Generalized pattern frequency in large permutations

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Abstract

In the study of permutations, generalized patterns extend classical patterns by adding the requirement that certain adjacent integers in a pattern must be adjacent in the permutation.

For any generalized pattern π_0^* of length k with $1 \leqslant b \leqslant k$ blocks, we prove that for all $\mu > 0$, there exists $0 < c = c(k, \mu) < 1$ so that whenever $n \geqslant n_0(k, \mu, c)$, all but $c^n n!$ many $\pi \in S_n$ admit $(1 \pm \mu)\frac{1}{k!}\binom{n}{b}$ occurrences of π_0^*. Up to the choice of c, this result is best possible for all π_0^* with $k \geqslant 2$.

We also give a lower bound on avoidance of the generalized pattern 12-34, which answers a question of S. Elizalde [8] (2006).

Keywords: generalized patterns; pattern avoidance; Azuma's inequality; Chernoff's inequality; Sharkovsky's Theorem

1 Introduction

Pattern and generalized pattern avoidance in permutations is a well-studied area (see, e.g.,[1–5, 7, 8, 10, 11]). Fix $1 \leqslant k \leqslant n$ and $\pi_0 \in S_k$ and let $\pi \in S_n$. An occurrence

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of a *pattern* π_0 in π is a sequence of integers $1 \leqslant \ell_1 < \cdots < \ell_k \leqslant n$ so that, for all $1 \leqslant i \neq j \leqslant k$,

$$\pi(\ell_i) < \pi(\ell_j) \quad \iff \quad \pi_0(i) < \pi_0(j). \tag{1}$$

In order to define generalized patterns, take a classical pattern $\pi_0 = (a_1, \ldots, a_k) = (\pi_0(1), \ldots, \pi_0(k))$, and fix $\pi_0^* = (a_1, \varepsilon_1, a_2, \varepsilon_2, \ldots, \varepsilon_{k-1}, a_k)$ where, for each $1 \leqslant i \leqslant k-1$, ε_i is either a dash '$-$' or the empty string. Then, $\pi \in S_n$ admits π_0^* as a *generalized pattern* if it contains an occurrence $1 \leqslant \ell_1 < \cdots < \ell_k \leqslant n$ of the classical pattern π_0 satisfying that,

$$\text{whenever } \varepsilon_i \neq -, \text{ then } \ell_{i+1} = \ell_i + 1. \tag{2}$$

More explicitly, suppose, for some positive integer sequence $\mathbf{q} = (q_1, \ldots, q_b)$, for which $q_1 + \cdots + q_b = k$, that

$$\begin{aligned} \pi_0^* = \pi_0^{\mathbf{q}} &= \big(a_1, \ldots, a_{q_1}, -, a_{q_1+1}, \ldots, a_{q_1+q_2}, -, \ldots, -, a_{k-q_b+1}, \ldots, a_k\big) \\ &= (A_1, -, A_2, -, \ldots, -, A_b). \end{aligned} \tag{3}$$

Then, for some integers $1 \leqslant \hat{\ell}_1 < \cdots < \hat{\ell}_b \leqslant n$,

$$\begin{aligned} (\ell_1, \ldots, \ell_k) &= \Big(\hat{\ell}_1, \ldots, \hat{\ell}_1 + q_1 - 1, \;\; \hat{\ell}_2, \ldots, \hat{\ell}_2 + q_2 - 1, \;\; \ldots, \;\; \hat{\ell}_b, \ldots, \hat{\ell}_b + q_b - 1\Big) \\ &= (L_1, \ldots, L_b). \end{aligned} \tag{4}$$

We shall refer to the subsequences $A_1, \ldots, A_b$ and $L_1, \ldots, L_b$ as *blocks*.

As an illustrative example, we note that the permutation $(3, 5, 2, 4, 1) = 35241$ contains the classical pattern 132 (realized uniquely by the 3, 5, and 4 occuring in that order). However, 35241 does not contain the generalized pattern 1-32, since the 5 and 4 are not adjacent.

Let $f_{\pi_0^*}(\pi)$ denote the frequency of the generalized pattern π_0^* in π, and set $F_{\pi_0}(\pi) = f_{\pi_0^*}(\pi)$ in the case that $\mathbf{q} = (1, \ldots, 1)$ (i.e., classical patterns). In this notation, the celebrated result of Marcus and Tardos [13] (cf. Klazar [11]) asserts $F_{\pi_0}(\pi) \geqslant 1$ for all but C^n permutations $\pi \in S_n$, where $C = C(\pi_0) > 1$ and n is sufficiently large. The first author [7] proved that F_{π_0} is concentrated about its mean: $F_{\pi_0}(\pi) = (1 \pm o(1)) \frac{1}{k!}\binom{n}{k}$ for all but $o(n!)$ permutations $\pi \in S_n$. Our main result shows, more generally, that $f_{\pi_0^*}$ is also concentrated about its mean, and we provide a sharp estimate for the error $o(n!)$ of concentration.

Theorem 1. *For every $k \geqslant 1$ and for all $\mu > 0$, there exists $0 < c < 1$ so that, for all sufficiently large integers n, the following holds. For every $\pi_0 \in S_k$ and for every sequence π_0^* with b blocks as in (3), all but $c^n n!$ many $\pi \in S_n$ satisfy $f_{\pi_0^*}(\pi) = (1 \pm \mu) \frac{1}{k!}\binom{n}{b}$.*

Remark 2. In Section 2, we offer two proofs of Theorem 1. The first, based on martingales, is fairly short. The second gives more detail, using a 'quasi-random' property (see Lemma 6) typical of random permutations. Lemma 6 extends some results from [7] and may be of independent interest.

Up to the choice of $0 < c < 1$, Theorem 1 is best possible for all π_0^* with $k \geqslant 2$. In particular, we prove the following result.

Proposition 3. *Fix $k \geqslant 2$, $b \geqslant 1$, and $\pi_0 = (a_1, \dots, a_k) \in S_k$. Let π_0^* be any sequence, as in (3), with b blocks. Then, there exists $0 < \gamma_0 < 1$ so that, for all $0 < \gamma < \gamma_0$, there exist infinitely many integers n for which at least $\gamma^n n!$ permutations $\pi \in S_n$ satisfy $f_{\pi_0^*}(\pi) < \gamma n^b$.*

We prove Proposition 3 in Section 3.

Proposition 3 can often be strengthened. Indeed, S. Elizalde [8] proved the following strong and quite general result (in [8], see Proposition 4.3).

Theorem 4 (Elizalde [8])**.** *Let π_0^* be a sequence, as in (3), having a block A_i of length at least 3. Then, there exists $0 < c < 1$ so that for all $n \geqslant k$, at least $c^n n!$ permutations $\pi \in S_n$ satisfy $f_{\pi_0^*}(\pi) = 0$.*

Elizalde [8] also considered to what extent Theorem 4 can be extended to sequences π_0^* whose every block has length at most two. He showed that, in general, Theorem 4 can't be extended to every such π_0^*. To describe these results, let $A_n(\pi_0^*)$ denote the set of permutations $\pi \in S_n$ for which $f_{\pi_0^*}(\pi) = 0$, and let $\alpha_n(\pi_0^*) = |A_n(\pi_0^*)|$. For $(1, -, 2, 3, -, 4) =$ 1–23–4, Elizalde showed (see Corollary 6.2 in [8])

$$\lim_{n\to\infty} \left(\frac{\alpha_n(1\text{–}23\text{–}4)}{n!} \right)^{1/n} = 0. \tag{5}$$

He asked (see Section 7 of [8]):

$$\textit{does } \lim_{n\to\infty} \left(\frac{\alpha_n(12\text{–}34)}{n!} \right)^{1/n} = 0\,? \tag{6}$$

We answer this question in the negative.

Theorem 5. *For odd integers n,*

$$\alpha_n(12\text{–}34) \geqslant \left(\frac{1}{2} - o(1) \right)^n n!.$$

We prove Theorem 5 in Section 4, and also consider some related problems.

2 Proofs of Theorem 1

For both of the following proofs, fix a positive integer k and fix $\mu > 0$.

2.1 The martingale proof

Let

$$c = \exp\left\{ -\frac{\mu^2}{9k^4 k!^2} \right\} \tag{7}$$

and let n be a sufficiently large integer wherever needed. Fix π_0^* with b blocks as in (3). We show that all but $c^n n!$ many $\pi \in S_n$ satisfy $f_{\pi_0^*}(\pi) = (1 \pm \mu)\frac{1}{k!}\binom{n}{b}$.

To that end, let $\pi \in S_n$ be chosen uniformly at random. We use the 'exposure process' to define the following sequence of random variables. Set

$$X_0 = \mathbb{E}[f_{\pi_0^*}(\pi)], \text{ where from (4), we have } (1 - o(1))\frac{1}{k!}\binom{n}{b} \leqslant \mathbb{E}[f_{\pi_0^*}(\pi)] \leqslant \frac{1}{k!}\binom{n}{b}. \quad (8)$$

For $r \in [n] = \{1, \ldots, n\}$, let $\pi_{[r]}$ denote the restriction $\pi : [r] \to [n]$. Set

$$X_r = \mathbb{E}\left[f_{\pi_0^*}(\pi) \middle| \pi_{[r]} \right],$$

so that $X_n = f_{\pi_0^*}(\pi)$ is the variable we wish to estimate. Then, $X_0, X_1, \ldots, X_n$ is the Doob martingale for the function $f_{\pi_0^*}$, to which we will apply Azuma's inequality.

For that purpose, observe that for each $0 \leqslant r \leqslant n-1$,

$$|X_{r+1} - X_r| \leqslant k\binom{n}{b-1}. \quad (9)$$

To see this, note that the element $r+1$ belongs to between zero and $k\binom{n}{b-1}$ occurrences $(\ell_1, \ldots, \ell_k)$ of π_0^* in π. Indeed, if $\ell_i = r+1$ belongs to block $L_{i'}$ (see (4)), then all of $L_{i'}$ is determined by $r+1 = \ell_i$ and $\mathbf{q} = \mathbf{q}(\pi_0^*)$. Thus, it remains to determine $L_1, \ldots, L_{i'-1}, L_{i'+1}, \ldots, L_b$, or equivalently, $\hat{\ell}_1, \ldots, \hat{\ell}_{i'-1}, \hat{\ell}_{i'+1}, \ldots, \hat{\ell}_b$, of which there are at most $\binom{n}{b-1}$.

Applying Azuma's inequality with $t = (\mu/2)X_0$ and using (8) and (9), we have

$$\begin{aligned}\mathbb{P}\big[|X_n - X_0| \geqslant t\big] &\leqslant 2\exp\left\{-\frac{t^2}{2\sum_{r=0}^{n-1}(X_{r+1}-X_r)^2}\right\} \leqslant \exp\left\{-\frac{\mu^2\frac{1}{k!^2}\binom{n}{b}^2}{8nk^2\binom{n}{b-1}^2}(1-o(1))\right\} \\ &= \exp\left\{-\frac{\mu^2 n}{8k!^2k^2b^2}(1-o(1))\right\} \leqslant \exp\left\{-\frac{\mu^2 n}{8k!^2k^4}(1-o(1))\right\} \leqslant \exp\left\{-\frac{\mu^2 n}{9k!^2k^4}\right\} \overset{(7)}{=} c^n.\end{aligned}$$

Thus, with probability $1 - c^n$,

$$f_{\pi_0^*}(\pi) = \left(1 \pm \frac{\mu}{2}\right)\mathbb{E}[f_{\pi_0^*}(\pi)] \overset{(8)}{=} \left(1 \pm \frac{\mu}{2}\right)(1 \pm o(1))\frac{1}{k!}\binom{n}{b} = (1 \pm \mu)\frac{1}{k!}\binom{n}{b},$$

as desired.

2.2 The quasi-random proof

To present Lemma 6, we need a few concepts. For integers $n > t \geqslant j \geqslant 1$, define $I_j = [(j-1)\lfloor n/t \rfloor + 1, j\lfloor n/t \rfloor]$ and $R = [n] \setminus \bigcup_{j=1}^{t} I_j$. We call $[n] = I_1 \cup I_2 \cup \cdots \cup I_t \cup R$ the *t-partition* $\boldsymbol{P}_t$ of $[n]$. Now, fix $\pi \in S_n$, and consider partitions $\boldsymbol{P}_s = I_1 \cup \cdots \cup I_s \cup R_s$ and $\boldsymbol{P}_t = E_1 \cup \cdots \cup E_t \cup R_t$ of $[n]$, where $n > t \geqslant s \geqslant q \geqslant 1$. For a set X, we will write

$(X)_m$ for the family of m-permutations of X, and we write $(|X|)_m$ for $|(X)_m|$, when $|X|$ is finite. For $\mathbf{i} = (i_1, \dots, i_q) \in ([s])_q$ and $j \in [t]$, let

$$E_{\mathbf{i}j}(\pi) = \Big\{ \hat{\ell} \in E_j : \hat{\ell} + q - 1 \in E_j \text{ and } \pi(\hat{\ell} + m - 1) \in I_{i_m} \text{ for all } m \in \{1, \dots, q\} \Big\}. \quad (10)$$

For $\zeta > 0$, and $(\mathbf{i}, j) \in ([s])_q \times [t]$, we say $\pi \in S_n$ is $(\mathbf{i}, j, \zeta, q)$-*typical (w.r.t.* $(\boldsymbol{P}_s, \boldsymbol{P}_t)$*)* if

$$|E_{\mathbf{i}j}(\pi)| \geqslant (1 - \zeta) \frac{1}{(s)_q} |E_j| = (1 - \zeta) \frac{1}{(s)_q} \left\lfloor \frac{n}{t} \right\rfloor \quad (11)$$

and say $\pi \in S_n$ is (ζ, q)-*typical (w.r.t.* $(\boldsymbol{P}_s, \boldsymbol{P}_t)$*)* if it is $(\mathbf{i}, j, \zeta, q)$-typical for all $(\mathbf{i}, j) \in ([s])_q \times [t]$.

Lemma 6. *For all $\zeta > 0$ and integers $q \geqslant 1$, there exists an integer s_0 so that for all integers $s \geqslant s_0$, there exists an integer t_0 so that for all integers $t \geqslant t_0$, there exists $c_0 > 0$ so that for all sufficiently large integers n, all but* $\exp\{-c_0 n\} n!$ *permutations $\pi \in S_n$ are (ζ, q)-typical w.r.t. $(\boldsymbol{P}_s, \boldsymbol{P}_t)$.*

Lemma 6 follows by a standard (albeit tedious) probablistic analysis, which we give in Section 5.

To show that Lemma 6 implies Theorem 1, define auxiliary constants $\delta, \zeta > 0$ so that

$$\delta = \frac{\mu}{k!} \qquad \text{and} \qquad (1 - 2\zeta)^{k+2} > 1 - \delta. \quad (12)$$

For $q \in [k]$, let $s_0(q)$ be the constant guaranteed by Lemma 6. Fix an integer s so that

$$s \geqslant \max\{s_0(1), \dots, s_0(k)\} \qquad \text{and} \qquad \binom{s}{k} \geqslant \frac{s^k}{k!}(1 - 2\zeta). \quad (13)$$

For $q \in [k]$, let $t_0(q)$ be the constant guaranteed by Lemma 6. Fix an integer t with

$$t \geqslant \max\big\{t_0(1), \dots, t_0(k)\big\} \qquad \text{and so that for all } b \in [k], \qquad \binom{t}{b} \geqslant \frac{t^b}{b!}(1 - 2\zeta). \quad (14)$$

For $q \in \{1, \dots, k\}$, let $c_0(q) > 0$ be the constant guaranteed by Lemma 6. Define

$$c_0 = \min\big\{c_0(1), \dots, c_0(k)\big\} \qquad \text{and} \qquad c = \exp\{-c_0/4\}. \quad (15)$$

In all that follows, let n be a sufficiently large integer.

Fix a permutation $\pi_0 \in S_k$, and let $\pi_0^* = \pi_0^{\mathbf{q}} = (A_1, -, \dots, -, A_b)$ be given as in (3) where $\mathbf{q} = (q_1, \dots, q_b)$. Apply Lemma 6 (cf. (13)–(15)) to conclude that all but

$$\Big(\exp\big\{ - c_0(q_1) n\big\} + \cdots + \exp\big\{ - c_0(q_b) n\big\}\Big) n! \leqslant k \exp\{-c_0 n\} n! \leqslant \exp\Big\{-\frac{c_0}{2} n\Big\} n!$$

permutations $\pi \in S_n$ are (ζ, q_x)-typical w.r.t. $(\boldsymbol{P}_s, \boldsymbol{P}_t)$ for all $x \in [b]$. For such a $\pi \in S_n$, we show

$$f_{\pi_0^{\mathbf{q}}}(\pi) \geqslant (1 - \delta) \frac{1}{k!} \binom{n}{b} \overset{(12)}{>} (1 - \mu) \frac{1}{k!} \binom{n}{b}. \quad (16)$$

Indeed, fix indices $1 \leqslant i_1 < \cdots < i_k \leqslant s$ and $1 \leqslant j_1 < \cdots < j_b \leqslant t$. For $x \in [b]$, recall the block

$$A_x = \left(a_{q_1+\cdots+q_{x-1}+1}, \ldots, a_{q_1+\cdots+q_x}\right) = \left(\pi_0(q_1 + \cdots + q_{x-1} + 1), \ldots, \pi_0(q_1 + \cdots + q_x)\right)$$

of $\pi_0^{\mathbf{q}}$ (cf. (3)). Consider the injection defined by, for each $x \in [b]$,

$$j_x \longmapsto \mathbf{i}_x \stackrel{\text{def}}{=} \left(i_a\right)_{a \in A_x} = \left(i_{a_{q_1+\cdots+q_{x-1}+1}}, \ldots, i_{a_{q_1+\cdots+q_x}}\right) \tag{17}$$

$$= \left(i_{\pi_0(q_1+\cdots+q_{x-1}+1)}, \ldots, i_{\pi_0(q_1+\cdots+q_x)}\right). \tag{18}$$

For each $x \in [b]$, arbitrarily select $\hat{\ell}_x \in E_{\mathbf{i}_x j_x}(\pi)$ (cf. (10)). We claim that the sequence

$$(L_1, L_2, \ldots, L_b), \qquad \text{where for each } x \in [b], \qquad L_x = \left(\hat{\ell}_x, \hat{\ell}_x + 1, \ldots, \hat{\ell}_x + q_x - 1\right), \tag{19}$$

is exactly an occurrence in π of the generalized pattern $\pi_0^{\mathbf{q}}$. The sequence $(L_1, \ldots, L_b)$ clearly satisfies (2), since each L_m is consecutive, and since $(L_1, L_2, \ldots, L_b)$ precisely mimics the block structure of $\pi_0^{\mathbf{q}} = (A_1, -, A_2, -, \ldots, -, A_b)$ (cf. (3)). It remains to check, therefore, that $(L_1, \ldots, L_b)$ is an occurrence of the classical pattern π_0 in π, i.e., that $(L_1, \ldots, L_b)$ satisfies (1).

Indeed, rewrite the sequence $(L_1, \ldots, L_b)$ as

$$(\ell_1, \ldots, \ell_k) = \left(\ell_1, \ldots, \ell_{q_1}, \ell_{q_1+1}, \ldots, \ell_{q_1+q_2}, \ldots\ldots, \ell_{k-q_b+1}, \ldots, \ell_k\right)$$

$$\text{so that for } x \in [b], \qquad L_x = \left(\ell_{q_1+\cdots+q_{x-1}+1}, \ldots, \ell_{q_1+\cdots+q_x}\right). \tag{20}$$

Comparing (19) and (20), we see that a term of the sequence $(L_1, \ldots, L_b)$ is determined by a choice of indices $1 \leqslant x \leqslant b$ and $1 \leqslant w \leqslant q_x$, and written simultaneously as

$$\hat{\ell}_x + w - 1 = \ell_{q_1+\cdots+q_{x-1}+w}. \tag{21}$$

(Such a term necessarily belongs to the block L_x.) Observe from (10) and (17) that

$$\pi(\hat{\ell}_x + w - 1) \in I_{i(x,w)}, \quad \text{where} \quad i(x,w) = i_{\pi_0(q_1+\cdots+q_{x-1}+w)}. \tag{22}$$

Now, fix two terms (cf. (21)) of the sequence $(L_1, \ldots, L_b)$:

$$\hat{\ell}_x + w - 1 = \ell_{q_1+\cdots+q_{x-1}+w} \qquad \text{and} \qquad \hat{\ell}_y + z - 1 = \ell_{q_1+\cdots+q_{y-1}+z},$$

where $1 \leqslant x, y \leqslant b$, $1 \leqslant w \leqslant q_x$ and $1 \leqslant z \leqslant q_y$. From (22), we conclude

$$\pi\left(\ell_{q_1+\cdots+q_{x-1}+w}\right) < \pi\left(\ell_{q_1+\cdots+q_{y-1}+z}\right) \iff \max I_{i(x,w)} < \min I_{i(y,z)}$$
$$\iff i(x,w) < i(y,z) \stackrel{(22)}{\iff} i_{\pi_0(q_1+\cdots+q_{x-1}+w)} < i_{\pi_0(q_1+\cdots+q_{y-1}+z)}$$
$$\iff \pi_0(q_1 + \cdots + q_{x-1} + w) < \pi_0(q_1 + \cdots + q_{y-1} + z),$$

as required by (1). (For the last step, recall the ordering $1 \leqslant i_1 < \cdots < i_k \leqslant s$ of the fixed indices.)

Now, the discussion above implies that

$$f_{\pi_0^{\mathbf{q}}}(\pi) \geqslant \sum\sum \left\{ \prod_{x=1}^{b} \left|E_{\mathbf{i}_x j_x}(\pi)\right| : 1 \leqslant i_1 < \cdots < i_k \leqslant s,\, 1 \leqslant j_1 < \cdots < j_b \leqslant t \right\}. \quad (23)$$

Since $\pi \in S_n$ is (ζ, q)-typical w.r.t. $(\boldsymbol{P}_s, \boldsymbol{P}_t)$ for every $q \in \{q_1, \ldots, q_b\}$, we have, for each $x \in [b]$,

$$\left|E_{\mathbf{i}_x j_x}(\pi)\right| \geqslant (1-\zeta)\frac{1}{(s)_{q_x}}\left\lfloor \frac{n}{t} \right\rfloor \geqslant (1-2\zeta)\frac{n}{t(s)_{q_x}} \geqslant (1-2\zeta)\frac{n}{ts^{q_x}}.$$

Returning to (23),

$$\frac{f_{\pi_0^{\mathbf{q}}}(\pi)}{\binom{s}{k}\binom{t}{b}} \geqslant (1-2\zeta)^b \left(\frac{n}{t}\right)^b \prod_{x=1}^{b} \frac{1}{s^{q_x}} = (1-2\zeta)^b \left(\frac{n}{t}\right)^b \frac{1}{s^{q_1+\cdots+q_b}}$$
$$= (1-2\zeta)^b \left(\frac{n}{t}\right)^b \frac{1}{s^k} \geqslant (1-2\zeta)^k \left(\frac{n}{t}\right)^b \frac{1}{s^k},$$

and so (16) follows from

$$f_{\pi_0^{\mathbf{q}}}(\pi) \geqslant \binom{s}{k}\binom{t}{b}(1-2\zeta)^k \left(\frac{n}{t}\right)^b \frac{1}{s^k} \overset{(13),\,(14)}{\geqslant} (1-2\zeta)^{k+2}\frac{1}{k!}\left(\frac{n^b}{b!}\right) \overset{(12)}{\geqslant} (1-\delta)\frac{1}{k!}\binom{n}{b}.$$

The corresponding upper bound $f_{\pi_0^{\mathbf{q}}}(\pi) \leqslant (1+\mu)\frac{1}{k!}\binom{n}{b}$ follows, in fact, from the lower bound. Indeed, first conclude (16) for every permutation $p \in S_k$ and $p^* = p^{\mathbf{q}}$. Thus, all but

$$k! \exp\left\{-\frac{c_0}{2}n\right\} n! < \exp\left\{-\frac{c_0}{4}n\right\} n! \overset{(15)}{=} c_1^n n!$$

permutations $\pi \in S_n$ satisfy, for every $p \in S_k$, $f_{p^{\mathbf{q}}}(\pi) \geqslant (1-\delta)\frac{1}{k!}\binom{n}{b}$. Fix such a $\pi \in S_n$. Observe that every $1 \leqslant \ell_1 < \cdots < \ell_k \leqslant n$ of the form in (19) and (20) defines a generalized pattern $p^{\mathbf{q}}$ of some $p \in S_k$. (Indeed, if $\pi(\{\ell_1, \ldots, \ell_k\}) = \{\lambda_1, \ldots, \lambda_k\}$, define $p(i) = j$ if and only if $\pi(\ell_i) = \lambda_j$.) Thus,

$$\binom{n}{b} \geqslant \sum_{p \in S_k} f_{p^{\mathbf{q}}}(\pi) = f_{\pi_0^{\mathbf{q}}}(\pi) + \sum_{\pi_0 \neq p \in S_k} f_{p^{\mathbf{q}}}(\pi) \geqslant f_{\pi_0^{\mathbf{q}}}(\pi) + (k!-1)(1-\delta)\frac{1}{k!}\binom{n}{b}$$
$$\Longrightarrow \qquad f_{\pi_0^{\mathbf{q}}}(\pi) \leqslant \left(\frac{1}{k!} + \delta - \frac{\delta}{k!}\right)\binom{n}{b} \leqslant (1+\delta k!)\frac{1}{k!}\binom{n}{b} \overset{(12)}{=} (1+\mu)\frac{1}{k!}\binom{n}{b}.$$

3 Proof of Proposition 3

Fix $k \geqslant 2$, $b \geqslant 1$, and $\pi_0 = (a_1, \ldots, a_k) \in S_k$. Fix any sequence π_0^*, as in (3), with b blocks. If π_0^* has a block of length at least 3, then let $0 < c = c(\pi_0^*) < 1$ be the constant guaranteed by Theorem 4, and set $\gamma_0 = c/2$. Otherwise, set $\gamma_0 = 1/2$. Fix $0 < \gamma < \gamma_0$, and write $g = \lfloor 1/\gamma \rfloor$, where we note that $\gamma < 1/2$ implies $g \geqslant 2$. For a sufficiently large integer n which is divisible by g, we guarantee at least $\gamma^n n!$ permutations $\pi \in S_n$ with $f_{\pi_0^*}(\pi) < \gamma n^b$.

Our proof is based on cases, depending on the structure of the sequence π_0^*. Clearly, we get the following case entirely for free on account of Theorem 4.

Case 0 (π_0^* has a block of length at least 3)**.** Theorem 4 guarantees at least $c^n n! > \gamma^n n!$ permutations $\pi \in S_n$ with $f_{\pi_0^*}(\pi) = 0 < \gamma n^b$.

To handle all other cases, we require the following considerations. For $0 \leqslant s \leqslant g-1$, write $I_s = [s(n/g)+1, (s+1)(n/g)]$ and $R_s = \{m \in [n] : m \equiv s \pmod g\}$. Then $[n] = I_0 \cup \cdots \cup I_{g-1}$ and $[n] = R_0 \cup \cdots \cup R_{g-1}$ are partitions of $[n]$ into parts of common size n/g. Consider the following four classes of permutations:

$$\begin{aligned}
S_{n,1} &= \{\pi \in S_n : \pi(I_s) = I_s,\ \forall\, 0 \leqslant s \leqslant g-1\}, \\
S_{n,2} &= \{\pi \in S_n : \pi(I_s) = I_{g-1-s},\ \forall\, 0 \leqslant s \leqslant g-1\}, \\
S_{n,3} &= \{\pi \in S_n : \pi(R_s) = I_{g-s},\ \forall\, 0 \leqslant s \leqslant g-1\} \qquad (\text{take } I_g = I_0), \\
S_{n,4} &= \{\pi \in S_n : \pi(R_s) = I_{s-1},\ \forall\, 0 \leqslant s \leqslant g-1\} \qquad (\text{take } I_{-1} = I_{g-1}).
\end{aligned}$$

Clearly, $|S_{n,1}| = |S_{n,2}| = |S_{n,3}| = |S_{n,4}| = ((n/g)!)^g$, where by Stirling's formula,

$$\left(\left(\frac{n}{g}\right)!\right)^g > \frac{1}{2}\left(\sqrt{2\pi(n/g)}\left(\frac{n}{eg}\right)^{(n/g)}\right)^g > \frac{\sqrt{\gamma}}{2}(2\pi\gamma n)^{\frac{g-1}{2}} \times \gamma^n\sqrt{2\pi n}\left(\frac{n}{e}\right)^n > \gamma^n n!.$$

We mention, in advance, that in the following four cases below, Case i will be handled by the family $S_{n,i}$, for $1 \leqslant i \leqslant 4$. We also mention that Cases 1 and 2 are not always disjoint from Case 0, nor are they always disjoint from each other. (It seemed easiest to preserve generality in the cases.)

We now consider when π_0^* has $b \geqslant 2$ blocks. In particular, suppose $a_i = \pi_0(i)$ and $a_j = \pi_0(j)$, $1 \leqslant i < j \leqslant k$, belong to blocks $A_{i'}$ and $A_{j'}$, respectively, where $A_{i'} \neq A_{j'}$.

Case 1 ($b \geqslant 2$, $a_i > a_j$)**.** Fix $\pi \in S_{n,1}$, and consider an occurence $1 \leqslant \ell_1 < \cdots < \ell_k \leqslant n$ of the generalized pattern π_0^* in π. Consider the terms $\ell_i < \ell_j$. From (1), since $\pi_0(i) = a_i > a_j = \pi_0(j)$, we have $\pi(\ell_i) > \pi(\ell_j)$. We therefore claim that, for some $1 \leqslant s \leqslant g$, we have $\ell_i, \ell_j \in I_s$. Indeed, if $\ell_i \in I_{s_i}$ and $\ell_j \in I_{s_j}$ for some $s_i < s_j$, then $\pi(\ell_i) < \pi(\ell_j)$ on account of $\pi \in S_{n,1}$, a contradiction. We also recall from (3) and (4), that ℓ_i belongs to block $L_{i'}$ and ℓ_j belongs to block $L_{j'}$ (since a_i belongs to block $A_{i'}$ and a_j belongs to block $A_{j'}$). Finally, recall from (4) that $L_{i'}$ begins with $\hat{\ell}_{i'}$ and $L_{j'}$ begins with $\hat{\ell}_{j'}$. Then, since $\ell_i, \ell_j \in I_s$, we have that $\hat{\ell}_{j'} \in I_s$ and $\hat{\ell}_{i'} \in I_{s-1} \cup I_s$. (If $\hat{\ell}_{i'} \in I_{s-1}$, it occurs very near the right boundary.) Clearly, there are at most $|I_s| = n/g$ choices for $\hat{\ell}_{j'}$. It is easy to check that there are fewer than n/g choices for $\hat{\ell}_{i'}$. Clearly, there are at most n^{b-2} choices for any remaining $\hat{\ell}_1, \ldots, \hat{\ell}_b$ in (4). Thus, $f_{\pi_0^*}(\pi) < n^{b-2}\sum_{s=1}^{g}(n/g)^2 \leqslant \gamma n^b$.

Case 2 ($b \geqslant 2$, $a_i < a_j$)**.** Fix $\pi \in S_{n,2}$. All details of Case 1 are repeated identically save the following: Now, $\pi(\ell_i) < \pi(\ell_j)$, which similarly implies that $\ell_i, \ell_j \in I_s$ for some $1 \leqslant s \leqslant g$. Indeed, $\ell_i \in I_{s_i}$ and $\ell_j \in I_{s_j}$ for some $s_i < s_j$ would imply $\pi(\ell_i) > \pi(\ell_j)$, on account of $\pi \in S_{n,2}$.

The only cases in the proof of Proposition 3 not covered by Cases 1 and 2 involve generalized patterns π_0^* with $b = 1$ block. (These are relatively rare, since there are only $k!$ such, while there are $2^{k-1}k!$ generalized patterns of $[k]$.) If $k \geqslant 3$ and $b = 1$, then π_0^* has (is) a block of length at least 3, which is included in Case 0. If $k = 2$ and $b = 1$, then $\pi_0^* = 12$ or $\pi_0^* = 21$, where these cases are entirely symmetric.

Case 3 ($\pi_0^* = 12$)**.** Fix $\pi \in S_{n,3}$, and consider an occurence $1 \leqslant \ell < \ell + 1 \leqslant n$ of the generalized pattern 12 in π. From (1), we have that $\pi(\ell) < \pi(\ell + 1)$. As such, $\pi \in S_{n,3}$ implies that $\ell \equiv 0 \pmod g$. Consequently, we have only $n/g \leqslant \gamma n$ choices for ℓ.

Case 4 ($\pi_0^* = 21$)**.** Fix $\pi \in S_{n,4}$. An occurence $1 \leqslant \ell < \ell + 1 \leqslant n$ of 21 in π results in $\pi(\ell) > \pi(\ell+1)$. Since $\pi \in S_{n,4}$, it must be that $\ell \equiv 0 \pmod g$, resulting in only $n/g \leqslant \gamma n$ choices for ℓ.

4 Proof of Theorem 5

Consider the following concept, which has a clear resemblance to patterns. For $\pi \in S_n$, call a pair $1 < i < j < n$ a *stretching pair* if $\pi(i) < i < j < \pi(j)$. We shall use stretching pairs to prove Theorem 5, although stretching pairs are interesting in their own right, as we discuss in Section 4.2.

4.1 Stretching pairs and Theorem 5

We establish a few initial considerations. First, let $C_{n+1} \subset S_{n+1}$ denote the set of $(n+1)$-cycles of S_{n+1}, and write each $\pi \in C_{n+1}$ in cyclic notation: $\pi = (n + 1\ a_1\ \ldots\ a_n)$, i.e., $\pi(a_i) = a_{i+1}$ for $0 \leqslant i \leqslant n$ and $a_0 = a_{n+1} = n + 1$. Consider the bijection $\phi : C_{n+1} \to S_n$ given by, for each $\pi = (n + 1\ a_1\ \ldots\ a_n) \in C_{n+1}$,

$$p = \phi(\pi) = (a_1, \ldots, a_n), \text{ that is, } p(i) = a_i \text{ for each } 1 \leqslant i \leqslant n. \tag{24}$$

We prove that

$$\pi \in C_{n+1} \textit{ admits a stretching pair } 1 \leqslant \pi(i) < i < j < \pi(j) \neq n + 1$$
$$\textit{if and only if } p = \phi(\pi) \textit{ admits } 21\text{–}34 \textit{ or } 34\text{–}21 \textit{ as a generalized pattern.} \tag{25}$$

Before we prove (25), we note that 21–34 is not the same as 12–34, which Theorem 5 considers. However, Elizalde proved (see Proposition 5.3 from [8]) that

$$\alpha_n(12\text{–}34) = \alpha_n(21\text{–}34), \tag{26}$$

and so we shall be able to use (25).

Proof of (25). Suppose first that $p = f(\pi) = (a_1, \ldots, a_n) \in S_n$ admits 21–34 or 34–21 as a generalized pattern. If $a_k, a_{k+1}, a_\ell, a_{\ell+1}$ is a copy of 21–34, where $1 < k + 1 < \ell < n$, then $a_{k+1} < a_k < a_\ell < a_{\ell+1}$, and so $\pi(i) = a_{k+1} < a_k = i < j = a_\ell < a_{\ell+1} = \pi(j) \leqslant n$ is a stretching pair of π. If $a_k, a_{k+1}, a_\ell, a_{\ell+1}$ is a copy of 34–21, then $a_{\ell+1} < a_\ell < a_k < a_{k+1}$, and so $\pi(i) = a_{\ell+1} < a_\ell = i < j = a_k < a_{k+1} = \pi(j) \leqslant n$ is a stretching pair of π. Assume now that $\pi = (n+1\ a_1\ \dots\ a_n) \in C_{n+1}$ admits a stretching pair $1 \leqslant \pi(i) < i < j < \pi(j) \leqslant n$. If $\pi = (n+1\ a_1\ \dots\ i\ \pi(i)\ \dots\ j\ \pi(j)\ \dots\ a_n)$, then for some $1 < k+1 < \ell < n$, $p = f(\pi)$ has $i = a_k$, $\pi(i) = a_{k+1}$, $j = a_\ell$ and $\pi(j) = a_{\ell+1}$, where $a_{k+1} < a_k < a_\ell < a_{\ell+1}$ gives a copy of 21–34. If $\pi = (n+1\ a_1\ \dots\ j\ \pi(j)\ \dots\ i\ \pi(i)\ \dots\ a_n)$, then for some $1 < k+1 < \ell < n$, $p = f(\pi)$ has $j = a_k$, $\pi(j) = a_{k+1}$, $i = a_\ell$ and $\pi(i) = a_{\ell+1}$, where $a_{\ell+1} < a_\ell < a_k < a_{k+1}$ gives a copy of 34–21. □

Now, define S'_{n+1} to be the family of $\pi \in S_{n+1}$ satisfying $(n+1)/2 < \pi(i) \leqslant n+1$ if, and only if, $1 \leqslant i \leqslant (n+1)/2$. Clearly, S'_{n+1} admits no stretching pairs. Set $C'_{n+1} = C_{n+1} \cap S'_{n+1}$, and observe that $C'_{n+1} \neq \emptyset$ if, and only if, n is odd. As such, if n is both odd and sufficiently large, Stirling's formula implies

$$\left|C'_{n+1}\right| = \frac{2}{n+1}\left(\left(\frac{n+1}{2}\right)!\right)^2 \geqslant \left(\frac{1}{2} - o(1)\right)^n n!.$$

It then follows from (25) that $\phi(C'_{n+1})$ avoids 21–34 and 34–21, and so

$$\begin{aligned}\alpha_n(12\text{–}34) \overset{(26)}{=} \alpha_n(21\text{–}34) &\geqslant |A_n(21\text{–}34) \cap A_n(34\text{–}21)| \\ &\geqslant \left|\phi(C'_{n+1})\right| = |C'_{n+1}| \geqslant \left(\frac{1}{2} - o(1)\right)^n n!,\end{aligned}$$

which proves Theorem 5.

4.2 A corollary of Theorem 1 for stretching pairs

Stretching pairs are motivated by considerations in dynamical systems. Namely, the occurrence of a stretching pair within a periodic orbit of a continuous interval map implies what is called 'turbulence' (see [3, 12] for details). These considerations are closely related to the celebrated theorem of Sharkovsky [14]. From this point of view, the second author [12] considered which n-cycles $\pi \in C_n$ admit stretching pairs, and proved that all but $o(n-1)!$ of them do. Theorem 1 allows us to sharpen this result in the following way.

Corollary 7. *For all $\delta > 0$, there exists $0 < c < 1$ so that for all sufficiently large integers n, all but $c^n(n-1)!$ cyclic permutations $\pi \in C_n$ admit $\frac{1}{12}\binom{n}{2}(1 \pm \delta)$ stretching pairs.*

Proof of Corollary 7. Let $\delta > 0$ be given. Set $k = 4$ and $\mu = \delta/2$, and let $0 < c_1 < 1$ be the constant guaranteed by Theorem 1. Define c to be any constant satisfying $c_1 < c < 1$, and let n be sufficiently large. For an n-cycle $\pi \in C_n$, write $\sigma(\pi)$ for the number of stretching pairs of π, and write $\sigma'(\pi)$ for the number of stretching pairs $1 \leqslant \pi(i) < i < j < \pi(j) \neq n$. Note that $\sigma'(\pi) \leqslant \sigma(\pi) \leqslant \sigma'(\pi) + n$, since if $1 \leqslant \pi(i) < i < j < \pi(j) = n$, then $j = \pi^{-1}(n)$ is fixed and there are at most $j - 1 \leqslant n$ choices for i. Note, moreover, that it follows from (25) that, for $p = \phi(\pi) \in S_{n-1}$, $\sigma'(\pi) = f_{21\text{-}34}(p) + f_{34\text{-}21}(p)$. Theorem 1 ensures that all but $2c_1^{n-1}(n-1)! < c^n(n-1)!$ permutations $p \in S_{n-1}$ satisfy

$$f_{21\text{-}34}(p) = (1 \pm \mu)\frac{1}{4!}\binom{n-1}{2} \quad \text{and} \quad f_{34\text{-}21}(p) = (1 \pm \mu)\frac{1}{4!}\binom{n-1}{2}.$$

For each such permutation $p \in S_{n-1}$, the corresponding n-cycle $\pi = \phi^{-1}(p) \in C_n$ satisfies

$$\sigma(\pi) = (1 \pm \mu)\frac{1}{4!}\binom{n-1}{2} + (1 \pm \mu)\frac{1}{4!}\binom{n-1}{2} \pm n \tag{27}$$

$$= (1 \pm \mu \pm o(1))\frac{1}{12}\binom{n}{2} = (1 \pm \delta)\frac{1}{12}\binom{n}{2}, \tag{28}$$

which proves Corollary 7. □

5 Proof of Lemma 6

Fix $\zeta > 0$ and integer $q \geqslant 1$. Define auxiliary constant

$$\zeta_0 = \zeta/4. \tag{29}$$

Define $s_0 = s_0(q, \zeta_0)$ to be the least integer s for which

$$(s)_q \geqslant (1 - 2\zeta_0)s^q. \tag{30}$$

Let $s \geqslant s_0$ be given. Define

$$t_0 = \left\lceil 4q8^q s^{2q}\zeta_0^{-2} \right\rceil. \tag{31}$$

Let integer $t \geqslant t_0$ be given. Define

$$c_0 = \frac{\zeta_0^2}{3qt2^{q+3}s^q}. \tag{32}$$

Let n be a sufficiently large integer, and fix $(\mathbf{i}_0, j_0) \in ([s])_q \times [t]$. We prove

$$\text{all but } \exp\{-2c_0 n\}n! \text{ permutations } \pi \in S_n \text{ are } (\mathbf{i}_0, j_0, \zeta, q)\text{-typical w.r.t. } (\boldsymbol{P}_s, \boldsymbol{P}_t). \tag{33}$$

Applying (33) to all $(\mathbf{i}, j) \in ([s])_q \times [t]$ and noting $s^q t \exp\{-2c_0 n\} < \exp\{-c_0 n\}$ yields Lemma 6.

We now outline our approach for proving (33) (and reduce the $\hat{\ell}$ notation in (10) to ℓ). Define equivalence relation $\sim$ on E_{j_0}: $\ell \sim \ell' \iff q \,|\, (\ell - \ell')$. Thus, for an integer $0 \leqslant r < q$, we may write

$$E_{j_0}^{(r)} = \left\{ \ell \in E_{j_0} : \ell \sim (j_0 - 1)\left\lfloor \frac{n}{t} \right\rfloor + 1 + r \right\} \qquad \text{so that} \qquad E_j = E_j^{(0)} \cup \cdots \cup E_j^{(q-1)} \tag{34}$$

is a partition. A key observation for later in the proof (cf. Claim 8) will be that

$$[\ell, \ell + 1 - q] \cap [\ell', \ell' + q - 1] = \emptyset \quad \text{whenever } \ell \neq \ell' \in E_j^{(r)}. \tag{35}$$

For some final notation, we shall write, for a permutation $\pi \in S_n$,

$$E^{(r)}_{\mathbf{i}_0 j_0}(\pi) = E_{\mathbf{i}_0 j_0}(\pi) \cap E^{(r)}_{j_0} \qquad \text{so that} \qquad E_{\mathbf{i}_0 j_0}(\pi) = E^{(0)}_{\mathbf{i}_0 j_0}(\pi) \cup \cdots \cup E^{(q-1)}_{\mathbf{i}_0 j_0}(\pi) \tag{36}$$

is a partition. We shall prove that, for a fixed $0 \leqslant r < q$,

$$\textit{all but } \exp\{-3c_0 n\} n! \textit{ permutations } \pi \in S_n \textit{ satisfy that } \left| E^{(r)}_{\mathbf{i}_0 j_0}(\pi) \right| \geqslant (1-\zeta) \frac{1}{q(s)_q} |E_{j_0}|. \tag{37}$$

Note that (37) implies (33) since then all but $q \exp\{-3c_0 n\} n! < \exp\{-2c_0 n\} n!$ many $\pi \in S_n$ satisfy

$$|E_{\mathbf{i}_0 j_0}(\pi)| \overset{(36)}{=} \sum_{r=0}^{q-1} \left| E^{(r)}_{\mathbf{i}_0 j_0}(\pi) \right| \geqslant (1-\zeta) \frac{1}{(s)_q} |E_{j_0}|.$$

To prove (37), let $\pi \in S_n$ be chosen uniformly at random. Then, $Y = Y^{(r)}_{\mathbf{i}_0 j_0} = |E^{(r)}_{\mathbf{i}_0 j_0}(\pi)|$ is a random variable whose mean we evaluate. To that end, recall from (10) that for an element $\ell \in E_{j_0}$ to be an element of $E_{\mathbf{i}_0 j_0}(\pi)$, we require that $\ell \leqslant j_0 \lfloor n/t \rfloor - q + 1$, where we will write $n_t = \lfloor n/t \rfloor$ and $n_s = \lfloor n/s \rfloor$. As such, delete the last $q-1$ elements from E_{j_0}, and write

$$\tilde{E}_{j_0} \overset{\text{def}}{=} \big[(j_0 - 1)n_t + 1,\ j_0 n_t - q + 1\big], \quad \tilde{E}^{(r)}_{j_0} = E^{(r)}_{j_0} \cap \tilde{E}_{j_0},$$
$$\text{and} \quad n_{t,q} \overset{\text{def}}{=} \left| \tilde{E}^{(r)}_{j_0} \right| = \left\lfloor \frac{n_t - q + 1}{q} \right\rfloor = \left\lfloor \frac{n_t + 1}{q} \right\rfloor - 1. \tag{38}$$

Now, for $\ell \in \tilde{E}^{(r)}_{j_0}$, define indicator random variable Y_ℓ by (cf. $\mathbf{i}_0 = (i_1, \ldots, i_q)$)

$$Y_\ell = \begin{cases} 1 & \text{if } \pi(\ell + m - 1) \in I_{i_m} \ \forall\, m \in [q], \\ 0 & \text{otherwise,} \end{cases} \qquad \Longrightarrow \qquad Y = \sum \left\{ Y_\ell : \ell \in \tilde{E}^{(r)}_{j_0} \right\}$$

$$\text{so that} \quad \mathbb{E}[Y_\ell] = \frac{(n-q)! \prod_{m=1}^q |I_{i_m}|}{n!} = \frac{n_s^q}{(n)_q} \implies \mathbb{E}[Y] = \frac{|\tilde{E}^{(r)}_{j_0}| \prod_{m=1}^q |I_{i_m}|}{(n)_q} = \frac{n_s^q n_{t,q}}{(n)_q}. \tag{39}$$

Following the method of Bernstein for the Chernoff inequality (cf. [10]), for $u = \log(1 - \zeta_0) = \log_e(1 - \zeta_0)$, the Markov inequality implies

$$\mathbb{P}[Y \leqslant \mathbb{E}[Y](1-\zeta_0)] = \mathbb{P}\left[e^{uY} \geqslant \exp\{u\mathbb{E}[Y](1-\zeta_0)\}\right]$$
$$\leqslant \exp\{-u\mathbb{E}[Y](1-\zeta_0)\} \mathbb{E}\left[e^{uY}\right] \overset{(39)}{=} \exp\left\{ -u \frac{n_s^q n_{t,q}}{(n)_q}(1-\zeta_0) \right\} \mathbb{E}\left[e^{uY}\right]. \tag{40}$$

While we do not have mutual independence among the Y_ℓ's, we will prove the following.

Claim 8.

$$\mathbb{E}\left[e^{uY}\right] = \mathbb{E}\left[\prod_{\ell \in \tilde{E}^{(r)}_{j_0}} e^{uY_\ell} \right] \leqslant \left(1 + q\frac{(4s)^q}{t}\right)^{n_{t,q}} \prod_{\ell \in \tilde{E}^{(r)}_{j_0}} \mathbb{E}\left[e^{uY_\ell}\right]$$
$$\overset{(39)}{=} \left(\left(1 + q\frac{(4s)^q}{t}\right) \left(1 + \frac{n_s^q}{(n)_q}(e^u - 1)\right) \right)^{n_{t,q}} \leqslant \exp\left\{ n_{t,q} \left(q\frac{(4s)^q}{t} + \frac{n_s^q}{(n)_q}(e^u - 1) \right) \right\}.$$

We shall defer the proof of Claim 8 in order first to finish the proof of (37).

Applying Claim 8 to (40), together with the Taylor series bound $-u(1-\zeta_0)+e^u-1 \leqslant -\zeta_0^2/2$,

$$\begin{aligned}
\mathbb{P}[Y \leqslant \mathbb{E}[Y](1-\zeta_0)] &\leqslant \exp\left\{n_{t,q}\left(q\frac{(4s)^q}{t}+\frac{n_s^q}{(n)_q}\Big(-u(1-\zeta_0)+e^u-1\Big)\right)\right\} \\
&\leqslant \exp\left\{n_{t,q}\left(q\frac{(4s)^q}{t}-\frac{\zeta_0^2 n_s^q}{2(n)_q}\right)\right\} \leqslant \exp\left\{n_{t,q}\left(q\frac{(4s)^q}{t}-\frac{\zeta_0^2}{2}\left(\frac{n_s}{n}\right)^q\right)\right\} \\
&\leqslant \exp\left\{n_{t,q}\left(q\frac{(4s)^q}{t}-\frac{\zeta_0^2}{2^{q+1}s^q}\right)\right\} \qquad (\text{since } n_s=\lfloor n/s\rfloor \geqslant n/(2s)) \\
&\overset{(31)}{\leqslant} \exp\left\{n_{t,q}\left(-\frac{\zeta_0^2}{2^{q+2}s^q}\right)\right\} \leqslant \exp\left\{-\left(\frac{\zeta_0^2}{qt2^{q+3}s^q}\right)n\right\} \overset{(32)}{=} \exp\{-3c_0 n\}.
\end{aligned}$$

(The last inequality above follows from $n_{t,q}=\lfloor (n_t+1)/q\rfloor-1 \geqslant n/(2tq)$.) In other words, with probability $1-\exp\{-3c_0n\}$, the randomly chosen permutation $\pi \in S_n$ satisfies

$$\begin{aligned}
Y = \left|E^{(r)}_{\mathbf{i}_0 j_0}(\pi)\right| &\geqslant \mathbb{E}[Y](1-\zeta_0) \overset{(39)}{=} \frac{n_s^q n_{t,q}}{(n)_q}(1-\zeta_0) \overset{(38)}{=} (1-\zeta_0)(1-o(1))\frac{1}{qs^q}\left\lfloor\frac{n}{t}\right\rfloor \\
&\overset{(30)}{\geqslant} (1-2\zeta_0)^2\frac{1}{q(s)_q}\left\lfloor\frac{n}{t}\right\rfloor \geqslant (1-4\zeta_0)\frac{1}{q(s)_q}\left\lfloor\frac{n}{t}\right\rfloor \overset{(29)}{=} (1-\zeta)\frac{1}{q(s)_q}\left\lfloor\frac{n}{t}\right\rfloor.
\end{aligned}$$

5.1 Proof of Claim 8

Write $\tilde{E}^{(r)}_{j_0}$ as $\ell_1 < \cdots < \ell_{n_{t,q}}$ (cf. (38)) so that

$$\mathbb{E}\left[\prod_{\ell \in \tilde{E}^{(r)}_{j_0}} e^{uY_\ell}\right] = \sum_{(y_1,\ldots,y_{n_{t,q}})\in\{0,1\}^{n_{t,q}}} \mathbb{P}\left[\bigwedge_{i=1}^{n_{t,q}} Y_{\ell_i}=y_i\right]\prod_{i=1}^{n_{t,q}} e^{uy_i}. \tag{41}$$

Fix $(y_1,\ldots,y_{n_{t,q}}) \in \{0,1\}^{n_{t,q}}$ so that

$$\mathbb{P}\left[\bigwedge_{i=1}^{n_{t,q}} Y_{\ell_i}=y_i\right] = \mathbb{P}\left[Y_{\ell_{n_{t,q}}}=y_{n_{t,q}} \,\middle|\, \bigwedge_{j=1}^{n_{t,q}-1} Y_{\ell_j}=y_j\right]\cdot \mathbb{P}\left[\bigwedge_{j=1}^{n_{t,q}-1} Y_{\ell_j}=y_j\right].$$

We claim that

$$\mathbb{P}\left[Y_{\ell_{n_{t,q}}}=y_{n_{t,q}} \,\middle|\, \bigwedge_{j=1}^{n_{t,q}-1} Y_{\ell_j}=y_j\right] = \mathbb{P}\left[Y_{\ell_{n_{t,q}}}=y_{n_{t,q}}\right]\left(1 \pm q\frac{(4s)^q}{t}\right). \tag{42}$$

If so, iteratively applying (42) to (41) yields Claim 8.

To see (42), recall the observation in (35). Thus,

$$\mathbb{P}\left[Y_{\ell_{n_{t,q}}}=1\Big|\bigwedge_{j=1}^{n_{t,q}-1}Y_{\ell_j}=y_j\right]\leqslant\frac{(n-qn_{t,q})!\prod_{m=1}^{q}|I_{i_m}|}{(n-q(n_{t,q}-1))!}=\frac{n_s^q}{(n-qn_{t,q}+q)_q},\text{ and}$$

$$\mathbb{P}\left[Y_{\ell_{n_{t,q}}}=1\Big|\bigwedge_{j=1}^{n_{t,q}-1}Y_{\ell_j}=y_j\right]\geqslant\frac{(n-qn_{t,q})!\prod_{m=1}^{q}(|I_{i_m}|-q(n_{t,q}-1))}{(n-q(n_{t,q}-1))!}$$
$$=\frac{(n_s-qn_{t,q}+q)^q}{(n-qn_{t,q}+q)_q}.$$

For the upper bound, we use (39) (and $qn_{t,q}\leqslant n_t$ (cf. (38)) to infer

$$\frac{n_s^q}{(n-qn_{t,q}+q)_q}=\mathbb{P}\left[Y_{\ell_{n_{t,q}}}=1\right]\cdot\frac{(n)_q}{(n-qn_{t,q}+q)_q}\leqslant\mathbb{P}\left[Y_{\ell_{n_{t,q}}}=1\right]\left(\frac{n}{n-n_t}\right)^q$$
$$\leqslant\mathbb{P}\left[Y_{\ell_{n_{t,q}}}=1\right]\left(1-\frac{1}{t}\right)^{-q}\leqslant\mathbb{P}\left[Y_{\ell_{n_{t,q}}}=1\right]\left(1+\frac{2}{t}\right)^q\leqslant\mathbb{P}\left[Y_{\ell_{n_{t,q}}}=1\right]\left(1+q\frac{4^q}{t}\right).$$

For the lower bound, we similarly infer

$$\frac{(n_s-qn_{t,q}+q)^q}{(n-qn_{t,q}+q)_q}\geqslant\frac{(n_s-qn_{t,q})^q}{(n)_q}=\mathbb{P}\left[Y_{\ell_{n_{t,q}}}=1\right]\left(\frac{n_s-qn_{t,q}}{n_s}\right)^q$$
$$\geqslant\mathbb{P}\left[Y_{\ell_{n_{t,q}}}=1\right]\left(\frac{n_s-n_t}{n_s}\right)^q\geqslant\mathbb{P}\left[Y_{\ell_{n_{t,q}}}=1\right]\left(1-2\frac{s}{t}\right)^q$$
$$\geqslant\mathbb{P}\left[Y_{\ell_{n_{t,q}}}=1\right]\left(1-q\frac{(4s)^q}{t}\right).$$

This proves (42) when $y_{n,t}=1$. Otherwise, (using $\mathbb{P}[Y_{\ell_{n_{t,q}}}=1]\leqslant 1/2\leqslant\mathbb{P}[Y_{\ell_{n_{t,q}}}=0]$) we have

$$\mathbb{P}\left[Y_{\ell_{n_{t,q}}}=0\Big|\bigwedge_{j=1}^{n_{t,q}-1}Y_{\ell_j}=y_j\right]=1-\mathbb{P}\left[Y_{\ell_{n_{t,q}}}=1\Big|\bigwedge_{j=1}^{n_{t,q}-1}Y_{\ell_j}=y_j\right]$$
$$=\mathbb{P}\left[Y_{\ell_{n_{t,q}}}=0\right]\left(1\pm q\frac{(4s)^q}{t}\right),$$

where we used $\mathbb{P}[Y_{\ell_{n_{t,q}}}=1]\leqslant 1/2\leqslant\mathbb{P}[Y_{\ell_{n_{t,q}}}=0]$.

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