m + 1$, then we have $l' = 0$ and so there is a blue $\mathcal{P}_{m+1}^3$. For $2q' = m - 1$, the assertion holds by an argument similar to the case 2. Now let $2q' = m - 3$. If $l' = 0$, then $|T| = 2$, so $T = \{u, v\}$ and hence $\mathcal{P}_{l''} \cup \{\{v_{2n-2}, v_2, y''\}, \{v_{2n-2}, v, u\}, \{u, v_1, v_{2n-1}\}\}$ is a blue $\mathcal{P}_m^3$ (note that $\{v_{2n-3}, v_{2n-2}, v_{2n-1}\} \cap V(\mathcal{P}_{l''}) = \emptyset$). If $l' > 0$, then $|T| = 1$, so $T = \{u\}$ and hence $\mathcal{P}_{l''} \cup \{\{v_{2n-2}, v_2, y''\}, \{v_{2n-2}, x', u\}\} \cup \mathcal{P}_{l'} \cup \{\{y', v_1, v_{2n-1}\}\}$ is a blue $\mathcal{P}_m^3$ and the proof is completed. ■

Theorem 6. *For every* $n \geqslant \lfloor \frac{5m}{4} \rfloor$,

$$R(\mathcal{P}^3_n, \mathcal{P}^3_m) = 2n + \left\lfloor \frac{m+1}{2} \right\rfloor.$$

Proof. We prove the theorem by induction on $m+n$. The proof of the case $m = n = 1$ is trivial. Suppose that for $m' + n' < m + n$ with $n' \geqslant \lfloor \frac{5m'}{4} \rfloor$, $R(\mathcal{P}^3_{n'}, \mathcal{P}^3_{m'}) = 2n' + \lfloor \frac{m'+1}{2} \rfloor$. Now, let $n \geqslant \lfloor \frac{5m}{4} \rfloor$ and let $\mathcal{K}^3_{2n+\lfloor \frac{m+1}{2} \rfloor}$ be 2-edge colored red and blue. We may assume there is no red copy of $\mathcal{P}^3_n$ and no blue copy of $\mathcal{P}^3_m$. Consider the following cases.

Case 1. $n = \lfloor \frac{5m}{4} \rfloor$.

Since $R(\mathcal{P}^3_{n-1}, \mathcal{P}^3_{m-1}) = 2(n-1) + \lfloor \frac{m}{2} \rfloor < 2n + \lfloor \frac{m+1}{2} \rfloor$ by induction hypothesis, then either there is a $\mathcal{P}^3_{n-1} \subseteq \mathcal{F}_{red}$ or a $\mathcal{P}^3_{m-1} \subseteq \mathcal{F}_{blue}$. If we have a red copy of $\mathcal{P}^3_{n-1}$, then by Lemma 5 we have a $\mathcal{P}^3_m \subseteq \mathcal{F}_{blue}$. Now assume that there is a blue copy of $\mathcal{P}^3_{m-1}$. Lemma 4 implies that $\mathcal{P}^3_{n-1} \subseteq \mathcal{F}_{red}$ and using Lemma 5 we have $\mathcal{P}^3_m \subseteq \mathcal{F}_{blue}$, a contradiction.

Case 2. $n > \lfloor \frac{5m}{4} \rfloor$.

In this case, $n - 1 \geqslant \lfloor \frac{5m}{4} \rfloor$ and since $R(\mathcal{P}^3_{n-1}, \mathcal{P}^3_m) = 2(n-1) + \lfloor \frac{m+1}{2} \rfloor < 2n + \lfloor \frac{m+1}{2} \rfloor$, by induction hypothesis we have a $\mathcal{P}^3_{n-1} \subseteq \mathcal{F}_{red}$. Using Lemma 5 we have a $\mathcal{P}^3_m \subseteq \mathcal{F}_{blue}$ and it completes the proof. ∎

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