A new determinant expression for the weighted Bartholdi zeta function of a digraph

Iwao Sato*, Hideo Mitsuhasi

Oyama National College of Technology
Oyama, Tochigi 323-0806, Japan

isato@oyama-ct.ac.jp

Hideaki Morita

Division of System Engineering for Mathematics
Muroran Institute of Technology
Muroran, Hokkaido 050-8585, Japan

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Abstract

We consider the weighted Bartholdi zeta function of a digraph D, and give a new determinant expression of it. Furthermore, we treat a weighted L-function of D, and give a new determinant expression of it. As a corollary, we present determinant expressions for the Bartholdi edge zeta functions of a graph and a digraph.

Key words: zeta function, digraph covering, L-function

1 Introduction

Zeta functions of graphs started from zeta functions of regular graphs by Ihara [7]. In [7], he showed that their reciprocals are explicit polynomials. A zeta function of a regular graph G associated with a unitary representation of the fundamental group of G was developed by Sunada [12,13]. Hashimoto [6] generalized Ihara's result on the zeta function of a regular graph to an irregular graph, and showed that its reciprocal is again a polynomial by a determinant containing the edge matrix. Bass [2] presented another determinant expression for the Ihara zeta function of an irregular graph by using its adjacency matrix.

Stark and Terras [11] gave an elementary proof of Bass' Theorem, and discussed three different zeta functions of any graph. Furthermore, various proofs of Bass' Theorem were given by Foata and Zeilberger [4], Kotani and Sunada [8].

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For two variable zeta function of a graph, Bartholdi [1] defined and gave a determinant expression of the Bartholdi zeta function of a graph. Mizuno and Sato [9] presented a decomposition formula for the Bartholdi zeta function of a regular covering of a graph.

As a digraph version of the Bartholdi zeta function, Choe, Kwak, Park and Sato [3] defined the weighted Bartholdi zeta function of a digraph, and presented its determinant expression.

As a multi-variable zeta function of a graph, Stark and Terras [11] defined the edge zeta function of a graph. Watanabe and Fukumizu [14] presented a determinant expression for the edge zeta function of a graph G with n vertices by $n \times n$ matrices.

In this paper, we present a new determinant expression of the weighted Bartholdi zeta function of a digraph D by using the method of Watanabe and Fukumizu [14]:

Main Theorem.

Let D be a connected digraph with n vertices and m arcs, and let $\mathbf{W} = \mathbf{W}(D)$ be a weighted matrix of D. Then the reciprocal of the weighted Bartholdi zeta function of D is given by

$$\zeta(D, w, u, t)^{-1} = \det(\mathbf{I}_n + (1-u)t^2\tilde{\mathbf{D}} - t\tilde{\mathbf{A}}_1 - t\tilde{\mathbf{A}}_0) \prod_{i=1}^{m_1}(1 - w(f_i)w(f_i^{-1})(1-u)^2t^2),$$

where $\tilde{\mathbf{D}}$, $\tilde{\mathbf{A}}_1$ and $\tilde{\mathbf{A}}_0$ are defined in Section 3, and $f_1^{\pm 1}, \ldots, f_{m_1}^{\pm 1}$ are symmetric arcs of D.

Furthermore, we present a new decomposition formula for the weighted Bartholdi zeta function of a group covering of D, and a new determinant expression for the weighted Bartholdi L-function of D.

2 Preliminaries

Graphs and digraphs treated here are finite. Let $G = (V(G), E(G))$ be a connected graph (possibly multiple edges and loops) with the set $V(G)$ of vertices and the set $E(G)$ of unoriented edges uv joining two vertices u and v. For $uv \in E(G)$, an arc (u, v) is the oriented edge from u to v. Set $D(G) = \{(u, v), (v, u) \mid uv \in E(G)\}$. For $e = (u, v) \in D(G)$, set $u = o(e)$ and $v = t(e)$. Furthermore, let $e^{-1} = (v, u)$ be the *inverse* of $e = (u, v)$.

A *path P of length n* in G is a sequence $P = (e_1, \cdots, e_n)$ of n arcs such that $e_i \in D(G)$, $t(e_i) = o(e_{i+1})(1 \leqslant i \leqslant n-1)$, where indices are treated *mod n*. Set $\mid P \mid = n$, $o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, P is called an *$(o(P), t(P))$-path*. We say that a path $P = (e_1, \cdots, e_n)$ has a *backtracking* or a *bump* at $t(e_i)$ if $e_{i+1}^{-1} = e_i$ for some $i (1 \leqslant i \leqslant n-1)$. A (v, w)-path is called a *v-cycle* (or *v-closed path*) if $v = w$.

We introduce an equivalence relation between cycles. Two cycles $C_1 = (e_1, \cdots, e_m)$ and $C_2 = (f_1, \cdots, f_m)$ are called *equivalent* if there exists k such that $f_j = e_{j+k}$ for all j. The inverse cycle of C is in general not equivalent to C. Let $[C]$ be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *power* of B. A cycle C is *reduced* if C has no backtracking. Furthermore, a cycle C is *prime* if it is not a power of a strictly smaller cycle.

The *Ihara zeta function* of a graph G is a function of $u \in \mathbf{C}$ with $|u|$ sufficiently small, defined by

$$\mathbf{Z}(G,t) = \prod_{[C]} (1 - t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G(see [7]).

Let m be the number of edges of G. Furthermore, let two $m \times m$ matrices $\mathbf{B} = (\mathbf{B}_{e,f})_{e,f \in A(D)}$ and $\mathbf{J}_0 = (\mathbf{J}_{e,f})_{e,f \in A(D)}$ be defined as follows:

$$\mathbf{B}_{e,f} = \begin{cases} 1 & \text{if } t(e) = o(f), \\ 0 & \text{otherwise} \end{cases}, \mathbf{J}_{e,f} = \begin{cases} 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbf{B} - \mathbf{J}_0$ is called the *edge matrix* of G.

Theorem 1 (Hashimoto; Bass)**.** *Let G be a connected graph with n vertices and m edges. Then the reciprocal of the Ihara zeta function of G is given by*

$$\mathbf{Z}(G,t)^{-1} = \det(\mathbf{I}_{2m} - t(\mathbf{B} - \mathbf{J}_0)) = (1-t^2)^{m-n} \det(\mathbf{I} - t\mathbf{A}(G) + t^2(\mathbf{D} - \mathbf{I})),$$

where $\mathbf{A}(G)$ is the adjacency matrix of G, and $\mathbf{D} = (d_{ij})$ is the diagonal matrix with $d_{ii} = \deg v_i$ where $V(G) = \{v_1, \cdots, v_n\}$.

Then the *Bartholdi zeta function* of G is defined by

$$\zeta_G(u,t) = \zeta(G,u,t) = \prod_{[C]} (1 - u^{cbc(C)} t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of G(see [1]).

Theorem 2 (Bartholdi)**.** *Let G be a connected graph with n vertices and m unoriented edges. Then the reciprocal of the Bartholdi zeta function of G is given by*

$$\begin{aligned}\zeta(G,u,t)^{-1} &= \det(\mathbf{I}_{2m} - t(\mathbf{B} - (1-u)\mathbf{J}_0)) \\ &= (1-(1-u)^2t^2)^{m-n} \det(\mathbf{I} - t\mathbf{A}(G) + (1-u)(\mathbf{D} - (1-u)\mathbf{I})t^2).\end{aligned}$$

In the case of $u = 0$, Theorem 2 implies Theorem 1.

We now state the weighted Bartholdi zeta function of a digraph. Let $D = (V(D), A(D))$ be a connected digraph with the set $V(D)$ of vertices and the set $A(D)$ of arcs. Furthermore, let D have n vertices $v_1, \ldots, v_n$ and m arcs. Then we consider an $n \times n$ matrix $\mathbf{W} = \mathbf{W}(D) = (w_{ij})_{1 \leqslant i,j \leqslant n}$ with ij entry nonzero complex number w_{ij} if $(v_i, v_j) \in A(D)$, and $w_{ij} = 0$ otherwise. The matrix $\mathbf{W} = \mathbf{W}(D)$ is called the *weighted matrix* of D. Furthermore, let $w(v_i, v_j) = w_{ij}$, $v_i, v_j \in V(D)$ and $w(e) = w_{ij}, e = (v_i, v_j) \in A(D)$. For each path $P = (e_1, \cdots, e_r)$ of G, the *norm* $w(P)$ of P is defined as follows: $w(P) = w(e_1) \cdots w(e_r)$.

The *cyclic bump count* $cbc(C)$ of a cycle $C = (e_1, \cdots, e_n)$ of G is

$$cbc(C) = | \{ i = 1, \cdots, n \mid e_i = e_{i+1}^{-1} \} |,$$

where $e_{n+1} = e_1$. Then the *weighted Bartholdi zeta function* of D is a function of $u, t \in \mathbf{C}$ with $|u|, |t|$ sufficiently small, defined by

$$\zeta(D, w, u, t) = \prod_{[C]} (1 - w(C) u^{cbc(C)} t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of D.

If $w = \mathbf{1}$, i.e., $w(v_i, v_j) = 1$ for each $(v_i, v_j) \in A(D)$, then the weighted Bartholdi zeta function of D is the Bartholdi zeta function of D. If $D = D_G$ is the symmetric digraph corresponding to a graph G, and $w = \mathbf{1}$, then the weighted Bartholdi zeta function of D_G is the Bartholdi zeta function of G. If $D = D_G$, $w = \mathbf{1}$ and $u = 0$, then the weighted Bartholdi zeta function of G is the Ihara zeta function of G.

Two $m \times m$ matrices $\mathbf{B}_w = (\mathbf{B}^w_{e,f})_{e,f \in A(D)}$ and $\mathbf{J}_w = (\mathbf{J}^w_{e,f})_{e,f \in A(D)}$ are defined as follows:

$$\mathbf{B}^w_{e,f} = \begin{cases} w(e) & \text{if } t(e) = o(f), \\ 0 & \text{otherwise} \end{cases}, \mathbf{J}^w_{e,f} = \begin{cases} w(e) & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we define two $n \times n$ matrices $\mathbf{W}_1 = \mathbf{W}_1(D) = (a_{uv})$ and $\mathbf{W}_0$ as follows:

$$a_{uv} = \begin{cases} w(u, v) & \text{if both } (u, v) \text{ and } (v, u) \in A(D), \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathbf{W}_0 = \mathbf{W}_0(D) = \mathbf{W}(D) - \mathbf{W}_1.$$

Let an $n \times n$ matrix $\mathbf{S} = (s_{xy})$ is the diagonal matrix defined by

$$s_{xx} =| \{e \in A(D) \mid o(e) = x, e^{-1} \in A(D)\} | .$$

Theorem 3 (Choe, Kwak, Park and Sato)**.** *Let D be a connected digraph, and let $\mathbf{W} = \mathbf{W}(D)$ be a weighted matrix of D. Furthermore, let $m_1 =| \{e \in A(D) \mid e^{-1} \in A(D)\} | /2$. Then the reciprocal of the weighted Bartholdi zeta function of D is given by*

$$\zeta(D, w, u, t)^{-1} = \det(\mathbf{I}_m - (\mathbf{B}_w - (1 - u)\mathbf{J}_w)t),$$

where $n =| V(D) |$ and $m =| A(D) |$.

Furthermore, if $w(e^{-1}) = w(e)^{-1}$ for each $e \in A(D)$ such that $e^{-1} \in A(D)$, then

$$\zeta(D, w, u, t)^{-1} = (1 - (1 - u)^2 t^2)^{m_1 - n}$$

$$\times \quad \det(\mathbf{I}_n - t\mathbf{W}_1(D) - (1 - (1 - u)^2 t^2)t\mathbf{W}_0(D) + (1 - u)t^2(\mathbf{S} - (1 - u)\mathbf{I}_n)).$$

If $D = D_G$, $w = \mathbf{1}$ and $u = 0$, then Theorem 2 implies Theorem 1.

Now, we proceed to the edge zeta function of a graph G with m edges. Let G be a connected graph and $D(G) = \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{2m}\}(e_{m+i} = e_i^{-1}(1 \leqslant i \leqslant m))$. We introduce $2m$ variables $z_1, \ldots, z_{2m}$, and set $g(C) = z_{i_1} \cdots z_{i_k}$ for each cycle $C =$

$(e_{i_1}, \dots, e_{i_k})$ of G. Set $z_{e_i} = z_i (1 \leqslant i \leqslant 2m)$ and $\mathbf{z} = (z_1, \dots, z_{2m})$. Then the *edge zeta function* $\zeta_G(\mathbf{z})$ of G is defined by

$$\zeta_G(\mathbf{z}) = \prod_{[C]} (1 - g(C))^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G.

Theorem 4 (Stark and Terras). *Let G be a connected graph with m edges. Then*

$$\zeta_G(\mathbf{z})^{-1} = \det(\mathbf{I}_{2m} - (\mathbf{B} - \mathbf{J}_0)\mathbf{U}),$$

where

$$\mathbf{U} = \begin{bmatrix} z_1 & & & & & 0 \\ & \ddots & & & & \\ & & z_m & & & \\ & & & z_{m+1} & & \\ & & & & \ddots & \\ 0 & & & & & z_{2m} \end{bmatrix}.$$

Let G be a graph with n vertices. Then we define an $n \times n$ matrix $\widehat{\mathbf{A}} = (a_{xy})$ as follows:

$$a_{xy} = \begin{cases} z_{(x,y)}/(1 - z_{(x,y)} z_{(y,x)}) & \text{if } (x,y) \in D(G), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, an $n \times n$ matrix $\widehat{\mathbf{D}} = (d_{xy})$ is the diagonal matrix defined by

$$d_{xx} = \sum_{o(e)=x} \frac{z_e z_{e^{-1}}}{1 - z_e z_{e^{-1}}}.$$

Theorem 5 (Watanabe and Fukumizu). *Let G be a connected graph with n vertices and m edges. Then*

$$\zeta_G(\mathbf{z})^{-1} = \det(\mathbf{I}_n + \widehat{\mathbf{D}} - \widehat{\mathbf{A}}) \prod_{i=1}^{m} (1 - z_{f_i} z_{f_i^{-1}}),$$

where $D(G) = \{f_1, f_1^{-1}, \dots, f_m f_m^{-1}\}$.

In Section 2, we present a new determinant expression of the weighted Bartholdi zeta function of a digraph D by using the method of Watanabe and Fukumizu [14]. In Section 3, we present a new decomposition formula for the weighted Bartholdi zeta function of a group covering of D. In Section 4, we present a new determinant expression for the weighted Bartholdi L-function of D. In Section 5, we define the Bartholdi edge zeta functions of graphs and digraphs, and present their determinant expressions as corollaries of Theorem 6.

3 Weighted Bartholdi zeta functions of digraphs

We present a new determinant expression of the weighted Bartholdi zeta function of a digraph.

Let D be a connected digraph with n vertices $v_1, \cdots, v_n$ and m arcs, and $\mathbf{W} = \mathbf{W}(D)$ a weighted matrix of D. Then we define two $n \times n$ matrices $\tilde{\mathbf{A}}_1 = \tilde{\mathbf{A}}_1(D) = (a_{xy})$ and $\tilde{\mathbf{A}}_0 = \tilde{\mathbf{A}}_0(D) = (b_{xy})$ as follows:

$$a_{xy} = \begin{cases} w(x,y)/(1-w(x,y)w(y,x)(1-u)^2t^2) & \text{if both } (x,y) \text{ and } (y,x) \in A(D), \\ 0 & \text{otherwise} \end{cases}$$

and

$$b_{xy} = \begin{cases} w(x,y) & \text{if } (x,y) \in A(D) \text{ and } (y,x) \notin A(D), \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, an $n \times n$ matrix $\tilde{\mathbf{D}} = \tilde{\mathbf{D}}(D) = (d_{xy})$ is the diagonal matrix defined by

$$d_{xx} = \sum_{o(e)=x, e^{-1} \in A(D)} \frac{w(e)w(e^{-1})}{1-w(e)w(e^{-1})(1-u)^2t^2}.$$

Let $\mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$ be the block diagonal sum of square matrices $\mathbf{M}_1, \cdots, \mathbf{M}_s$. A new determinant expression for $\zeta(D, w, u, t)$ is given as follows:

Theorem 6. *Let D be a connected digraph, and let $\mathbf{W} = \mathbf{W}(D)$ be a weighted matrix of D. Then the reciprocal of the weighted Bartholdi zeta function of D is given by*

$$\zeta(D,w,u,t)^{-1} = \det(\mathbf{I}_n + (1-u)t^2\tilde{\mathbf{D}} - t\tilde{\mathbf{A}}_1 - t\tilde{\mathbf{A}}_0) \prod_{i=1}^{m_1} (1-w(f_i)w(f_i^{-1})(1-u)^2t^2),$$

where $n = \mid V(D) \mid$, $m = \mid A(D) \mid$ and $f_1^{\pm 1}, \ldots, f_{m_1}^{\pm 1}$ are symmetric arcs of D.

Proof. Let $V(D) = \{v_1, \cdots, v_n\}$ and, let $A(D) = \{e_1, \cdots, e_{m_0}, f_1, \cdots, f_{m_1}, f_1^{-1}, \cdots, f_{m_1}^{-1}\}$ such that $e_i^{-1} \notin A(D) (1 \leqslant i \leqslant m_0)$. Note that $m = m_0 + 2m_1$.

Arrange arcs of D as follows:

$$e_1, \cdots, e_{m_0}, f_1, f_1^{-1}, \cdots, f_{m_1}, f_{m_1}^{-1}.$$

Let

$$\mathbf{U} = \begin{bmatrix} w(e_1) & & & & & 0 \\ & \ddots & & & & \\ & & w(e_{m_0}) & & & \\ & & & w(f_1) & & \\ & & & & w(f_1^{-1}) & \\ 0 & & & & & \ddots \end{bmatrix}.$$

Then we have

$$\mathbf{U}\mathbf{B} = \mathbf{B}_w \textit{ and } \mathbf{U}\mathbf{J}_0 = \mathbf{J}_w.$$

Thus,

$$\mathbf{B}_w - (1-u)\mathbf{J}_w = \mathbf{U}(\mathbf{B} - (1-u)\mathbf{J}_0).$$

By Theorem 2, it follows that

$$\zeta(D, w, u, t)^{-1} = \det(\mathbf{I}_m - t\mathbf{U}(\mathbf{B} - (1-u)\mathbf{J}_0)).$$

Now, let $\mathbf{K} = (\mathbf{K}_{ev})\ _{e \in A(D); v \in V(D)}$ be the $m \times n$ matrix defined as follows:

$$\mathbf{K}_{ev} := \begin{cases} 1 & \text{if } o(e) = v, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we define the $m \times n$ matrix $\mathbf{L} = (\mathbf{L}_{ev})_{e \in A(D); v \in V(D)}$ as follows:

$$\mathbf{L}_{ev} := \begin{cases} 1 & \text{if } t(e) = v, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\mathbf{L}^t \mathbf{K} = \mathbf{B}.$$

Thus,

$$\begin{aligned} & \det(\mathbf{I}_m - t\mathbf{U}(\mathbf{B} - (1-u)\mathbf{J}_0)) \\ = \ & \det(\mathbf{I}_m - t\mathbf{U}(\mathbf{L}^t\mathbf{K} - (1-u)\mathbf{J}_0)) = \det(\mathbf{I}_m - t\mathbf{U}\mathbf{L}^t\mathbf{K} + (1-u)t\mathbf{U}\mathbf{J}_0). \end{aligned}$$

But, we have

$$\mathbf{I}_m + (1-u)t\mathbf{U}\mathbf{J}_0 = \mathbf{I}_{m_0} \oplus (\oplus_{j=1}^{m_1} \begin{bmatrix} 1 & (1-u)tw(f_j) \\ (1-u)tw(f_j^{-1}) & 1 \end{bmatrix}). \tag{1}$$

Since $|u|, |t|$ are sufficiently small, we have

$$\det(\begin{bmatrix} 1 & (1-u)tw(f_j) \\ (1-u)tw(f_j^{-1}) & 1 \end{bmatrix}) = 1 - (1-u)^2 t^2 w(f_j) w(f_j^{-1}) \neq 0 \ (1 \leqslant j \leqslant m_1).$$

Thus, $\mathbf{I}_m + (1-u)t\mathbf{U}\mathbf{J}_0$ is invertible. Therefore,

$$\begin{aligned} & \det(\mathbf{I}_m - t\mathbf{U}(\mathbf{B} - (1-u)\mathbf{J}_0)) \\ = \ & \det(\mathbf{I}_m - t\mathbf{U}\mathbf{L}^t\mathbf{K}(\mathbf{I}_m + (1-u)t\mathbf{U}\mathbf{J}_0)^{-1}) \det(\mathbf{I}_m + (1-u)t\mathbf{U}\mathbf{J}_0). \end{aligned}$$

But, if $\mathbf{A}$ and $\mathbf{B}$ are a $m \times n$ and $n \times m$ matrices, respectively, then we have

$$\det(\mathbf{I}_m - \mathbf{A}\mathbf{B}) = \det(\mathbf{I}_n - \mathbf{B}\mathbf{A}). \tag{2}$$

Thus, we have

$$\begin{aligned} & \det(\mathbf{I}_m - t\mathbf{U}(\mathbf{B} - (1-u)\mathbf{J}_0)) \\ = \ & \det(\mathbf{I}_n - t\ {}^t\mathbf{K}(\mathbf{I}_m + (1-u)t\mathbf{U}\mathbf{J}_0)^{-1}\mathbf{U}\mathbf{L}) \det(\mathbf{I}_m + (1-u)t\mathbf{U}\mathbf{J}_0). \end{aligned}$$

Next, we have

$$\det(\mathbf{I}_m + (1-u)t\mathbf{U}\mathbf{J}_0) = \prod_{i=1}^{m_1}(1 - w(f_i)w(f_i^{-1})(1-u)^2t^2).$$

Furthermore, the $m \times n$ matrix $\mathbf{UL} = (c_{ev})_{e \in A(D); v \in V(D)}$ is given as follows:

$$c_{ev} := \begin{cases} w(e) & \text{if } t(e) = v, \\ 0 & \text{otherwise.} \end{cases}$$

But, we have

$$(\mathbf{I}_m + (1-u)t\mathbf{U}\mathbf{J}_0)^{-1} = \mathbf{I}_{m_0} \oplus (\oplus_{j=1}^{m_1} \begin{bmatrix} 1/x_j & -(1-u)tw(f_j)/x_j \\ -(1-u)tw(f_j^{-1})/x_j & 1/x_j \end{bmatrix}),$$

where $x_i = 1 - w(f_i)w(f_i^{-1})(1-u)^2t^2$ $(1 \leqslant i \leqslant m_1)$.

Now, for a symmetric arc $(x,y) \in A(D)$,

$$({}^t\mathbf{K}(\mathbf{I}_m + (1-u)t\mathbf{U}\mathbf{J}_0)^{-1}\mathbf{UL})_{xy} = w(x,y)/(1 - w(x,y)w(y,x)(1-u)^2t^2).$$

For a nonsymmetric arc $(x,y) \in A(D)$,

$$({}^t\mathbf{K}(\mathbf{I}_m + (1-u)t\mathbf{U}\mathbf{J}_0)^{-1}\mathbf{UL})_{xy} = w(x,y).$$

Furthermore, if $x = y$, then

$$({}^t\mathbf{K}(\mathbf{I}_m + (1-u)t\mathbf{U}\mathbf{J}_0)^{-1}\mathbf{UL})_{xx} = -\sum_{o(e)=x, e^{-1} \in A(D)} \frac{(1-u)tw(e)w(e^{-1})}{1 - w(e)w(e^{-1})(1-u)^2t^2}.$$

Thus,

$$\det(\mathbf{I}_n - t\,{}^t\mathbf{K}(\mathbf{I}_m + (1-u)t\mathbf{U}\mathbf{J}_0)^{-1}\mathbf{UL}) = \det(\mathbf{I}_n + (1-u)t^2\tilde{\mathbf{D}} - t\tilde{\mathbf{A}}_1 - t\tilde{\mathbf{A}}_0).$$

Therefore, it follows that

$$\zeta(D,w,u,t)^{-1} = \det(\mathbf{I}_n + (1-u)t^2\tilde{\mathbf{D}} - t\tilde{\mathbf{A}}_1 - t\tilde{\mathbf{A}}_0)\prod_{i=1}^{m_1}(1 - w(f_i)w(f_i^{-1})(1-u)^2t^2).$$

□

By Theorem 5, we obtain the second identity of Theorem 2.

Corollary 1 (Choe, Kwak, Park and Sato)**.** *Let D be a connected digraph, and let $\mathbf{W} = \mathbf{W}(D)$ be a weighted matrix of D. Furthermore, assume that $w(e^{-1}) = w(e)^{-1}$ for each $e \in A(D)$ such that $e^{-1} \in A(D)$. Then the reciprocal of the weighted Bartholdi zeta function of D is given by*

$$\zeta(D,w,u,t)^{-1} = (1 - (1-u)^2t^2)^{m_1 - n}$$

$$\times \quad \det(\mathbf{I}_n - t\mathbf{W}_1(D) - (1 - (1-u)^2t^2)t\mathbf{W}_0(D) + (1-u)t^2(\mathbf{S} - (1-u)\mathbf{I}_n)).$$

where $n =\mid V(D) \mid$ and $m =\mid A(D) \mid$.

Proof. Since $w(e^{-1}) = w(e)^{-1}$ for each symmetric arc $e \in A(D)$, we have $w(e^{-1})$ $w(e)^{-1} = 1$. Then we have

$$\tilde{\mathbf{D}} = \frac{1}{1-(1-u)^2t^2}\mathbf{S},\ \tilde{\mathbf{A}}_1 = \frac{1}{1-(1-u)^2t^2}\mathbf{W}_1(D).$$

Furthermore, $\tilde{\mathbf{A}}_0 = \mathbf{W}_0(D)$. Thus,

$$\begin{aligned}\zeta(D,w,u,t)^{-1} &= (1-(1-u)^2t^2)^{m_1}\det\Big(\mathbf{I}_n - t/(1-(1-u)^2t^2)\mathbf{W}_1(D) \\ &\quad - t\mathbf{W}_0(D) + (1-u)t^2/(1-(1-u)^2t^2)\mathbf{S}\Big) \\ &= (1-(1-u)^2t^2)^{m_1-n}\det\Big(\mathbf{I}_n - t\mathbf{W}_1(D) - (1-(1-u)^2t^2)t\mathbf{W}_0(D) \\ &\quad + (1-u)t^2(\mathbf{S}-(1-u)\mathbf{I}_n)\Big).\end{aligned}$$

□

4 Weighted Bartholdi zeta functions of group coverings of digraphs

We can generalize the notion of a Γ-covering of a graph to a simple digraph. Let D be a connected digraph and Γ a finite group. Then a mapping $\alpha : A(D) \longrightarrow \Gamma$ is called a *pseudo ordinary voltage assignment* if $\alpha(v,u) = \alpha(u,v)^{-1}$ for each $(u,v) \in A(D)$ such that $(v,u) \in A(D)$. The pair (D,α) is called an *ordinary voltage digraph*. The *derived digraph* D^α of the ordinary voltage digraph (D,α) is defined as follows: $V(D^\alpha) = V(D) \times \Gamma$ and $((u,h),(v,k)) \in A(D^\alpha)$ if and only if $(u,v) \in A(D)$ and $k = h\alpha(u,v)$. The digraph D^α is called a Γ-*covering* of D. Note that a Γ-covering of the symmetric digraph corresponding to a graph G is a Γ-covering of G(see [5]).

Let D be a connected digraph, Γ a finite group and $\alpha : A(D) \longrightarrow \Gamma$ a pseudo ordinary voltage assignment. In the Γ-covering D^α, set $v_g = (v,g)$ and $e_g = (e,g)$, where $v \in V(D), e \in A(D), g \in \Gamma$. For $e = (u,v) \in A(D)$, the arc e_g emanates from u_g and terminates at $v_{g\alpha(e)}$.

Let $\mathbf{W} = \mathbf{W}(D)$ be a weighted matrix of D. Then we define the *weighted matrix* $\tilde{\mathbf{W}} = \mathbf{W}(D^\alpha) = (\tilde{w}(u_g,v_h))$ of D^α *derived from* $\mathbf{W}$ as follows:

$$\tilde{w}(u_g,v_h) := \begin{cases} w(u,v) & \text{if } (u,v) \in A(D) \text{ and } h = g\alpha(u,v), \\ 0 & \text{otherwise.}\end{cases}$$

If $\mathbf{M}_1 = \mathbf{M}_2 = \cdots = \mathbf{M}_s = \mathbf{M}$, then we write $s \circ \mathbf{M} = \mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$. The *Kronecker product* $\mathbf{A} \bigotimes \mathbf{B}$ of matrices $\mathbf{A}$ and $\mathbf{B}$ is considered as the matrix $\mathbf{A}$ having the element a_{ij} replaced by the matrix $a_{ij}\mathbf{B}$.

Theorem 7. *Let D be a connected digraph with n vertices and m arcs, Γ a finite group, $\alpha : A(D) \longrightarrow \Gamma$ a pseudo ordinary voltage assignment and $\mathbf{W} = \mathbf{W}(D)$ a weighted* *matrix of D. Set $m_1 = \mid \{e \in A(D) \mid e^{-1} \in A(D)\} \mid /2$ and $\mid \Gamma \mid = r$. Furthermore, let $\rho_1 = 1, \rho_2, \cdots, \rho_k$ be the irreducible representations of Γ, and d_i the degree of ρ_i for each i, where $d_1 = 1$. For $g \in \Gamma$, the matrix $\mathbf{A}_{1,g} = (a^{(g)}_{xy})$ is defined as follows:*

$$a^{(g)}_{xy} := \begin{cases} w(x,y)/(1 - w(x,y)w(y,x)(1-u)^2t^2) & \textit{if } (x,y),(y,x) \in A(D) \textit{ and } \alpha(x,y) = g, \\ 0 & \textit{otherwise.} \end{cases}$$

Furthermore, the matrix $\mathbf{A}_{0,g} = (b^{(g)}_{xy})$ is defined as follows:

$$b^{(g)}_{xy} := \begin{cases} w(x,y) & \textit{if } (x,y) \in A(D), (y,x) \notin A(D) \textit{ and } \alpha(x,y) = g, \\ 0 & \textit{otherwise.} \end{cases}$$

Suppose that the Γ-covering D^α of D is connected. Then the reciprocal of the weighted Bartholdi zeta function of D^α is

$$\zeta(D^\alpha, \tilde{w}, u, t)^{-1} = \prod_{i=1}^{m_1} (1 - w(f_i)w(f_i^{-1})(1-u)^2t^2)^r$$

$$\times \prod_{i=1}^{k} \{\det(\mathbf{I}_{nd_i} - t\sum_{h\in\Gamma} \rho_i(h) \bigotimes \mathbf{A}_{1,h} - t\sum_{h\in\Gamma} \rho_i(h) \bigotimes \mathbf{A}_{0,h} + (1-u)t^2(\mathbf{I}_{d_i} \bigotimes \tilde{\mathbf{D}}(D)))\}^{d_i},$$

where $f_1^{\pm 1}, \ldots, f_{m_1}^{\pm 1}$ are symmetric arcs of D.

Proof . Let $V(D) = \{v_1, \cdots, v_n\}$ and $\Gamma = \{1 = g_1, g_2, \cdots, g_r\}$. Arrange vertices of D^α in n blocks: $(v_1, 1), \cdots, (v_n, 1); (v_1, g_2), \cdots, (v_n, g_2); \cdots; (v_1, g_r), \cdots, (v_n, g_r)$. We consider the three matrices $\tilde{\mathbf{A}}_1(D^\alpha)$, $\tilde{\mathbf{W}}_0(D^\alpha)$ and $\tilde{\mathbf{D}}(D^\alpha)$ under this order. By Theorem 5, we have

$$\zeta(D^\alpha, \tilde{w}, u, t)^{-1} = \det(\mathbf{I}_{\nu m} - t\tilde{\mathbf{A}}_1(D^\alpha) - t\tilde{\mathbf{A}}_0(D^\alpha) + (1-u)t^2\tilde{\mathbf{D}}(D^\alpha))$$

$$\cdot \prod_{i=1}^{m_1} (1 - w(f_i)w(f_i^{-1})(1-u)^2t^2)^r.$$

For $h \in \Gamma$, the matrix $\mathbf{P}_h = (p^{(h)}_{ij})$ is defined as follows:

$$p^{(h)}_{ij} = \begin{cases} 1 & \text{if } g_i h = g_j, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $p^{(h)}_{ij} = 1$, i.e., $g_j = g_i h$. Then $((u, g_i), (v, g_j)) \in A(D^\alpha)$ if and only if $(u, v) \in A(D)$ and $g_j = g_i \alpha(u, v)$, i.e., $\alpha(u, v) = g_i^{-1} g_j = g_i^{-1} g_i h = h$. Thus we have

$$\tilde{\mathbf{A}}_0(D^\alpha) = \sum_{h\in\Gamma} \mathbf{P}_h \bigotimes \mathbf{A}_{0,h} \textit{ and } \tilde{\mathbf{A}}_1(D^\alpha) = \sum_{h\in\Gamma} \mathbf{P}_h \bigotimes \mathbf{A}_{1,h}.$$

Let ρ be the right regular representation of Γ. Furthermore, let $\rho_1 = 1, \rho_2, \cdots, \rho_k$ be the irreducible representations of Γ, and d_i the degree of ρ_i for each i, where $d_1 = 1$. Then we have $\rho(h) = \mathbf{P}_h$ for $h \in \Gamma$. Furthermore, there exists a nonsingular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\rho(h)\mathbf{P} = (1) \oplus d_2 \circ \rho_2(h) \oplus \cdots \oplus d_k \circ \rho_k(h)$ for each $h \in \Gamma$(see [10]). Putting $\mathbf{B} = (\mathbf{P}^{-1} \bigotimes \mathbf{I}_n)(\tilde{\mathbf{A}}_1(D^\alpha) + \tilde{\mathbf{A}}_0(D^\alpha))(\mathbf{P} \bigotimes \mathbf{I}_n)$, we have

$$\mathbf{B} = \sum_{h\in\Gamma} \{(1) \oplus d_2 \circ \rho_2(h) \oplus \cdots \oplus d_k \circ \rho_k(h)\} \bigotimes (\mathbf{A}_{1,h} + \mathbf{A}_{0,h}).$$

Note that $\tilde{\mathbf{A}}_i(D) = \sum_{h\in\Gamma} \mathbf{A}_{i,h}$ $(i = 0, 1)$ and $1 + d_2^2 + \cdots + d_k^2 = r$. Therefore it follows that

$$\zeta(D^\alpha, \tilde{w}, u, t)^{-1} = \prod_{j=1}^{m_1} (1 - w(f_j)w(f_j^{-1})(1-u)^2t^2)^r$$

$$\times \prod_{i=1}^{k} \det(\mathbf{I}_{nd_i} - t\sum_{h\in\Gamma} \rho_i(h) \bigotimes \mathbf{A}_{1,h} - t\sum_{h\in\Gamma} \rho_i(h) \bigotimes \mathbf{A}_{0,h} + (1-u)t^2(\mathbf{I}_{d_i} \bigotimes \tilde{\mathbf{D}}(D)))^{d_i}.$$

□

5 L-functions of digraphs

Let D be a connected digraph with m arcs, Γ a finite group, $\alpha : A(D) \longrightarrow \Gamma$ a pseudo ordinary voltage assignment and $\mathbf{W} = \mathbf{W}(D)$ a weighted matrix of D. For each path $P = (e_1, \cdots, e_l)$ of D, set $\alpha(P) = \alpha(e_1) \cdots \alpha(e_l)$ and $w(P) = w(e_1) \cdots w(e_l)$. Furthermore, let ρ be a representation of Γ and d its degree.

The *weighted Bartholdi L-function* of D associated with ρ and α is defined by

$$\zeta_D(w, u, t, \rho, \alpha) = \prod_{[C]} \det(\mathbf{I}_d - w(C)\rho(\alpha(C))u^{cbc(C)}t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of D.

Two $md \times md$ matrices $\mathbf{B}_w^\rho = (\mathbf{B}_{e,f})_{e,f\in A(D)}$ and $\mathbf{J}_w^\rho = (\mathbf{J}_{e,f})_{e,f\in A(D)}$ are defined as follows:

$$\mathbf{B}_{e,f} = \begin{cases} w(e)\rho(\alpha(e)) & \text{if } t(e) = o(f), \\ \mathbf{0}_d & \text{otherwise} \end{cases}, \mathbf{J}_{e,f} = \begin{cases} w(e)\rho(\alpha(e)) & \text{if } f = e^{-1}, \\ \mathbf{0}_d & \text{otherwise.} \end{cases}$$

A determinant expression for the weighted Bartholdi L-function of D associated with ρ and α was given by Choe, Kwak, Park and Sato [3]. Let $1 \leqslant i, j \leqslant n$. Then the (i, j)-*block* $\mathbf{F}_{ij}$ of a $dn \times dn$ matrix $\mathbf{F}$ is the submatrix of $\mathbf{F}$ consisting of $d(i-1)+1, \ldots, di$ rows and $d(j-1)+1, \ldots, dj$ columns.

Theorem 8 (Choe, Kwak, Park and Sato)**.** *Let D be a connected digraph with m arcs, Γ a finite group, $\alpha : A(D) \longrightarrow \Gamma$ a pseudo ordinary voltage assignment and $\mathbf{W} = \mathbf{W}(D)$ a weighted matrix of D. Furthermore, let ρ be a representation of Γ, and d the degree of ρ. Then the reciprocal of the weighted Bartholdi L-function of D associated with ρ and α is*

$$\zeta_D(w, u, t, \rho, \alpha)^{-1} = \det(\mathbf{I}_{md} - (\mathbf{B}_w^\rho - (1-u)\mathbf{J}_w^\rho)t).$$

A new determinant expression for the weighted Bartholdi L-function of D associated with ρ and α is given as follows:

Theorem 9. *Let D be a connected digraph, and let $\mathbf{W} = \mathbf{W}(D)$ be a weighted matrix of D. Then the reciprocal of the weighted Bartholdi L-function of D is given by*

$$\zeta_D(w,u,t,\rho,\alpha)^{-1} = \prod_{i=1}^{m_1}(1 - w(f_i)w(f_i^{-1})(1-u)^2t^2)^d$$

$$\times \det(\mathbf{I}_{nd} + (1-u)t^2\mathbf{I}_d \bigotimes \tilde{\mathbf{D}}(D) - t\sum_{g\in\Gamma}\rho(g)\bigotimes \mathbf{A}_{1,g} - t\sum_{g\in\Gamma}\rho(g)\bigotimes \mathbf{A}_{0,g}),$$

where $n =\mid V(D) \mid$, $m =\mid A(D) \mid$ and $f_1^{\pm 1}, \dots, f_{m_1}^{\pm 1}$ are symmetric arcs of D.

Proof. Let $V(D) = \{v_1, \cdots, v_n\}$ and, let $A(D) = \{e_1, \cdots, e_{m_0}, f_1, \cdots, f_{m_1}, f_1^{-1}, \cdots, f_{m_1}^{-1}\}$ such that $e_i^{-1} \notin A(D) (1 \leqslant i \leqslant m_0)$. Note that $m = m_0 + 2m_1$.

Arrange arcs of D as follows:

$$e_1, \cdots, e_{m_0}, f_1, f_1^{-1}, \cdots, f_{m_1}, f_{m_1}^{-1}.$$

Let

$$\mathbf{U} = \begin{bmatrix} w(e_1) & & & & & 0 \\ & \ddots & & & & \\ & & w(e_{m_0}) & & & \\ & & & w(f_1) & & \\ & & & & w(f_1^{-1}) & \\ & & & & & \ddots \end{bmatrix}.$$

Furthermore, let two $md \times md$ matrices $\mathbf{B}_\rho = (\mathbf{B}^\rho_{e,f})_{e,f\in A(D)}$ and $\mathbf{J}_\rho = (\mathbf{J}^\rho_{e,f})_{e,f\in A(D)}$ be defined as follows:

$$\mathbf{B}^\rho_{e,f} = \begin{cases} \rho(\alpha(e)) & \text{if } t(e) = o(f), \\ \mathbf{0}_d & \text{otherwise} \end{cases}, \mathbf{J}^\rho_{e,f} = \begin{cases} \rho(\alpha(e)) & \text{if } f = e^{-1}, \\ \mathbf{0}_d & \text{otherwise.} \end{cases}$$

Then we have

$$(\mathbf{U}\bigotimes \mathbf{I}_d)\mathbf{B}_\rho = \mathbf{B}^\rho_w \textit{ and } (\mathbf{U}\bigotimes \mathbf{I}_d)\mathbf{J}_\rho = \mathbf{J}^\rho_w.$$

Thus,

$$\mathbf{B}^\rho_w - (1-u)\mathbf{J}^\rho_w = (\mathbf{U}\bigotimes \mathbf{I}_d)(\mathbf{B}_\rho - (1-u)\mathbf{J}_\rho).$$

By Theorem 7, it follows that

$$\zeta_D(w,u,t,\rho,\alpha)^{-1} = \det(\mathbf{I}_{md} - t(\mathbf{U}\bigotimes \mathbf{I}_d)(\mathbf{B}_\rho - (1-u)\mathbf{J}_\rho)).$$

Now, let $\mathbf{K} = (\mathbf{K}_{ev})_{e\in A(D); v\in V(D)}$ be the $md \times nd$ matrix defined as follows:

$$\mathbf{K}_{ev} := \begin{cases} \mathbf{I}_d & \text{if } o(e) = v, \\ \mathbf{0}_d & \text{otherwise.} \end{cases}$$

Furthermore, we define the $md \times nd$ matrix $\mathbf{L} = (\mathbf{L}_{ev})_{e \in A(D); v \in V(D)}$ as follows:

$$\mathbf{L}_{ev} := \begin{cases} \rho(\alpha(e)) & \text{if } t(e) = v, \\ \mathbf{0}_d & \text{otherwise.} \end{cases}$$

Set $\mathbf{U}_d = \mathbf{U} \bigotimes \mathbf{I}_d$. Then we have

$$\mathbf{L}^t \mathbf{K} = \mathbf{B}_\rho.$$

Thus,

$$\begin{aligned} & \det(\mathbf{I}_{md} - t\mathbf{U}_d(\mathbf{B}_\rho - (1-u)\mathbf{J}_\rho)) \\ = \ & \det(\mathbf{I}_{md} - t\mathbf{U}_d(\mathbf{L}^t\mathbf{K} - (1-u)\mathbf{J}_\rho)) = \det(\mathbf{I}_{md} - t\mathbf{U}_d\mathbf{L}^t\mathbf{K} + (1-u)t\mathbf{U}_d\mathbf{J}_\rho). \end{aligned}$$

But, we have

$$\begin{aligned} & \mathbf{I}_{md} + (1-u)t\mathbf{U}_d\mathbf{J}_\rho \\ = \mathbf{I}_{m_0 d} \oplus (\oplus_{j=1}^{m_1} & \begin{bmatrix} \mathbf{I}_d & (1-u)tw(f_j)\rho(\alpha(f_j)) \\ (1-u)tw(f_j^{-1})\rho(\alpha(f_j^{-1})) & \mathbf{I}_d \end{bmatrix}). \end{aligned} \tag{3}$$

Since $|u|, |t|$ are sufficiently small, we have

$$\begin{aligned} & \det(\begin{bmatrix} \mathbf{I}_d & (1-u)tw(f_j)\rho(\alpha(f_j)) \\ (1-u)tw(f_j^{-1})\rho(\alpha(f_j^{-1})) & \mathbf{I}_d \end{bmatrix}) \\ & = (1-(1-u)^2t^2w(f_j)w(f_j^{-1}))^d \neq 0 \ (1 \leqslant j \leqslant m_1). \end{aligned}$$

Thus, $\mathbf{I}_{md} + (1-u)t\mathbf{U}_d\mathbf{J}_\rho$ is invertible. Therefore,

$$\begin{aligned} & \det(\mathbf{I}_{md} - t\mathbf{U}_d(\mathbf{B}_\rho - (1-u)\mathbf{J}_\rho)) \\ = \ & \det(\mathbf{I}_{md} - t\mathbf{U}_d\mathbf{L}^t\mathbf{K}(\mathbf{I}_{md} + (1-u)t\mathbf{U}_d\mathbf{J}_\rho)^{-1}) \det(\mathbf{I}_{md} + (1-u)t\mathbf{U}_d\mathbf{J}_\rho). \end{aligned}$$

By (2), we have

$$\begin{aligned} & \det(\mathbf{I}_{md} - t\mathbf{U}_d(\mathbf{B}_\rho - (1-u)\mathbf{J}_\rho)) \\ = \ & \det(\mathbf{I}_{nd} - t\ {}^t\mathbf{K}(\mathbf{I}_{nd} + (1-u)t\mathbf{U}_d\mathbf{J}_\rho)^{-1}\mathbf{U}_d\mathbf{L}) \det(\mathbf{I}_{md} + (1-u)t\mathbf{U}_d\mathbf{J}_\rho). \end{aligned}$$

Next, we have

$$\det(\mathbf{I}_{md} + (1-u)t\mathbf{U}_d\mathbf{J}_\rho) = \prod_{i=1}^{m_1}(1 - w(f_i)w(f_i^{-1})(1-u)^2t^2)^d.$$

Furthermore, the $md \times nd$ matrix $\mathbf{U}_d\mathbf{L} = (c_{ev})_{e \in A(D); v \in V(D)}$ is given as follows:

$$c_{ev} := \begin{cases} w(e)\rho(\alpha(e)) & \text{if } t(e) = v, \\ 0 & \text{otherwise.} \end{cases}$$

But, we have

$$(\mathbf{I}_{md} + (1-u)t\mathbf{U}_d\mathbf{J}_\rho)^{-1}$$

$$= \mathbf{I}_{m_0d} \oplus (\oplus_{j=1}^{m_1} \begin{bmatrix} 1/x_j\mathbf{I}_d & -(1-u)tw(f_j)/x_j\rho(\alpha(f_j)) \\ -(1-u)tw(f_j^{-1})/x_j\rho(\alpha(f_j^{-1})) & 1/x_j\mathbf{I}_d \end{bmatrix}).$$

where $x_i = 1 - w(f_i)w(f_i^{-1})(1-u)^2t^2$ $(1 \leqslant i \leqslant m_1)$.

But, for a symmetric arc $(x,y) \in A(D)$,

$$({}^t\mathbf{K}(\mathbf{I}_{md} + (1-u)t\mathbf{U}_d\mathbf{J}_\rho)^{-1}\mathbf{U}_d\mathbf{L})_{xy} = w(x,y)/(1-w(x,y)w(y,x)(1-u)^2t^2)\rho(\alpha(x,y)).$$

For a nonsymmetric arc $(x,y) \in A(D)$,

$$({}^t\mathbf{K}(\mathbf{I}_{md} + (1-u)t\mathbf{U}_d\mathbf{J}_\rho)^{-1}\mathbf{U}_d\mathbf{L})_{xy} = w(x,y)\rho(\alpha(x,y)).$$

Furthermore, if $x = y$, then

$$({}^t\mathbf{K}(\mathbf{I}_{md} + (1-u)t\mathbf{U}_d\mathbf{J}_\rho)^{-1}\mathbf{U}_d\mathbf{L})_{xx} = -\sum_{o(e)=x, e^{-1}\in A(D)} \frac{(1-u)tw(e)w(e^{-1})}{1-w(e)w(e^{-1})(1-u)^2t^2}\mathbf{I}_d.$$

Thus,

$$\det(\mathbf{I}_{nd} - t\,{}^t\mathbf{K}(\mathbf{I}_{nd} + (1-u)t\mathbf{U}_d\mathbf{J}_\rho)^{-1}\mathbf{U}_d\mathbf{L})$$

$$= \det(\mathbf{I}_{nd} + (1-u)t^2\tilde{\mathbf{D}}(D) \bigotimes \mathbf{I}_d - t\sum_{g\in\Gamma}\mathbf{A}_{1,g} \bigotimes \rho(g) - t\sum_{g\in\Gamma}\mathbf{A}_{0,g} \bigotimes \rho(g)),$$

Therefore, it follows that

$$\zeta_D(w,u,t,\rho,\alpha)^{-1} = \prod_{i=1}^{m_1}(1-w(f_i)w(f_i^{-1})(1-u)^2t^2)^d$$

$$\times \det(\mathbf{I}_{nd} + (1-u)t^2\mathbf{I}_d \bigotimes \tilde{\mathbf{D}}(D) - t\sum_{g\in\Gamma}\rho(g) \bigotimes \mathbf{A}_{1,g} - t\sum_{g\in\Gamma}\rho(g) \bigotimes \mathbf{A}_{0,g}),$$

□

By Theorems 6,8, the following result holds.

Corollary 2 (Choe, Kwak, Park and Sato)**.** *Let D be a connected digraph, Γ a finite group, $\alpha : A(D) \longrightarrow \Gamma$ a pseudo ordinary voltage assignment and $\mathbf{W} = \mathbf{W}(D)$ a weighted matrix of D. Then we have*

$$\zeta(D^\alpha, \tilde{w}, u, t) = \prod_\rho \zeta_D(w,u,t,\rho,\alpha)^{\deg\rho},$$

where ρ runs over all inequivalent irreducible representations of Γ.

6 Bartholdi edge zeta function of a digraph

Let D be a connected digraph with m arcs $e_1, \ldots, e_m$. Furthermore, let $z_1, \ldots z_m$ be m variables. Set $z_{e_i} = z_i (1 \leqslant i \leqslant m)$ and $\mathbf{z} = (z_1, \ldots, z_m)$. Then the *Bartholdi edge zeta function* $\zeta(D, w, u)$ of D is defined by

$$\zeta(D, \mathbf{z}, u) = \prod_{[C]} (1 - g(C) u^{cbc(C)})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of D. If $D = D_G$ is the symmetric digraph of a graph G, then the Bartholdi edge zeta function $\zeta(D_G, \mathbf{z}, u)$ of D_G is called the *Bartholdi edge zeta function* $\zeta(G, \mathbf{z}, u)$ of G.

Now, set $|V(D)| = n$. Then we define an $n \times n$ matrix $\mathbf{A}'_1 = \mathbf{A}'_1(D) = (a_{xy})$ as follows:

$$a_{xy} = \begin{cases} z_{(x,y)}/(1 - z_{(x,y)} z_{(y,x)} (1-u)^2) & \text{if both } (x,y) \text{ and } (y,x) \in A(D), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, an $n \times n$ matrix $\mathbf{D}' = \mathbf{D}'(D) = (d_{xy})$ is the diagonal matrix defined by

$$d_{xx} = \sum_{o(e)=x, e^{-1} \in A(D)} \frac{z_e z_{e^{-1}}}{1 - z_e z_{e^{-1}} (1-u)^2}.$$

Substituting $t = 1$ in Theorem 5, we obtain the following result.

Corollary 3. *Let D be a connected digraph with m arcs and let $\mathbf{z} = (z_1, \ldots, z_m)$ be m variables. Then the reciprocal of the Bartholdi edge zeta function of D is given by*

$$\zeta(D, \mathbf{z}, u)^{-1} = \det(\mathbf{I}_n + (1-u)\mathbf{D}' - \mathbf{A}'_1(D) - \tilde{\mathbf{A}}_0) \prod_{i=1}^{m_1} (1 - z_{f_i} z_{f_i^{-1}} (1-u)^2),$$

where $n =| V(D) |$ and $f_1^{\pm 1}, \ldots, f_{m_1}^{\pm 1}$ are symmetric arcs of D.

If $D = D_G$, then

Corollary 4. *Let G be a connected graph with m edges and let $\mathbf{z} = (z_1, \ldots, z_{2m})$ be $2m$ variables.and let $\mathbf{W} = \mathbf{W}(G)$ be a weighted matrix of G. Then the reciprocal of the Bartholdi edge zeta function of G is given by*

$$\zeta(G, \mathbf{z}, u)^{-1} = \det(\mathbf{I}_n + (1-u)\mathbf{D}' - \mathbf{A}'_1(G) - \tilde{\mathbf{A}}_0) \prod_{i=1}^{m} (1 - z_{f_i} z_{f_i^{-1}} (1-u)^2),$$

where $n =| V(G) |$ and $D(G) = \{f_1^{\pm 1}, \ldots, f_m^{\pm 1}\}$.

7 Example

Finally, we give an example. Let D be the digraph with three vertices v_1, v_2, v_3 and five arcs (v_1, v_2), $(v_2, v_1), (v_2, v_3), (v_3, v_2), (v_3, v_1)$. Furthermore, let

$$\mathbf{W}(D) = \begin{bmatrix} 0 & a & 0 \\ b & 0 & c \\ d & e & 0 \end{bmatrix}.$$

Then we have $n = 3, m = 5, m_1 = 2$. By Theorem 5, we have

$$\begin{aligned} & \zeta(D, w, u, t)^{-1} \\ = \; & (1 - ab(1-u)^2t^2)(1 - ce(1-u)^2t^2) \det(\mathbf{I}_3 - t\tilde{\mathbf{A}}_1 - t\tilde{\mathbf{A}}_0 + (1-u)t^2\tilde{\mathbf{D}}) \\ = \; & AB \det\left(\begin{bmatrix} 1 + abF/A & -at/A & 0 \\ -bt/A & 1 + abF/A + ceF/B & -ct/B \\ -dt & -et/B & 1 + ceF/B \end{bmatrix} \right) \\ = \; & 1 - (ab + ce)u^2t^2 + abce(u^4 - u^2)t^4 - acdt^3, \end{aligned}$$

where $A = 1 - ab(1-u)^2t^2$, $B = 1 - ce(1-u)^2t^2$ and $F = (1-u)t^2$.

Let $\Gamma = Z_3 = \{1, \tau, \tau^2\}(\tau^3 = 1)$ be the cyclic group of order 3, and let $\alpha : A(D) \longrightarrow Z_3$ be the pseudo ordinary voltage assignment such that $\alpha(v_1, v_2) = \tau$, $\alpha(v_2, v_1) = \tau^2$ and $\alpha(v_2, v_3) = \alpha(v_3, v_2) = \alpha(v_3, v_1) = 1$. The characters of $\mathbf{Z}_3$ are given as follows: $\chi_i(\tau^j) = (\xi^i)^j$, $0 \leqslant i, j \leqslant 2$, where $\xi = \frac{-1+\sqrt{-3}}{2}$.

Now, we present the weighted Bartholdi L-function $\zeta_D(w, u, t, \chi_1, \alpha)$ of D associated with χ_1 and α. Theorem 8 implies that

$$\begin{aligned} \zeta_D(w, u, t, \chi_1, \alpha)^{-1} &= AB \det(\mathbf{I}_3 - t\sum_{i=0}^{2} \chi_1(\tau^i)\mathbf{A}_{1,\tau^i} - t\sum_{i=0}^{2} \chi_1(\tau^i)\mathbf{A}_{0,\tau^i} + (1-u)t^2\tilde{\mathbf{D}}) \\ &= AB \det\left(\begin{bmatrix} 1 + abF/A & -at\xi/A & 0 \\ -bt\xi^2/A & 1 + abF/A + ceF/B & -ct/B \\ -dt & -et/B & 1 + ceF/B \end{bmatrix} \right) \\ &= 1 - (ab + ce)u^2t^2 + abce(u^4 - u^2)t^4 - acdt^3\xi. \end{aligned}$$

Similarly, we have

$$\zeta_D(w, u, t, \chi_2, \alpha)^{-1} = 1 - (ab + ce)u^2t^2 + abce(u^4 - u^2)t^4 - acdt^3\xi^2.$$

By Corollary 2, it follows that

$$\begin{aligned} \zeta(D^\alpha, \tilde{w}, u, t)^{-1} &= \zeta(D, w, u, t)^{-1}\zeta_D(w, u, t, \chi_1, \alpha)^{-1}\zeta_D(w, u, t, \chi_2, \alpha)^{-1} \\ &= (1 - (ab + ce)u^2t^2 + abce(u^4 - u^2)t^4)^3 - a^3c^3d^3t^9. \end{aligned}$$

If $w(e^{-1}) = w(e)^{-1}$ for each symmetric arc $e \in A(D)$, then

$$\zeta(D, w, u, t)^{-1} = 1 - 2u^2t^2 + (u^4 - u^2)t^4 - acdt^3,$$

$$\zeta_D(w, u, t, \chi_i, \alpha)^{-1} = 1 - 2u^2t^2 + (u^4 - u^2)t^4 - acdt^3\xi^i \ (i = 1, 2)$$

and

$$\zeta(D^\alpha, \tilde{w}, u, t)^{-1} = (1 - 2u^2t^2 + (u^4 - u^2)t^4)^3 - a^3c^3d^3t^9.$$

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