Generalized pattern frequency
in large permutations

Joshua Cooper ∗
Department of Mathematics
University of South Carolina
Columbia, SC, U.S.A.
cooper@math.sc.edu

Erik Lundberg
Department of Mathematics
Purdue University
West Lafayette, IN, U.S.A.
elundber@math.purdue.edu

Brendan Nagle †
Department of Mathematics and Statistics
University of South Florida
Tampa, FL, U.S.A.
bnagle@usf.edu

Submitted: Sep 7, 2012; Accepted: Jan 29, 2013; Published: Feb 5, 2013
Mathematics Subject Classifications: 05A05, 05A16

Abstract
In the study of permutations, generalized patterns extend classical patterns by
adding the requirement that certain adjacent integers in a pattern must be adjacent
in the permutation.

For any generalized pattern $\pi^*_0$ of length $k$ with $1 \leq b \leq k$ blocks, we prove that
for all $\mu > 0$, there exists $0 < c = c(k, \mu) < 1$ so that whenever $n \geq n_0(k, \mu, c)$, all
but $c^n n!$ many $\pi \in S_n$ admit $(1 \pm \mu) \frac{1}{k!} \binom{n}{b}$ occurrences of $\pi^*_0$. Up to the choice of $c$,
this result is best possible for all $\pi^*_0$ with $k \geq 2$.

We also give a lower bound on avoidance of the generalized pattern 12-34, which

Keywords: generalized patterns; pattern avoidance; Azuma’s inequality; Chern-
off’s inequality; Sharkovsky’s Theorem

1 Introduction

Pattern and generalized pattern avoidance in permutations is a well-studied area (see,
e.g.,[1–5, 7, 8, 10, 11]). Fix $1 \leq k \leq n$ and $\pi_0 \in S_k$ and let $\pi \in S_n$. An occurrence

---

*The first author was partially supported by NSF grant DMS-1001370.
†The third author was partially supported by NSF grant DMS-1001781
of a pattern \( \pi_0 \) in \( \pi \) is a sequence of integers \( 1 \leq \ell_1 < \cdots < \ell_k \leq n \) so that, for all \( 1 \leq i \neq j \leq k \),
\[
\pi(\ell_i) < \pi(\ell_j) \iff \pi_0(i) < \pi_0(j).
\] (1)

In order to define generalized patterns, take a classical pattern \( \pi_0 = (a_1, \ldots, a_k) = (\pi_0(1), \ldots, \pi_0(k)) \), and fix \( \pi_0^q = (a_1, \varepsilon_1, a_2, \varepsilon_2, \ldots, \varepsilon_{k-1}, a_k) \) where, for each \( 1 \leq i \leq k-1 \), \( \varepsilon_i \) is either a dash ‘−’ or the empty string. Then, \( \pi \in S_n \) admits \( \pi_0^q \) as a generalized pattern if it contains an occurrence \( 1 \leq \ell_1 < \cdots < \ell_k \leq n \) of the classical pattern \( \pi_0 \) satisfying that,
\[
\text{whenever } \varepsilon_i \neq \varepsilon_j, \text{ then } \ell_{i+1} = \ell_i + 1.
\] (2)

More explicitly, suppose, for some positive integer sequence \( q = (q_1, \ldots, q_b) \), for which \( q_1 + \cdots + q_b = k \), that
\[
\pi_0^q = \pi_0^q = (a_1, \ldots, a_{q_1}, -, a_{q_1+1}, \ldots, a_{q_1+q_2}, -, \ldots, -, a_{k-q_b+1}, \ldots, a_k)
\] (3)

Then, for some integers \( 1 \leq \ell_1 < \cdots < \ell_b \leq n \),
\[
(\ell_1, \ldots, \ell_k) = (\ell_1, \ldots, \ell_1 + q_1 - 1, \ell_2, \ldots, \ell_2 + q_2 - 1, \cdots, \ell_b, \ldots, \ell_b + q_b - 1)
\] (4)

We shall refer to the subsequences \( A_1, \ldots, A_b \) and \( L_1, \ldots, L_b \) as blocks.

As an illustrative example, we note that the permutation \((3, 5, 2, 4, 1) = 35241\) contains the classical pattern 132 (realized uniquely by the 3, 5, and 4 occurring in that order). However, 35241 does not contain the generalized pattern 1-32, since the 5 and 4 are not adjacent.

Let \( f_{\pi_0}^{\pi} \) denote the frequency of the generalized pattern \( \pi_0^q \) in \( \pi \), and set \( F_{\pi_0}(\pi) = f_{\pi_0}^{\pi} \) in the case that \( q = (1, \ldots, 1) \) (i.e., classical patterns). In this notation, the celebrated result of Marcus and Tardos [13] (cf. Klazar [11]) asserts \( F_{\pi_0}(\pi) \geq 1 \) for all but \( C^n \) permutations \( \pi \in S_n \), where \( C = C(\pi_0) > 1 \) and \( n \) is sufficiently large. The first author [7] proved that \( F_{\pi_0} \) is concentrated about its mean: \( F_{\pi_0}(\pi) = (1 \pm o(1)) \frac{1}{C} \binom{n}{k} \) for all but \( o(n!) \) permutations \( \pi \in S_n \). Our main result shows, more generally, that \( f_{\pi_0}^{\pi} \) is also concentrated about its mean, and we provide a sharp estimate for the error \( o(n!) \) of concentration.

**Theorem 1.** For every \( k \geq 1 \) and for all \( \mu > 0 \), there exists \( 0 < c < 1 \) so that, for all sufficiently large integers \( n \), the following holds. For every \( \pi_0 \in S_k \) and for every sequence \( \pi_0^q \) with \( b \) blocks as in (3), all but \( c^n n! \) many \( \pi \in S_n \) satisfy \( f_{\pi_0}(\pi) = (1 \pm \mu) \frac{1}{C} \frac{\binom{n}{k}}{k^b} \).

**Remark 2.** In Section 2, we offer two proofs of Theorem 1. The first, based on martingales, is fairly short. The second gives more detail, using a ‘quasi-random’ property (see Lemma 6) typical of random permutations. Lemma 6 extends some results from [7] and may be of independent interest.

Up to the choice of \( 0 < c < 1 \), Theorem 1 is best possible for all \( \pi_0^{\pi} \) with \( k \geq 2 \). In particular, we prove the following result.
Proposition 3. Fix $k \geq 2$, $b \geq 1$, and $\pi_0 = (a_1, \ldots, a_k) \in S_k$. Let $\pi^*_0$ be any sequence, as in (3), with $b$ blocks. Then, there exists $0 < \gamma_0 < 1$ so that, for all $0 < \gamma < \gamma_0$, there exist infinitely many integers $n$ for which at least $\gamma^n n!$ permutations $\pi \in S_n$ satisfy $f_{\pi^*_0}(\pi) < \gamma^n b$.

We prove Proposition 3 in Section 3.

Proposition 3 can often be strengthened. Indeed, S. Elizalde [8] proved the following strong and quite general result (in [8], see Proposition 4.3).

Theorem 4 (Elizalde [8]). Let $\pi^*_0$ be a sequence, as in (3), having a block $A_i$ of length at least 3. Then, there exists $0 < c < 1$ so that for all $n \geq \frac{k}{2}$, at least $c^n n!$ permutations $\pi \in S_n$ satisfy $f_{\pi^*_0}(\pi) = 0$.

Elizalde [8] also considered to what extent Theorem 4 can be extended to sequences $\pi^*_0$ whose every block has length at most two. He showed that, in general, Theorem 4 can’t be extended to every such $\pi^*_0$. To describe these results, let $A_n(\pi^*_0)$ denote the set of permutations $\pi \in S_n$ for which $f_{\pi^*_0}(\pi) = 0$, and let $\alpha_n(\pi^*_0) = |A_n(\pi^*_0)|$. For $(1, -, 2, 3, -, 4) = 1-23-4$, Elizalde showed (see Corollary 6.2 in [8])

$$\lim_{n \to \infty} \left( \frac{\alpha_n(1-23-4)}{n!} \right)^{1/n} = 0.$$  
(5)

He asked (see Section 7 of [8]):

$$\text{does } \lim_{n \to \infty} \left( \frac{\alpha_n(12-34)}{n!} \right)^{1/n} = 0?$$  
(6)

We answer this question in the negative.

Theorem 5. For odd integers $n$,

$$\alpha_n(12-34) \geq \left( \frac{1}{2} - o(1) \right)^n n!.$$  

We prove Theorem 5 in Section 4, and also consider some related problems.

2 Proofs of Theorem 1

For both of the following proofs, fix a positive integer $k$ and fix $\mu > 0$.

2.1 The martingale proof

Let

$$c = \exp \left\{ -\frac{\mu^2}{9k^4k^4} \right\}$$  
(7)
and let \( n \) be a sufficiently large integer wherever needed. Fix \( \pi_0^* \) with \( b \) blocks as in (3). We show that all but \( c^n! \) many \( \pi \in S_n \) satisfy \( f_{\pi_0^*}(\pi) = (1 + \mu) \frac{1}{k!} \binom{n}{b} \).

To that end, let \( \pi \in S_n \) be chosen uniformly at random. We use the ‘exposure process’ to define the following sequence of random variables. Set

\[
X_0 = \mathbb{E}[f_{\pi_0^*}(\pi)], \quad \text{where from (4), we have } (1 - o(1)) \frac{1}{k!} \binom{n}{b} \leq \mathbb{E}[f_{\pi_0^*}(\pi)] \leq \frac{1}{k!} \binom{n}{b}. \tag{8}
\]

For \( r \in [n] = \{1, \ldots, n\} \), let \( \pi_{[r]} \) denote the restriction \( \pi : [r] \rightarrow [n] \). Set

\[
X_r = \mathbb{E} \left[ f_{\pi_0^*}(\pi) \bigg| \pi_{[r]} \right],
\]

so that \( X_n = f_{\pi_0^*}(\pi) \) is the variable we wish to estimate. Then, \( X_0, X_1, \ldots, X_n \) is the Doob martingale for the function \( f_{\pi_0^*} \), to which we will apply Azuma’s inequality.

For that purpose, observe that for each \( 0 \leq r \leq n - 1 \),

\[
|X_{r+1} - X_r| \leq k \left( \frac{n}{b-1} \right). \tag{9}
\]

To see this, note that the element \( r + 1 \) belongs to between zero and \( k \binom{n}{b-1} \) occurrences \((\ell_1,\ldots,\ell_k)\) of \( \pi_0^* \) in \( \pi \). Indeed, if \( \ell_i = r + 1 \) belongs to block \( L_i \) (see (4)), then all of \( L_i \) is determined by \( r + 1 = \ell_i \) and \( q = q(\pi_0^*) \). Thus, it remains to determine \( L_1, \ldots, L_{i-1}, L_{i+1}, \ldots, L_b \), or equivalently, \( \ell_1, \ldots, \ell_{i-1}, \ell_{i+1}, \ldots, \ell_b \), of which there are at most \( \binom{n}{b-1} \).

Applying Azuma’s inequality with \( t = (\mu/2)X_0 \) and using (8) and (9), we have

\[
\mathbb{P}[|X_n - X_0| \geq t] \leq 2 \exp \left\{ -\frac{t^2}{2 \sum_{r=0}^{n-1} (X_{r+1} - X_r)^2} \right\} \leq \exp \left\{ -\frac{\mu^2}{8n k^2 (b-1)^2} (1 - o(1)) \right\}
\]

\[
= \exp \left\{ -\frac{\mu^2 n}{8k!^2 k^2 b^2} (1 - o(1)) \right\} \leq \exp \left\{ -\frac{\mu^2 n}{9k!^2 k^2} (1 - o(1)) \right\} \leq \exp \left\{ -\frac{\mu^2 n}{9k!^2 k^4} \right\} \leq c^n.
\]

Thus, with probability \( 1 - c^n \),

\[
f_{\pi_0^*}(\pi) = \left( 1 + \frac{\mu}{2} \right) \mathbb{E}[f_{\pi_0^*}(\pi)] \overset{(8)}{=} \left( 1 + \frac{\mu}{2} \right) (1 + o(1)) \frac{1}{k!} \binom{n}{b} = \left( 1 + \mu \right) \frac{1}{k!} \binom{n}{b},
\]

as desired.

### 2.2 The quasi-random proof

To present Lemma 6, we need a few concepts. For integers \( n > t \geq j \geq 1 \), define \( I_j = [(j-1)\lfloor n/t \rfloor + 1, j\lfloor n/t \rfloor] \) and \( R = [n] \setminus \bigcup_{j=1}^{t} I_j \). We call \([n]=I_1 \cup I_2 \cup \cdots \cup I_t \cup R\) the \( t \)-partition \( \mathcal{P}_t \) of \([n]\). Now, fix \( \pi \in S_n \), and consider partitions \( \mathcal{P}_s = I_1 \cup \cdots \cup I_s \cup R_s \) and \( \mathcal{P}_t = E_1 \cup \cdots \cup E_t \cup R_t \) of \([n]\), where \( n > t \geq s \geq q \geq 1 \). For a set \( X \), we will write
For $i = (i_1, \ldots, i_q) \in ([s])_q$ and $j \in [t]$, let

$$E_{ij} = \left\{ \hat{\ell} \in E_j : \hat{\ell} + q - 1 \in E_j \text{ and } \pi(\hat{\ell} + m - 1) \in I_{im} \text{ for all } m \in \{1, \ldots, q\} \right\}. \quad (10)$$

For $\zeta > 0$, and $(i, j) \in ([s])_q \times [t]$, we say $\pi \in S_n$ is $(i, j, \zeta, q)$-typical (w.r.t. $(P_s, P_t)$) if

$$|E_{ij}^\zeta(\pi)| \geq (1 - \zeta) \frac{1}{(s)_q} |E_j| = (1 - \zeta) \frac{1}{(s)_q} \left( \frac{n}{t} \right) \quad (11)$$

and say $\pi \in S_n$ is $(\zeta, q)$-typical (w.r.t. $(P_s, P_t)$) if it is $(i, j, \zeta, q)$-typical for all $(i, j) \in ([s])_q \times [t]$.

**Lemma 6.** For all $\zeta > 0$ and integers $q \geq 1$, there exists an integer $s_0$ so that for all integers $s \geq s_0$, there exists an integer $t_0$ so that for all integers $t \geq t_0$, there exists $c_0 > 0$ so that for all sufficiently large integers $n$, all but $\exp\{ -c_0 n \} n!$ permutations $\pi \in S_n$ are $(\zeta, q)$-typical w.r.t. $(P_s, P_t)$.

Lemma 6 follows by a standard (albeit tedious) probabilistic analysis, which we give in Section 5.

To show that Lemma 6 implies Theorem 1, define auxiliary constants $\delta, \zeta > 0$ so that

$$\delta = \frac{\mu}{k!} \quad \text{and} \quad (1 - 2\zeta)^{k+2} > 1 - \delta. \quad (12)$$

For $q \in [k]$, let $s_0(q)$ be the constant guaranteed by Lemma 6. Fix an integer $s$ so that

$$s \geq \max \{s_0(1), \ldots, s_0(k)\} \quad \text{and} \quad \binom{s}{k} \geq \frac{s^k}{k!} (1 - 2\zeta). \quad (13)$$

For $q \in [k]$, let $t_0(q)$ be the constant guaranteed by Lemma 6. Fix an integer $t$ with

$$t \geq \max \{t_0(1), \ldots, t_0(k)\} \quad \text{and so that for all } b \in [k], \quad \binom{t}{b} \geq \frac{t^b}{b!} (1 - 2\zeta). \quad (14)$$

For $q \in \{1, \ldots, k\}$, let $c_0(q) > 0$ be the constant guaranteed by Lemma 6. Define

$$c_0 = \min \{c_0(1), \ldots, c_0(k)\} \quad \text{and} \quad c = \exp\{ -c_0/4 \}. \quad (15)$$

In all that follows, let $n$ be a sufficiently large integer.

Fix a permutation $\pi_0 \in S_k$, and let $\pi_0^q = \pi_0^q = (A_1, -, \ldots, -, A_b)$ be given as in (3) where $q = (q_1, \ldots, q_b)$. Apply Lemma 6 (cf. (13)–(15)) to conclude that all but

$$\left( \exp \{ -c_0(q_1) n \} + \cdots + \exp \{ -c_0(q_b) n \} \right) n! \leq k \exp\{ -c_0 n \} n! \leq \exp \left\{ -\frac{c_0}{2} n \right\} n!$$

permutations $\pi \in S_n$ are $(\zeta, q_x)$-typical w.r.t. $(P_s, P_t)$ for all $x \in [b]$. For such a $\pi \in S_n$, we show

$$f_{s_0^q}(\pi) \geq (1 - \delta) \frac{1}{k!} \binom{n}{b} \quad (12) \quad \left( 1 - \mu \right) \frac{1}{k!} \binom{n}{b}. \quad (16)$$
Indeed, fix indices \(1 \leq i_1 < \cdots < i_k \leq s\) and \(1 \leq j_1 < \cdots < j_b \leq t\). For \(x \in [b]\), recall the block

\[
A_x = (a_{q_1+\cdots+q_{x-1}+1}, \ldots, a_{q_1+\cdots+q_x}) = (\pi_0(q_1 + \cdots + q_{x-1} + 1), \ldots, \pi_0(q_1 + \cdots + q_x))
\]
of \(\pi^q_0\) (cf. (3)). Consider the injection defined by, for each \(x \in [b]\),

\[
j_x \mapsto i_x \overset{\text{def}}{=} (i_a)_{a \in A_x} = \left(\hat{i}_{a_{q_1+\cdots+q_{x-1}+1}}, \ldots, \hat{i}_{a_{q_1+\cdots+q_x}}\right)
\]

(17)

\[
= (i_{\pi_0(q_1+\cdots+q_{x-1}+1)}, \ldots, i_{\pi_0(q_1+\cdots+q_x)}).
\]

(18)

For each \(x \in [b]\), arbitrarily select \(\hat{\ell}_x \in E_{i_xj_x}(\pi)\) (cf. (10)). We claim that the sequence

\[
(L_1, L_2, \ldots, L_b), \quad \text{where for each } x \in [b], \quad L_x = (\hat{\ell}_x, \hat{\ell}_x + 1, \ldots, \hat{\ell}_x + q_x - 1),
\]

(19)

is exactly an occurrence in \(\pi\) of the generalized pattern \(\pi^q_0\). The sequence \((L_1, \ldots, L_b)\) clearly satisfies (2), since each \(L_m\) is consecutive, and since \((L_1, L_2, \ldots, L_b)\) precisely mimics the block structure of \(\pi^q_0 = (A_1, -, A_2, -, \ldots, -, A_b)\) (cf. (3)). It remains to check, therefore, that \((L_1, \ldots, L_b)\) is an occurrence of the classical pattern \(\pi_0\) in \(\pi\), i.e., that \((L_1, \ldots, L_b)\) satisfies (1).

Indeed, rewrite the sequence \((L_1, \ldots, L_b)\) as

\[
(\ell_1, \ldots, \ell_k) = (\ell_1, \ldots, \ell_{q_1}, \ell_{q_1+1}, \ldots, \ell_{q_1+q_2}, \ldots, \ell_{k-q_n+1}, \ldots, \ell_k)
\]

so that for \(x \in [b]\),

\[
L_x = (\ell_{q_1+\cdots+q_{x-1}+1}, \ldots, \ell_{q_1+\cdots+q_x}).
\]

(20)

Comparing (19) and (20), we see that a term of the sequence \((L_1, \ldots, L_b)\) is determined by a choice of indices \(1 \leq x \leq b\) and \(1 \leq w \leq q_x\), and written simultaneously as

\[
\hat{\ell}_x + w - 1 = \ell_{q_1+\cdots+q_{x-1}+w}.
\]

(21)

(Such a term necessarily belongs to the block \(L_x\).) Observe from (10) and (17) that

\[
\pi(\hat{\ell}_x + w - 1) \in I_{i(x,w)}, \quad \text{where } i(x, w) = i_{\pi_0(q_1+\cdots+q_{x-1}+w)}.
\]

(22)

Now, fix two terms (cf. (21)) of the sequence \((L_1, \ldots, L_b)\):

\[
\hat{\ell}_x + w - 1 = \ell_{q_1+\cdots+q_{x-1}+w} \quad \text{and} \quad \hat{\ell}_y + z - 1 = \ell_{q_1+\cdots+q_{y-1}+z},
\]

where \(1 \leq x, y \leq b\), \(1 \leq w \leq q_x\) and \(1 \leq z \leq q_y\). From (22), we conclude

\[
\pi(\ell_{q_1+\cdots+q_{x-1}+w}) < \pi(\ell_{q_1+\cdots+q_{y-1}+z}) \iff \max I_{i(x,w)} < \min I_{i(y,z)}
\]

\[
\iff i(x, w) < i(y, z) \iff \pi_0(q_1+\cdots+q_{x-1}+w) < \pi_0(q_1+\cdots+q_{y-1}+z),
\]

as required by (1). (For the last step, recall the ordering \(1 \leq i_1 < \cdots < i_k \leq s\) of the fixed indices.)
Now, the discussion above implies that

\[
f_{\pi^0_0}(\pi) \geq \sum \sum \left\{ \prod_{x=1}^{b} |E_{k,x}(\pi)| : 1 \leq i_1 < \cdots < i_k \leq s, 1 \leq j_1 < \cdots < j_b \leq t \right\}.
\]

(23)

Since \( \pi \in S_n \) is \((\zeta, q)-\)typical w.r.t. \((P_s, P_t)\) for every \( q \in \{q_1, \ldots, q_b\} \), we have, for each \( x \in [b], \)

\[
|E_{k,x}(\pi)| \geq (1-\zeta) \frac{n}{(s)_q} \left\lceil \frac{n}{t} \right\rceil \geq (1-2\zeta) \frac{n}{t(s)_q} \geq (1-2\zeta) \frac{n}{t s^q e}.
\]

Returning to (23),

\[
\frac{f_{\pi^0_0}(\pi)}{\binom{s}{k} \binom{t}{b}} \geq (1-2\zeta)^b \left( \frac{n}{t} \right)^b \prod_{x=1}^{b} \frac{1}{s^q e} = (1-2\zeta)^b \left( \frac{n}{t} \right)^b \frac{1}{s^{q_1 + \cdots + q_b}}
\]

\[
= (1-2\zeta)^b \left( \frac{n}{t} \right)^b \frac{1}{s^b} \geq (1-2\zeta)^k \left( \frac{n}{t} \right)^b \frac{1}{s^k},
\]

and so (16) follows from

\[
f_{\pi^0_0}(\pi) \geq \binom{s}{k} \binom{t}{b} (1-2\zeta)^k \left( \frac{n}{t} \right)^b \frac{1}{s^k} \geq (1-2\zeta)^{k+2} \frac{1}{k!} \binom{b}{1} \left( \frac{n}{b!} \right)^{(12)} \geq (1-\delta) \frac{1}{k!} \binom{n}{b}.
\]

The corresponding upper bound \( f_{\pi^0_0}(\pi) \leq (1+\mu) \frac{1}{k!} \binom{n}{b} \) follows, in fact, from the lower bound. Indeed, first conclude (16) for every permutation \( p \in S_k \) and \( p^* = p^a \). Thus, all but

\[
k! \exp \left\{ -\frac{c_0}{2} n \right\} n! < \exp \left\{ -\frac{c_0}{4} n \right\} n! = c_1 n!
\]

permutations \( \pi \in S_n \) satisfy, for every \( p \in S_k \), \( f_{p^0}(\pi) \geq (1-\delta) \frac{1}{k!} \binom{n}{b} \). Fix such a \( \pi \in S_n \). Observe that every \( 1 \leq \ell_1 < \cdots < \ell_k \leq n \) of the form in (19) and (20) defines a generalized pattern \( p^a \) of some \( p \in S_k \). (Indeed, if \( \pi(\{\ell_1, \ldots, \ell_k\}) = \{\lambda_1, \ldots, \lambda_k\} \), define \( p(i) = j \) if and only if \( \pi(\ell_i) = \lambda_j \).) Thus,

\[
\binom{n}{b} \geq \sum_{p \in S_k} f_{p^0}(\pi) = f_{\pi^0_0}(\pi) + \sum_{\pi_0 \neq p \in S_k} f_{p^0}(\pi) \geq f_{\pi^0_0}(\pi) + (k!-1)(1-\delta) \frac{1}{k!} \binom{n}{b}
\]

\[
\implies f_{\pi^0_0}(\pi) \leq \left( \frac{1}{k!} + \delta - \frac{\delta}{k!} \right) \binom{n}{b} \leq (1 + \delta k! \frac{1}{k!} \binom{n}{b}) \leq (1 + \mu) \frac{1}{k!} \binom{n}{b}.
\]

3 Proof of Proposition 3

Fix \( k \geq 2, b \geq 1 \), and \( \pi_0 = (a_1, \ldots, a_k) \in S_k \). Fix any sequence \( \pi^0_0 \), as in (3), with \( b \) blocks. If \( \pi^0_0 \) has a block of length at least 3, then let \( 0 < c = c(\pi^0_0) < 1 \) be the constant guaranteed by Theorem 4, and set \( \gamma_0 = c/2 \). Otherwise, set \( \gamma_0 = 1/2 \). Fix \( 0 < \gamma < \gamma_0 \), and write \( g = [1/\gamma] \), where we note that \( \gamma < 1/2 \) implies \( g \geq 2 \). For a sufficiently large
integer \( n \) which is divisible by \( g \), we guarantee at least \( \gamma^n n! \) permutations \( \pi \in S_n \) with \( f_{\pi_0^*}(\pi) < \gamma n^b \).

Our proof is based on cases, depending on the structure of the sequence \( \pi_0^* \). Clearly, we get the following case entirely for free on account of Theorem 4.

**Case 0** (\( \pi_0^* \) has a block of length at least 3). Theorem 4 guarantees at least \( c^n n! > \gamma^n n! \) permutations \( \pi \in S_n \) with \( f_{\pi_0^*}(\pi) = 0 < \gamma n^b \).

To handle all other cases, we require the following considerations. For \( 0 \leq s \leq g - 1 \), write \( I_s = [s(n/g) + 1, (s + 1)(n/g)] \) and \( R_s = \{ m \in [n] : m \equiv s \pmod{g} \} \). Then \([n] = I_0 \cup \cdots \cup I_{g-1} \) and \([n] = R_0 \cup \cdots \cup R_{g-1} \) are partitions of \([n]\) into parts of common size \( n/g \). Consider the following four classes of permutations:

\[
\begin{align*}
S_{n,1} &= \{ \pi \in S_n : \pi(I_s) = I_s, \forall 0 \leq s \leq g - 1 \}, \\
S_{n,2} &= \{ \pi \in S_n : \pi(I_s) = I_{s-1}, \forall 0 \leq s \leq g - 1 \}, \\
S_{n,3} &= \{ \pi \in S_n : \pi(R_s) = R_s, \forall 0 \leq s \leq g - 1 \} \quad \text{(take } I_g = I_0 \text{)}, \\
S_{n,4} &= \{ \pi \in S_n : \pi(R_s) = R_{s-1}, \forall 0 \leq s \leq g - 1 \} \quad \text{(take } I_{g-1} = I_{g-1} \text{)}.
\end{align*}
\]

Clearly, \( |S_{n,1}| = |S_{n,2}| = |S_{n,3}| = |S_{n,4}| = ((n/g)!)^g \), where by Stirling’s formula,

\[
\left( \frac{n}{g} \right)! > \frac{1}{2} \left( \frac{\sqrt{2\pi(n/g)}}{n/e} \right)^{(n/g)} > \frac{\sqrt{\gamma}}{2} \left( 2\pi n \right)^{g-1/2} \times \gamma^n \sqrt{2\pi n} \left( \frac{n}{e} \right)^n > \gamma^n n!.
\]

We mention, in advance, that in the following four classes below, Case \( i \) will be handled by the family \( S_{n,i} \), for \( 1 \leq i \leq 4 \). We also mention that Cases 1 and 2 are not always disjoint from Case 0, nor are they always disjoint from each other. (It seemed easiest to preserve generality in the cases.)

We now consider when \( \pi_0^* \) has \( b \geq 2 \) blocks. In particular, suppose \( a_i = \pi_0(i) \) and \( a_j = \pi_0(j), 1 \leq i < j \leq k \), belong to blocks \( A_i \) and \( A_j \), respectively, where \( A_i \neq A_j \).

**Case 1** (\( b \geq 2, a_i > a_j \)). Fix \( \pi \in S_{n,1} \), and consider an occurrence \( 1 \leq \ell_1 < \cdots < \ell_k \leq n \) of the generalized pattern \( \pi_0^* \) in \( \pi \). Consider the terms \( \ell_i < \ell_j \). From (1), since \( \pi_0(i) = a_i > a_j = \pi_0(j) \), we have \( \pi(\ell_i) > \pi(\ell_j) \). We therefore claim that, for some \( 1 \leq s \leq g \), we have \( \ell_i, \ell_j \in I_s \). Indeed, if \( \ell_i \in I_{s_i} \) and \( \ell_j \in I_{s_j} \) for some \( s_i < s_j \), then \( \pi(\ell_i) < \pi(\ell_j) \) on account of \( \pi \in S_{n,1} \), a contradiction. We also recall from (3) and (4), that \( \ell_i \) belongs to block \( L_i \) and \( \ell_j \) belongs to block \( L_j \) (since \( a_i \) belongs to block \( A_i \) and \( a_j \) belongs to block \( A_j \)). Finally, recall from (4) that \( L_i \) begins with \( \hat{\ell}_i \) and \( L_j \) begins with \( \hat{\ell}_j \). Then, since \( \ell_i, \ell_j \in I_s \), we have that \( \hat{\ell}_i \in I_s \) and \( \hat{\ell}_i \in I_s - I_i \). (If \( \hat{\ell}_i \in I_{s-1} \), it occurs very near the right boundary.) Clearly, there are at most \( |I_s| = n/g \) choices for \( \hat{\ell}_i \). It is easy to check that there are fewer than \( n/g^2 \) choices for \( \hat{\ell}_j \). Clearly, there are at most \( n^{b-2} \) choices for any remaining \( \hat{\ell}_1, \ldots, \hat{\ell}_b \) in (4). Thus, \( f_{\pi_0^*}(\pi) < n^{b-2} \sum_{s=1}^g (n/g)^2 \leq \gamma n^b \).

**Case 2** (\( b \geq 2, a_i < a_j \)). Fix \( \pi \in S_{n,2} \). All details of Case 1 are repeated identically save the following: Now, \( \pi(\ell_i) < \pi(\ell_j) \), which similarly implies that \( \ell_i, \ell_j \in I_s \) for some \( 1 \leq s \leq g \). Indeed, \( \ell_i \in I_{s_i} \) and \( \ell_j \in I_{s_j} \) for some \( s_i < s_j \) would imply \( \pi(\ell_i) > \pi(\ell_j) \), on account of \( \pi \in S_{n,2} \).
The only cases in the proof of Proposition 3 not covered by Cases 1 and 2 involve generalized patterns \( \pi^*_0 \) with \( b = 1 \) block. (These are relatively rare, since there are only \( k! \) such, while there are \( 2^{k-1}k! \) generalized patterns of \([k]\).) If \( k \geq 3 \) and \( b = 1 \), then \( \pi^*_0 \) has (is) a block of length at least 3, which is included in Case 0. If \( k = 2 \) and \( b = 1 \), then \( \pi^*_0 = 12 \) or \( \pi^*_0 = 21 \), where these cases are entirely symmetric.

**Case 3 (\( \pi^*_0 = 12 \)).** Fix \( \pi \in S_{n,3} \), and consider an occurrence \( 1 \leq \ell < \ell + 1 \leq n \) of the generalized pattern 12 in \( \pi \). From (1), we have that \( \pi(\ell) < \pi(\ell + 1) \). As such, \( \pi \in S_{n,3} \) implies that \( \ell \equiv 0 \pmod{g} \). Consequently, we have only \( n/g \leq \gamma n \) choices for \( \ell \).

**Case 4 (\( \pi^*_0 = 21 \)).** Fix \( \pi \in S_{n,4} \). An occurrence \( 1 \leq \ell < \ell + 1 \leq n \) of 21 in \( \pi \) results in \( \pi(\ell) > \pi(\ell + 1) \). Since \( \pi \in S_{n,4} \), it must be that \( \ell \equiv 0 \pmod{g} \), resulting in only \( n/g \leq \gamma n \) choices for \( \ell \).

4 Proof of Theorem 5

Consider the following concept, which has a clear resemblance to patterns. For \( \pi \in S_n \), call a pair \( 1 < i < j < n \) a stretching pair if \( \pi(i) < i < j < \pi(j) \). We shall use stretching pairs to prove Theorem 5, although stretching pairs are interesting in their own right, as we discuss in Section 4.2.

4.1 Stretching pairs and Theorem 5

We establish a few initial considerations. First, let \( C_{n+1} \subset S_{n+1} \) denote the set of \((n + 1)\)-cycles of \( S_{n+1} \), and write each \( \pi \in C_{n+1} \) in cyclic notation: \( \pi = (n + 1 \ a_1 \ldots \ a_n) \), i.e., \( \pi(a_i) = a_{i+1} \) for \( 0 \leq i \leq n \) and \( a_0 = a_{n+1} = n + 1 \). Consider the bijection \( \phi : C_{n+1} \rightarrow S_n \) given by, for each \( \pi = (n + 1 \ a_1 \ldots \ a_n) \in C_{n+1} \),

\[
p = \phi(\pi) = (a_1, \ldots, a_n), \quad \text{that is, } p(i) = a_i \text{ for each } 1 \leq i \leq n.
\]  

(24)

We prove that

\[
\pi \in C_{n+1} \text{ admits a stretching pair } 1 \leq \pi(i) < i < j < \pi(j) \neq n + 1
\]

if and only if \( p = \phi(\pi) \) admits 21–34 or 34–21 as a generalized pattern.  

(25)

Before we prove (25), we note that 21–34 is not the same as 12–34, which Theorem 5 considers. However, Elizalde proved (see Proposition 5.3 from [8]) that

\[
\alpha_n(12–34) = \alpha_n(21–34),
\]

(26)

and so we shall be able to use (25).

**Proof of (25).** Suppose first that \( p = f(\pi) = (a_1, \ldots, a_n) \in S_n \) admits 21–34 or 34–21 as a generalized pattern. If \( a_k, a_{k+1}, a_\ell, a_{\ell+1} \) is a copy of 21–34, where \( 1 < k + 1 < \ell < n \), then \( a_{k+1} < a_k < a_\ell < a_{\ell+1} \), and so \( \pi(i) = a_{k+1} < a_k = i < j = a_\ell < a_{\ell+1} = \pi(j) \leq n \) is a
stretching pair of \( \pi \). If \( a_k, a_{k+1}, a_\ell, a_{\ell+1} \) is a copy of 34–21, then \( a_{\ell+1} < a_\ell < a_k < a_{k+1} \), and so \( \pi(i) = a_{\ell+1} < a_\ell = i < j = a_k < a_{k+1} = \pi(j) \leq n \) is a stretching pair of \( \pi \). Assume now that \( \pi = (n+1 \ a_1 \ \ldots \ \ a_n) \in C_{n+1} \) admits a stretching pair \( 1 \leq \pi(i) < i < j < \pi(j) \leq n \). If \( \pi = (n+1 \ a_1 \ \ldots \ \ i \ \ \pi(i) \ \ldots \ j \ \pi(j) \ \ldots \ a_n) \), then for some \( 1 < k+1 < \ell < n \), \( p = f(\pi) \) has \( i = a_k \), \( \pi(i) = a_{k+1} \), \( j = a_\ell \) and \( \pi(j) = a_{\ell+1} \), where \( a_{k+1} < a_k < a_\ell < a_{\ell+1} \) gives a copy of 21–34. If \( \pi = (n+1 \ a_1 \ \ldots \ j \ \pi(j) \ \ldots \ i \ \pi(i) \ \ldots \ a_n) \), then for some \( 1 < k+1 < \ell < n \), \( p = f(\pi) \) has \( j = a_k \), \( \pi(j) = a_{k+1} \), \( i = a_\ell \) and \( \pi(i) = a_{\ell+1} \), where \( a_{\ell+1} < a_\ell < a_k < a_{k+1} \) gives a copy of 34–21. \( \square \)

Now, define \( S'_{n+1} \) to be the family of \( \pi \in S_{n+1} \) satisfying \( (n+1)/2 < \pi(i) \leq n+1 \) if, and only if, \( 1 \leq i \leq (n+1)/2 \). Clearly, \( S'_{n+1} \) admits no stretching pairs. Set \( C'_{n+1} = C_{n+1} \cap S'_{n+1} \), and observe that \( C'_{n+1} \neq \emptyset \) if, and only if, \( n \) is odd. As such, if \( n \) is both odd and sufficiently large, Stirling’s formula implies

\[
|C_{n+1}'| = \frac{2}{n+1} \left( \frac{n+1}{2} \right)! \geq \left( \frac{1}{2} - o(1) \right)^n n!.
\]

It then follows from (25) that \( \phi(C_{n+1}) \) avoids 21–34 and 34–21, and so

\[
\alpha_n(12–34) \overset{(26)}{=} \alpha_n(21–34) \geq |A_n(21–34) \cap A_n(34–21)|
\]

\[
\geq |\phi(C_{n+1})| = |C_{n+1}'| \geq \left( \frac{1}{2} - o(1) \right)^n n!,
\]

which proves Theorem 5.

### 4.2 A corollary of Theorem 1 for stretching pairs

Stretching pairs are motivated by considerations in dynamical systems. Namely, the occurrence of a stretching pair within a periodic orbit of a continuous interval map implies what is called ‘turbulence’ (see [3, 12] for details). These considerations are closely related to the celebrated theorem of Sharkovsky [14]. From this point of view, the second author [12] considered which \( n \)-cycles \( \pi \in C_n \) admit stretching pairs, and proved that all but \( o(n-1)! \) of them do. Theorem 1 allows us to sharpen this result in the following way.

**Corollary 7.** For all \( \delta > 0 \), there exists \( 0 < c < 1 \) so that for all sufficiently large integers \( n \), all but \( c^n(n-1)! \) cyclic permutations \( \pi \in C_n \) admit \( \frac{1}{12} \binom{n}{2} (1 \pm \delta) \) stretching pairs.

**Proof of Corollary 7.** Let \( \delta > 0 \) be given. Set \( k = 4 \) and \( m = \delta/2 \), and let \( 0 < c_1 < 1 \) be the constant guaranteed by Theorem 1. Define \( c \) to be any constant satisfying \( c_1 < c < 1 \), and let \( n \) be sufficiently large. For an \( n \)-cycle \( \pi \in C_n \), write \( \sigma(\pi) \) for the number of stretching pairs of \( \pi \), and write \( \sigma'(\pi) \) for the number of stretching pairs \( 1 \leq \pi(i) < i < j < \pi(j) \neq n \). Note that \( \sigma'(\pi) \leq \sigma(\pi) \leq \sigma'(\pi) + n \), since if \( 1 \leq \pi(i) < i < j < \pi(j) = n \), then \( j = \pi^{-1}(n) \) is fixed and there are at most \( j-1 \leq n \) choices for \( i \). Note, moreover, that it follows
from (25) that, for $p = \phi(\pi) \in S_{n-1}$, $\sigma'(\pi) = f_{21-34}(p) + f_{34-21}(p)$. Theorem 1 ensures that all but $2c_1^{n-1}(n-1)! < c^n(n-1)!$ permutations $p \in S_{n-1}$ satisfy

$$ f_{21-34}(p) = (1 \pm \mu) \frac{1}{4!} \binom{n-1}{2} \quad \text{and} \quad f_{34-21}(p) = (1 \pm \mu) \frac{1}{4!} \binom{n-1}{2}. $$

For each such permutation $p \in S_{n-1}$, the corresponding $n$-cycle $\pi = \phi^{-1}(p) \in C_n$ satisfies

$$ \sigma(\pi) = (1 \pm \mu) \frac{1}{4!} \binom{n-1}{2} + (1 \pm \mu) \frac{1}{4!} \binom{n-1}{2} \pm n $$

which proves Corollary 7.

\[\square\]

5 Proof of Lemma 6

Fix $\zeta > 0$ and integer $q \geq 1$. Define auxiliary constant

$$ \zeta_0 = \zeta/4. $$

Define $s_0 = s_0(q, \zeta_0)$ to be the least integer $s$ for which

$$ (s)_q \geq (1 - 2\zeta_0)s^q. $$

Let $s \geq s_0$ be given. Define

$$ t_0 = \left\lceil 4q8^q2^q\zeta_0^{-2}\right\rceil. $$

Let integer $t \geq t_0$ be given. Define

$$ c_0 = \frac{\zeta_0^2}{3qt2^q+3s^q}. $$

Let $n$ be a sufficiently large integer, and fix $(i_0, j_0) \in ([s])_q \times [t]$. We prove

all but $\exp\{-2c_0n\}n!$ permutations $\pi \in S_n$ are $(i_0, j_0, \zeta, q)$-typical w.r.t. $(P_s, P_t)$.

Applying (33) to all $(i, j) \in ([s])_q \times [t]$ and noting $s^q\exp\{-2c_0n\} < \exp\{-c_0n\}$ yields Lemma 6.

We now outline our approach for proving (33) (and reduce the $\hat{\ell}$ notation in (10) to $\ell$). Define equivalence relation $\sim$ on $E_{j_0}$: $\ell \sim \ell' \iff q | (\ell - \ell')$. Thus, for an integer $0 \leq r < q$, we may write

$$ E_{j_0}^{(r)} = \{ \ell \in E_{j_0} : \ell \sim (j_0 - 1) \frac{n}{t} + 1 + r \} $$

so that $E_j = E_j^{(0)} \cup \cdots \cup E_j^{(q-1)}$ is a partition. A key observation for later in the proof (cf. Claim 8) will be that

$$ [\ell, \ell + 1 - q] \cap [\ell', \ell' + q - 1] = \emptyset \quad \text{whenever} \ \ell \neq \ell' \in E_j^{(r)}. $$
For some final notation, we shall write, for a permutation $\pi \in S_n$,

$$E_{k_0 j_0}^{(r)}(\pi) = E_{k_0 j_0}(\pi) \cap E_{j_0}^{(r)}$$

so that $E_{k_0 j_0}(\pi) = E_{k_0 j_0}^{(0)}(\pi) \cup \cdots \cup E_{k_0 j_0}^{(q-1)}(\pi)$ \hspace{1cm} (36)

is a partition. We shall prove that, for a fixed $0 \leq r < q$,

all but $\exp\{-3c_0 n\}n!$ permutations $\pi \in S_n$ satisfy that $|E_{k_0 j_0}^{(r)}(\pi)| \geq (1 - \zeta) \frac{1}{q(s)_q} |E_{j_0}|$. 

(37)

Note that (37) implies (33) since then all but $q \exp\{-3c_0 n\}n! < \exp\{-2c_0 n\}n!$ many $\pi \in S_n$ satisfy

$$|E_{k_0 j_0}(\pi)| \geq \sum_{r=0}^{q-1} |E_{k_0 j_0}^{(r)}(\pi)| \geq (1 - \zeta) \frac{1}{q(s)_q} |E_{j_0}|.$$ 

To prove (37), let $Y = Y_{k_0 j_0}^{(r)} = |E_{k_0 j_0}^{(r)}(\pi)|$ is a random variable whose mean we evaluate. To that end, recall from (10) that for an element $\ell \in E_{j_0}$ to be an element of $E_{k_0 j_0}(\pi)$, we require that $\ell \leq j_0/\lceil n/t \rceil - q + 1$, where we will write $n_t = \lfloor n/t \rfloor$ and $n_s = \lceil n/s \rceil$. As such, delete the last $q - 1$ elements from $E_{j_0}$, and write

$$\tilde{E}_{j_0} \coloneqq [(j_0 - 1)n_t + 1, j_0 n_t - q + 1], \quad \tilde{E}_{j_0}^{(r)} = E_{j_0}^{(r)} \cap \tilde{E}_{j_0},$$

and $n_{t,q} \coloneqq |E_{j_0}^{(r)}| = \left\lfloor \frac{n_t - q + 1}{q} \right\rfloor = \left\lfloor \frac{n_t + 1}{q} \right\rfloor - 1$. \hspace{1cm} (38)

Now, for $\ell \in \tilde{E}_{j_0}^{(r)}$, define indicator random variable $Y_{\ell}$ by (cf. $i_0 = (i_1, \ldots, i_q)$)

$$Y_{\ell} = \begin{cases} 1 & \text{if } \pi(\ell + m - 1) \in I_m \ \forall \ m \in [q], \\ 0 & \text{otherwise,} \end{cases} \quad \implies \quad Y = \sum_{\ell \in \tilde{E}_{j_0}^{(r)}} Y_{\ell}$$

so that $E[Y] = \frac{(n - q)!}{n!} \prod_{m=1}^{n} |I_m| = \frac{n^q}{(n)_q} \implies E[Y] = \frac{\tilde{E}_{j_0}^{(r)} |\prod_{m=1}^{n} |I_m|}{(n)_q} = \frac{n^q n_{t,q}}{(n)_q}$. \hspace{1cm} (39)

Following the method of Bernstein for the Chernoff inequality (cf. [10]), for $u = \log(1 - \zeta_0) = \log_e (1 - \zeta_0)$, the Markov inequality implies

$$\mathbb{P}[Y \leq E[Y](1 - \zeta_0)] = \mathbb{P} \left[ e^{u Y} \geq \exp \left\{ u E[Y](1 - \zeta_0) \right\} \right] \leq \exp \left\{ -u E[Y](1 - \zeta_0) \right\} E \left[ e^{u Y} \right] \overset{(39)}{=} \exp \left\{ -u \frac{n^q n_{t,q}}{(n)_q} (1 - \zeta_0) \right\} E \left[ e^{u Y} \right]. \hspace{1cm} (40)$$

While we do not have mutual independence among the $Y_{\ell}$’s, we will prove the following.

**Claim 8.**

$$E \left[ e^{u Y} \right] = E \left[ \prod_{\ell \in \tilde{E}_{j_0}^{(r)}} e^{u Y_{\ell}} \right] \leq \left( 1 + \frac{q(4s)_q}{t} \right)^{n_{t,q}} \prod_{\ell \in \tilde{E}_{j_0}^{(r)}} E \left[ e^{u Y_{\ell}} \right] \overset{(39)}{=} \left[ \left( 1 + \frac{q(4s)_q}{t} \right) \left( 1 + \frac{n^q}{(n)_q} (e^u - 1) \right) \right]^{n_{t,q}} \leq \exp \left\{ n_{t,q} \left( \frac{q(4s)_q}{t} + \frac{n^q}{(n)_q} (e^u - 1) \right) \right\}. \hspace{1cm}$$
We shall defer the proof of Claim 8 in order first to finish the proof of (37).

Applying Claim 8 to (40), together with the Taylor series bound $-u(1-\zeta_0) + e^n - 1 \le -\zeta_0^2/2$,

$$\Pr[Y \le E[Y](1-\zeta_0)] \le \exp\left\{ n_{t,q} \left( q \frac{(4s)^q}{t} + n_q^2 \left( -u(1-\zeta_0) + e^n - 1 \right) \right) \right\}$$

$$\le \exp\left\{ n_{t,q} \left( q \frac{(4s)^q}{t} - \frac{\zeta_0^2}{2(n)^q} \right) \right\} \le \exp\left\{ n_{t,q} \left( q \frac{(4s)^q}{t} - \frac{\zeta_0^2}{2(n)^q} \right) \right\}$$

$$\le \exp\left\{ n_{t,q} \left( q \frac{(4s)^q}{t} - \frac{\zeta_0^2}{2n^q+1} \right) \right\} \quad \text{(since } n_q = \lfloor n/s \rfloor \ge n/(2s))$$

$$\le \exp\left\{ n_{t,q} \left( -\frac{\zeta_0^2}{2n^q+2} \right) \right\} \le \exp\left\{ -\left( \frac{\zeta_0^2}{qn^q+3} \right) n \right\} \equiv \exp\{-3c_0n\}.$$  

(The last inequality above follows from $n_{t,q} = \lceil (n_t + 1)/q \rceil - 1 \ge n/(2tq)$.) In other words, with probability $1 - \exp\{-3c_0n\}$, the randomly chosen permutation $\pi \in S_n$ satisfies

$$Y = \left| E_{t,q}^{(r)}(\pi) \right| \ge E[Y](1-\zeta_0) \geq \sum_{(y_1, \ldots, y_{nt,q}) \in \{0,1\}^{nt,q}} \Pr[\bigwedge_{i=1}^{nt,q} Y_{\ell_i} = y_i] \prod_{i=1}^{nt,q} e^{uy_i}. \quad (31)$$

5.1 Proof of Claim 8

Write $E_{t,q}^{(r)}$ as $\ell_1 < \cdots < \ell_{nt,q}$ (cf. (38)) so that

$$\mathbb{E} \left[ \prod_{\ell \in E_{t,q}^{(r)}} e^{uY_{\ell}} \right] = \sum_{(y_1, \ldots, y_{nt,q}) \in \{0,1\}^{nt,q}} \Pr[\bigwedge_{i=1}^{nt,q} Y_{\ell_i} = y_i] \prod_{i=1}^{nt,q} e^{uy_i}. \quad (41)$$

Fix $(y_1, \ldots, y_{nt,q}) \in \{0,1\}^{nt,q}$ so that

$$\Pr[\bigwedge_{i=1}^{nt,q} Y_{\ell_i} = y_i] = \Pr[Y_{t,q} = y_{t,q}] \frac{\prod_{j=1}^{nt,q-1} (Y_{\ell_j} = y_j)} {\prod_{j=1}^{nt,q-1} (Y_{\ell_j} = y_j)}. \quad (30)$$

We claim that

$$\Pr[Y_{t,q} = y_{t,q} \prod_{j=1}^{nt,q-1} Y_{\ell_j} = y_j] = \Pr[Y_{t,q} = y_{t,q}] \left( 1 \pm q \left( \frac{4s}{t} \right)^q \right). \quad (42)$$

If so, iteratively applying (42) to (41) yields Claim 8.
To see (42), recall the observation in (35). Thus,

\[
\mathbb{P}\left[Y_{m_{i}, q} = 1 \bigg| \bigwedge_{j=1}^{n_{t_{i}, q}-1} Y_{\ell_{j}} = y_{j}\right]\leq \frac{(n - q n_{t_{i}, q})! \prod_{m=1}^{q} |I_{m}|}{(n - q(n_{t_{i}, q} - 1))!} = \frac{n_{s}^{q}}{(n - q n_{t_{i}, q} + q)_{q}},
\]

and

\[
\mathbb{P}\left[Y_{m_{i}, q} = 1 \bigg| \bigwedge_{j=1}^{n_{t_{i}, q}-1} Y_{\ell_{j}} = y_{j}\right] = \frac{(n - q n_{t_{i}, q} + q)_{q}}{(n - q n_{t_{i}, q} + q)_{q}}.
\]

For the upper bound, we use (39) (and \(q n_{t_{i}, q} \leq n_{t}\) (cf. (38)) to infer

\[
\frac{n_{s}^{q}}{(n - q n_{t_{i}, q} + q)_{q}} = \mathbb{P}\left[Y_{m_{i}, q} = 1 \bigg| Y_{m_{i}, q} = 1\right] \cdot \frac{(n)_{q}}{(n - q n_{t_{i}, q} + q)_{q}} \leq \mathbb{P}\left[Y_{m_{i}, q} = 1 \bigg| Y_{m_{i}, q} = 1\right] \left(\frac{n}{n - n_{t}}\right)^{q}
\]

\[
\leq \mathbb{P}\left[Y_{m_{i}, q} = 1 \bigg| Y_{m_{i}, q} = 1\right] \left(1 - \frac{1}{2^{q}}\right) \leq \mathbb{P}\left[Y_{m_{i}, q} = 1 \bigg| Y_{m_{i}, q} = 1\right] \left(1 + \frac{4^{q}}{2^{q}}\right),
\]

For the lower bound, we similarly infer

\[
\frac{(n_{s} - q n_{t_{i}, q} + q)_{q}}{(n - q n_{t_{i}, q} + q)_{q}} \geq \frac{(n_{s} - q n_{t_{i}, q})_{q}}{(n)_{q}} = \mathbb{P}\left[Y_{m_{i}, q} = 1 \bigg| Y_{m_{i}, q} = 1\right] \left(\frac{n_{s} - q n_{t_{i}, q}}{n_{s}}\right)^{q}
\]

\[
\geq \mathbb{P}\left[Y_{m_{i}, q} = 1 \bigg| Y_{m_{i}, q} = 1\right] \left(\frac{n_{s} - n_{t}}{n_{s}}\right)^{q} \geq \mathbb{P}\left[Y_{m_{i}, q} = 1 \bigg| Y_{m_{i}, q} = 1\right] \left(1 - \frac{2^{q}}{2^{q}}\right)^{q}
\]

\[
\geq \mathbb{P}\left[Y_{m_{i}, q} = 1 \bigg| Y_{m_{i}, q} = 1\right] \left(1 - \frac{4^{q} q}{2^{q} t}\right).
\]

This proves (42) when \(y_{n_{t}} = 1\). Otherwise, (using \(\mathbb{P}[Y_{m_{i}, q} = 1] \leq 1/2 \leq \mathbb{P}[Y_{m_{i}, q} = 0]\)) we have

\[
\mathbb{P}\left[Y_{m_{i}, q} = 0 \bigg| \bigwedge_{j=1}^{n_{t_{i}, q}-1} Y_{\ell_{j}} = y_{j}\right] = 1 - \mathbb{P}\left[Y_{m_{i}, q} = 1 \bigg| \bigwedge_{j=1}^{n_{t_{i}, q}-1} Y_{\ell_{j}} = y_{j}\right]
\]

\[
= \mathbb{P}[Y_{m_{i}, q} = 0] \left(1 \pm q \frac{4^{q}}{2^{q} t}\right),
\]

where we used \(\mathbb{P}[Y_{m_{i}, q} = 1] \leq 1/2 \leq \mathbb{P}[Y_{m_{i}, q} = 0]\).

**Acknowledgements**

We would like to thank the Referee for their careful reading of our work, and for helpful suggestions leading to an improved exposition.
References


