Residue reduced form of a rational function as an iterated Laurent series

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Abstract

Lipshitz showed that the diagonal of a D-finite power series is still D-finite, but his proof seems hard to implement. This paper may be regarded as the first step towards an efficient algorithm realizing Lipshitz's theory. We show that the idea of a reduced form may be a big saving for computing the D-finite functional equation. For the residue in one variable of a rational function, we develop an algorithm for computing its minimal algebraic functional equation.

Keywords: diagonal; residue; algebraic; D-finite

1 Introduction

The following rational power series

$$A(x,y,z)=\frac{1}{(1-x)^2(1-y)^2(1-z)^2(1-y-x)(1-z-y)(1-z-x)},$$

came from polyomino enumerations [2]. Recently the first author was asked to compute the diagonal $F(t)$ of $A(x,y,z)$, that is,

$$F(t)=\sum_{n\geqslant 0}[x^ny^nz^n]A(x,y,z)t^n=\operatorname*{CT}_{x,y}A(x,y,t/xy),$$

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where $[x^n y^n z^n]$ means to take the coefficient of $x^n y^n z^n$, $\mathrm{CT}_{x,y}$ means to take the constant term in x and y, and $A(x, y, t/xy)$ is regarded as a power series in t with coefficients Laurent series in x and y. This is our archetype.

The general theory of Lipshitz shows that the diagonal generating function $F(t)$ is D-finite [3], which means that there exist polynomials $a_0(t), a_1(t), \ldots, a_d(t)$ such that $a_d(t) \neq 0$ and

$$a_d(t)F^{(d)}(t) + a_{d-1}F^{(d-1)}(t) + \cdots + a_0F(t) = 0.$$

However Lipshitz's proof seems to give too large a degree bound to efficiently deliver the desired D-finite functional equation.

In this paper we show how the idea of reduced form can be used to develop an efficient algorithm realizing Lipshitz theory, at least in some special cases. The basic idea is as follows. Suppose we need to compute the residue $\mathrm{res}_x\, Q(x,t)$. Since the residue of a differential is 0, we can deal with $\mathrm{res}_x\, \overline{Q}(x,t)$ instead where $\overline{Q}(x,t) = Q(x,t) - \partial_x P(x,t)$ for a suitably chosen $P(x,t)$. Then $\overline{Q}(x,t)$ will have some nice minimal property leading to some nice consequences.

This idea works nicely when $Q(x,t)$ is a rational function (usually containing other variables), which is the main object of study in this paper. We use the field of iterated Laurent series of [7] to characterize the series expansion of $Q(x,t)$. See Section 3 for a brief description. Then the reduced form can be simplified further. As a consequence, a D-finite equation of small order for $\mathrm{res}_x\, Q(x,t)$ is easily constructed. It is also well-known that $\mathrm{res}_x\, Q(x,t)$ is in fact algebraic. We present an algorithm for computing the minimal algebraic functional equation for $\mathrm{res}_x\, Q(x)$ without using details of the roots of the denominator. As an example, we find that the diagonal generating function $F(t)$ is algebraic of degree 6, with coefficients polynomials in t of degree 61.

Many combinatorial problems can be reduced to the computation of the residue in several variables of certain rational functions. Given a rational function, it is a hard problem even for deciding the nullity of its residue. Lipshitz's theorem applies, but it is clearly not economic to use D-finiteness to characterize rational functions. Thus our algorithm may be taken as the initial step before applying Lipshitz's theorem. The potential for the idea of reduced form in the general case is discussed in Section 8.

2 Residue reduced form of a rational function

Let K be a field of characteristic 0 and x a variable. Given a rational function $Q(x) \in K(x)$, define $Q_1(x)$ to be residue equivalent to $Q(x)$ if there exists $R(x)$ such that $Q(x) = Q_1(x) + \partial_x R(x)$, where ∂_x is short for $\frac{\partial}{\partial x}$. Denote

$$Q(x) \underset{\mathrm{res}}{\equiv} Q_1(x) \Leftrightarrow Q(x) = Q_1(x) + \partial_x R(x) \text{ for some } R(x) \in K(x).$$

There are existing algorithms for finding $Q_1(x)$ and $R(x)$ simultaneously with $Q_1(x)$ minimal in some sense. Such a representation is useful when integrating $Q(x)$. In the context of residue computation, we are only interested in finding the minimal $Q_1(x)$, since the residue of $\partial_x R(x)$ is always 0, which explains the term residue equivalent. The following result is well-known.

Proposition 1. *Suppose $Q(x) \in K(x)$ has denominator $D_1(x)^{r_1} \cdots D_k(x)^{r_k}$, where the $D_i(x) \in K[x]$ are irreducible. Then there exist $p_1(x), \ldots, p_k(x) \in K[x]$ with $\deg p_i(x) < \deg D_i(x)$ such that*

$$Q(x) \underset{\text{res}}{\equiv} \frac{p_1(x)}{D_1(x)} + \cdots + \frac{p_k(x)}{D_k(x)}.$$

The right hand side is called the RRF (short for residue reduced form) of $Q(x)$.

Proof. By the partial fraction decomposition, we may write

$$Q(x) = P(x) + \frac{N_1(x)}{D_1(x)^{r_1}} + \cdots + \frac{N_k}{D_k(x)^{r_k}},$$

where $P(x)$ is a polynomial and $N_i(x)$ is a polynomial of degree less than $r_i \deg D_i(x)$. It is clear that $P(x) \equiv_{\text{res}} 0$ since $\int P(x)dx$ is a polynomial. Thus it is sufficient to show that

$$\frac{N_i(x)}{D_i(x)^{r_i}} \underset{\text{res}}{\equiv} \frac{p_i(x)}{D_i(x)} \tag{1}$$

holds for some $p_i(x) \in K[x]$. We prove this by induction on r_i. Equation (1) clearly holds when $r_i = 1$. Assume (1) holds for $r_i < r$. Consider

$$\partial_x \frac{L(x)}{D_i(x)^{r-1}} = \frac{\partial_x L(x)}{D_i(x)^{r-1}} - \frac{(r-1)L(x)\partial_x D_i(x)}{D_i(x)^r}.$$

Choose $L(x) = \text{rem}(\frac{1}{r-1}N_i(x)\alpha(x), D_i(x))$, where $\text{rem}(A, B)$ denotes the reminder of A divided by B and $\alpha(x)\partial_x D_i(x) + \beta(x)D_i(x) = 1$. Then $N_i(x) - (r-1)L(x)\partial_x D_i(x) = 0 \pmod{D_i(x)}$. This implies that

$$\frac{N_i(x)}{D_i(x)^r} + \partial_x \frac{L(x)}{D_i(x)^{r-1}} = \frac{\overline{N}(x)}{D_i(x)^{r'}}$$

for some $\overline{N}(x)$ and $r' < r$. Then by the induction hypothesis we have

$$\frac{N_i(x)}{D_i(x)^r} \underset{\text{res}}{\equiv} \frac{\overline{N}(x)}{D_i(x)^{r'}} \underset{\text{res}}{\equiv} \frac{p_i(x)}{D_i(x)}.$$

This completes the proof. □

Here is an outline of the algorithm for RRF.

RRF algorithm

INPUT: A rational function $Q(x)$.

OUTPUT: The RRF of $Q(x)$ in x.

1. As in the proof, write $Q(x)$ as: $Q(x) = P(x) + \sum_{i=1}^{k} \frac{N_i(x)}{D_i(x)^{r_i}}$.

2. For $i = 1, \dots, k$ do:

   let $r := r_i$, $P_i(x) := N_i(x)$, and find $\alpha(x)$ so that $\alpha(x)\partial_x D_i(x) + \beta(x) D_i(x) = 1$;

   while $r > 1$ do:

   find $P_i'(x)$ such that

$$\frac{P_i'(x)}{D_i(x)^{r'}} = \frac{P_i(x)}{D_i(x)^r} + \partial_x \frac{L(x)}{D_i(x)^{r-1}},$$

   where $L(x) := \mathrm{rem}(\frac{1}{r-1} P_i(x)\alpha(x), D_i(x))$;

   let $r := r'$, $P_i(x) := P_i'(x)$.

   $p_i(x) := \mathrm{rem}(P_i(x), D_i(x))$.

3. Return $\frac{p_1(x)}{D_1(x)} + \cdots + \frac{p_k(x)}{D_k(x)}$.

□

3 Residue reduced form in a field of iterated Laurent series

Let $\mathcal{K} = K((x_n))\cdots((x_1))$ be the field of iterated Laurent series and let x be one of the variables. Then a rational function $Q(x)$ has a unique series expansion in $\mathcal{K}$ and hence the residue $\mathrm{res}_x\, Q(x)$ is defined to be the residue of the series expansion of $Q(x)$. It should be noted that $Q(x)$ may have different residue in different field of iterated Laurent series, so we must specify the working field when taking residues. This treatment works for almost all residue calculations since $\mathcal{K}$ is maximal in some sense. In particular $\mathcal{K}$ includes the field $K(x_1, \dots, x_n)$ of rational functions as a subfield. See [7] for more details.

Our ultimate goal is to compute or characterize the residue $\mathrm{res}_x\, Q(x)$ of a rational function $Q(x)$. We need some basic facts for elements in $\mathcal{K}$. The field $\mathcal{K}$ defines a total order $0 < x_1 \ll x_2 \ll \cdots \ll x_n < 1$ on the variables, which induces a total group order on the group of monomials. Here we only need the following two facts.

1. A monomial $M = x_1^{k_1} \cdots x_n^{k_n} \neq 1$ is said to be *small* if $k_1 = \cdots = k_s = 0$, $k_{s+1} > 0$, otherwise it is said to be *large*. The initial term of a Laurent polynomial is the term with the largest monomial.

   For example, in the field $K((x))((y))((t))$, the initial term of $x + x^2y + t/x$ is x.

2. If $F \in \mathcal{K}$ and the initial term of $1-F$ is 1, then $(1-F)^{-1}$ and $\log(1-F)^{-1}$ are also in $\mathcal{K}$ and we have the following series expansion:

$$\frac{1}{1-F} = \sum_{m\geqslant 0} F^m, \quad \text{and} \quad \log\frac{1}{1-F} = \sum_{m\geqslant 1} \frac{1}{m} F^m.$$

Let $D(x)$ be an irreducible polynomial of degree d with initial term cM. Then $D(x)$ is said to be *crucial* if $1 \leqslant \deg_x cM \leqslant d-1$. Our main result is the following.

Theorem 2. *Suppose a rational function $Q(x)$ has denominator $D_1(x)^{r_1}\cdots D_k(x)^{r_k}$. Then there exist $p_i(x)$ and a constant C free of x such that it holds in $\mathcal{K}$ that*

$$Q(x) \underset{\text{res}}{\equiv} \frac{C}{x} + \sum_i \frac{p_i(x)}{D_i(x)},$$

where the sum ranges over all crucial irreducible factors D_i, and $\deg_x p_i(x) \leqslant \deg_x D_i(x) - 2$. The right hand side is called the ILSRRF (Iterated Laurent series residue reduced form) of $Q(x)$.

Proof. By Proposition 1, we may assume the RRF of $Q(x)$ is given by

$$Q(x) = \frac{p_1(x)}{D_1(x)} + \cdots + \frac{p_k(x)}{D_k(x)}.$$

By linearity, it is sufficient to consider $p(x)/D(x)$. Let cM be the initial term of $D(x)$, and denote by $d = \deg_x D(x)$, and $\ell = \deg_x cM$. Write $D(x) = a_0 + a_1x + \cdots + a_dx^d$, and $p(x) = b_0 + b_1x + \cdots + b_{d-1}x^{d-1}$. Note that $a_0 \neq 0$ unless $D(x) = x$, since $D(x)$ is irreducible. We can simplify by the following three cases.

i) If $\ell = 0$, then

$$\frac{p(x)}{D(x)} = \frac{p(x)}{a_0(1 + a_1x/a_0 + \cdots + a_dx^d/a_0)} = \sum_{n\geqslant 0} \frac{p(x)}{a_0}(-1)^n(a_1x/a_0 + \cdots + a_dx^d/a_0)^n$$

has only nonnegative powers in x. It follows that $\operatorname{res}_x p(x)/D(x) = 0$.

ii) If $\ell = d$, then

$$\begin{aligned}\frac{p(x)}{D(x)} &= \frac{p(x)}{a_dx^d(1 + \frac{a_{d-1}}{a_dx} + \cdots + \frac{a_0}{a_dx^d})}\\ &= \sum_{n\geqslant 0}(\frac{b_0}{a_dx^d} + \frac{b_1}{a_dx^{d-1}} + \cdots + \frac{b_{d-1}}{a_dx})(-1)^n(\frac{a_{d-1}}{a_dx} + \cdots + \frac{a_0}{a_dx^d})^n.\end{aligned}$$

It follows that $\operatorname{res}_x p(x)/D(x) = b_{d-1}/a_d$.

iii) If $0 < \ell < d$, then $D(x)$ is crucial and we do not have an expression as simple as above. However, since the initial term of $D(x)/a_\ell x^\ell$ is 1, $\log\frac{D(x)}{a_\ell x^\ell} \in \mathcal{K}$. Therefore

$$\operatorname*{res}_x \partial_x \log\frac{D(x)}{a_\ell x^\ell} = \operatorname*{res}_x \frac{\partial_x D(x)}{D(x)} - \ell = 0.$$

Thus if we write

$$\frac{p(x)}{D(x)} - \frac{b_{d-1}}{da_d}\frac{\partial_x D(x)}{D(x)} = \frac{\overline{p}(x)}{D(x)},$$

then $\overline{p}(x)$ has degree no more than $d-2$ and it follows that

$$\operatorname*{res}_x \frac{p(x)}{D(x)} = \operatorname*{res}_x \frac{\overline{p}(x)}{D(x)} + \frac{\ell b_{d-1}}{da_d}.$$

□

Based on the proof, we give an algorithm for computing the ILSRRF of a rational function.

ILSRRF algorithm

INPUT: A rational function $Q(x)$ and a working field $K((x_n))\cdots((x_1))$, x is one of the variables.

OUTPUT: The ILSRRF of $Q(x)$ in x.

1. By the RRF algorithm, let the RRF of $Q(x)$ be

$$Q(x) = \frac{p_1(x)}{D_1(x)} + \cdots + \frac{p_k(x)}{D_k(x)}.$$

2. Let $C := 0$;

   for $i = 1, \ldots, k$ do:

   $d := \deg_x D_i(x)$;

   $[x^d]D_i(x) := a_d$;

   $[x^{d-1}]p_i(x) := b_{d-1}$.

   find the initial term cM of $D_i(x)$ and let $l := \deg_x cM$;

   if $l = d$, then $C := C + b_{d-1}/(a_d x)$;

   else if $l > 0$, then $C := C + p_i(x)/D_i(x) - b_{d-1}\partial_x D_i(x)/(da_d D_i(x)) + lb_{d-1}/(da_d x)$.

3. Return C.

□

4 The D-finite functional equation

We follow the notations of the last section. Let t be another variable. An iterated Laurent series $G(t) \in \mathcal{K}$ is said to be D-finite if there exist polynomials $a_0(t), a_1(t), \dots, a_d(t)$ such that

$$a_d \partial_t^d G(t) + \cdots + a_1 \partial_t G(t) + a_0 G(t) = 0,$$

where a_d is nonzero. The smallest such d is called the order of the D-finiteness of $G(t)$. This slightly extends Stanley's concept of D-finite generating functions. See [6] for more information on the concepts of D-finite and P-recursive.

A direct consequence of Theorem 2 is the following.

Theorem 3. *Let $Q(x,t)$ be a rational function with irreducible denominator factors $D_1, \dots, D_k$. Then the residue $\operatorname{res}_x Q(x,t)$ in $\mathcal{K}$ is D-finite of order at most $1+\sum_i (\deg_x D_i - 1)$, where the sum ranges over all crucial D_i.*

Proof. Assume the crucial D_i's are $D_1, \dots, D_s$. Write $G(t) = \operatorname{res}_x Q(x,t)$. Let $\mathcal{L}$ be the linear space spanned by x^{-1} and $\{x^{i_j}/D_i(x,t) : 1 \leqslant i \leqslant s,\ 0 \leqslant i_j \leqslant \deg_x D_i(x,t) - 2\}$, where the coefficients are rational functions free of x. Then $\dim \mathcal{L} = d := 1 + \sum_{i=1}^s (\deg_x D_i(x,t) - 1)$.

By Theorem 2, there exists $Q_0(x,t) \in \mathcal{L}$ such that $G(t) = \operatorname{res}_x Q_0(x,t)$. Recursively define $Q_i(x,t)$ to be the ILSRRF of $\partial_t Q_{i-1}(x,t)$. Then it is clear that

$$\frac{\partial^i}{\partial t^i} G(t) = \operatorname*{res}_x Q_i(x,t), \text{ and } Q_i(x,t) \in \mathcal{L}.$$

Then by the method of undetermined coefficients we can find the a_i's not all zero such that

$$a_0 Q_0(x,t) + a_1 Q_1(x,t) \cdots + a_d Q_d(x,t) = 0,$$

which implies the D-finiteness of $G(t)$ through taking residues. □

Theorem 3 guarantees the existence of an annihilating operator $L = a_d \partial_t^d + \cdots + a_1 \partial_t + a_0$ for $\operatorname{res}_x Q(x,t)$, i.e., $L \operatorname{res}_x Q(x,t) = 0$. Suppose the minimal annihilating operator is M. It is not hard to show that M must be a right factor of L. Thus to find M, we need to search for all right factors of L. This is hard since factorization for differential operators is hard, due to the noncommutative relation $\partial_t t = 1 + t\partial_t$.

5 The algebraic functional equation

We first recall some known results. Let K be a field. An element α is algebraic over K if there is a polynomial $p(x) \in K[x]$ such that $p(\alpha) = 0$. Such a polynomial of minimal degree is called the minimal polynomial (we do not need the monic condition). Assume $p(x)$ is the minimal polynomial of α and $\deg p(x) = d$. Then it is clear that $p(x)$ is irreducible and $K(\alpha)$ is isomorphic to $K[x]/\langle p(x) \rangle$. Thus when regarded as a K-linear space, $K(\alpha)$ has the canonical basis $\{1, \alpha, \dots, \alpha^{d-1}\}$, and hence has dimension d. A direct consequence is the following.

Lemma 4. *Suppose α is algebraic of degree d. Then for any rational function $R(x)$, $R(\alpha)$ is algebraic of degree at most d.*

Proof. Define $P_0(x) = 1$, and define $P_i(x)$ with degree less than d to be the representation of $R^i(\alpha)$ as $P_i(\alpha)$ for $i = 0, 1, \dots, d$. Then $P_0(x), \dots, P_d(x)$ are K-linearly dependent. By the method of undetermined coefficients, we can find the $a_i \in K$ not all zero so that $a_0P_0(x)+a_1P_1(x)+\cdots+a_dP_d(x) = 0$. Then $q(x) = a_0+a_1x+\cdots a_dx^d$ satisfies $q(P(\alpha)) = 0$. This completes the proof. □

From the above proof that we can easily get an algorithm for computing an algebraic functional equation for $R(\alpha)$.

AFE algorithm

INPUT: A rational function $R(x)$ and an algebraic functional equation $Eq(x)$ for α with degree d.

OUTPUT: An algebraic functional equation for $R(\alpha)$.

1. Let $P(\alpha)$ be the result of rationalizing of the denominator of $R(\alpha)$.
2. By the method of undetermined coefficients, find the $a_i \in K$ not all zero such that $a_0 + a_1P(\alpha) + \cdots + a_dP(\alpha)^d = 0$.
3. Return $a_0 + a_1x + \cdots a_dx^d$.

□

Note that in the AFE algorithm, we do not need the explicit form of α.

The following result is stronger than Lemma 4.

Lemma 5. *Suppose α is algebraic of degree d. Let $R(x)$ be a rational function and $m(x)$ be the minimal polynomial of $R(\alpha)$. Then $\deg m(x)$ divides d.*

Proof. Denote by $b = \deg m(x)$. Suppose $M(x)$ is the minimal polynomial of α and $M(x)$ has roots $\{\alpha_i\}_{1\leqslant i\leqslant d}$. We claim that $m(x)$ is also the minimal polynomial of $R(\alpha_i)$ for all i. This is because $m(R(\alpha_i)) = 0$ if and only if $m(R(x)) \equiv 0 \pmod{M(x)}$.

Consider $q(x) = (x-R(\alpha_1))\cdots(x-R(\alpha_d))$. Since $q(x)$ is symmetric in the α_i, it follows that $q(x) \in K[x]$ and hence $m(x)$ divides $q(x)$. On the other hand, each irreducible factor of $q(x)$ must vanish at $R(\alpha_i)$ for some i and therefore be divisible by $m(x)$. It follows that $q(x)$ must be equal to $m(x)^{d/b}$ up to a constant factor. □

Another useful fact is that the set of algebraic elements form a field. In particular, we need the following.

Lemma 6. *If α and β are both algebraic then $\alpha + \beta$ is algebraic.*

Proof. Let $\gamma = \alpha + \beta$. Suppose the minimal polynomials of α and β are $p(x)$ and $q(x)$ with $\deg p(x) = d$ and $\deg q(x) = \ell$. We provide two ways to find $P(x) \in K[x]$ such that $P(\gamma) = 0$.

The classical way: It is clear that $\gamma^n = (\alpha+\beta)^n$ is a linear combination of $\alpha^i\beta^j$, where $0 \leqslant i < d, 0 \leqslant j < \ell$. Therefore $\{(\alpha+\beta)^n : 0 \leqslant n \leqslant d\ell\}$ is linearly dependent, and hence we can find $P(x) \in K[x]$ of degree at most $d\ell$ such that $P(\gamma) = 0$ by the method of undetermined coefficients.

An alternative way: Construct $P(x) = \prod_{i=1}^{d}\prod_{j=1}^{\ell}(x-\alpha_i-\beta_j)$, where the α_i are all the roots of $p(x)$ and the β_j are all the roots of $q(x)$. Then it is clear that $P(\alpha+\beta) = 0$, and $P(x)$ is symmetric in the α_i's and in the β_j's and hence belongs to $K[x]$. It is a matter of how to rewrite $P(x)$ without the α_i and β_j. We can write $P(x) = p(x-\beta_1)\cdots p(x-\beta_\ell)$. Indeed, if we let $\overline{p}(z) = p(x-z)$ then the roots of $\overline{p}(z)$ are $x-\alpha_i$, and $P(x)$ is just the resultant of $\overline{p}(z)$ and $q(z)$, which has a determinant representation of size $d+\ell$. □

AFE2 algorithm

INPUT: An algebraic functional equation $p(x)$ for α and an algebraic functional equation $q(x)$ for β.

OUTPUT: A minimal polynomial of $\alpha+\beta$.

1. Let $R(x)$ be the resultant of $p(x-z)$ and $q(z)$ with respect to z.
2. Find an irreducible factor $P(x)$ of $R(x)$ such that $P(\alpha+\beta) = 0$.
3. Return $P(x)$.

□

Remark 7. It should be noted that if $\beta_i = r_i(\beta)$ for some polynomials $r_i(x)$, then $P(x) = p(x-r_1(\beta))\cdots p(x-r_\ell(\beta))$, whose standard representation contains no β. For example, if $\ell = 2$ and we know that one root β_1 of $q(x)$ is of the form $c_1 + c_2\sqrt{\beta}$, then $P(x)$ can be written as $p(x-c_1-c_2\sqrt{\beta})p(x-c_1+c_2\sqrt{\beta})$, which contains no $\sqrt{\beta}$ after simplification.

Remark 8. Let $\alpha_i, i = 1,\ldots,d$ be the roots of $p(x)$. We claim that $\gamma = \alpha_1+\cdots+\alpha_s$ is algebraic of degree at most $\binom{d}{s}$. Let $P(x) = \prod_{|S|=s}\left(x-\sum_{i\in S}\alpha_i\right)$. Then $P(\gamma) = 0$, $\deg P(x) = \binom{d}{s}$, and $P(x)$ is symmetric in the α_i and hence belongs to $K[x]$.

The following theorem slightly generalizes the well-known result that the diagonal of a rational generating function in two variables is algebraic.

Theorem 9. *For any rational function $Q(x)$ in $\mathcal{K}$, the residue of $Q(x)$ in x is algebraic, and there is an algorithm to find the minimal polynomial of* $\operatorname{res}_x Q(x)$.

Proof. The proof is classical. Here we address how to find the minimal polynomial of $\operatorname{res}_x Q(x)$.

Assume the ILSRRF of $Q(x)$ is a linear combination of $p(x)/D(x)$ as described in Theorem 2. Since linear combinations of algebraic elements are algebraic, it is sufficient to show that $\operatorname{res}_x p(x)/D(x)$ is algebraic.

In some field extension, $D(x)$ can be factored as $a_d(x-\alpha_1)\cdots(x-\alpha_d)$. We have the following partial fraction decomposition:

$$\begin{aligned}\frac{p(x)}{D(x)} &= \sum_{i=1}^{d} \frac{p(x)}{D(x)/(x-\alpha_i)}\Big|_{x=\alpha_i} \cdot \frac{1}{x-\alpha_i} \\ &= \sum_{i=1}^{d} \frac{p(\alpha_i)}{\partial_x D(\alpha_i)} \cdot \frac{1}{x-\alpha_i}.\end{aligned}$$

Suppose the initial term of $D(x)$ is cM with $s=\deg_x cM$. By renaming the α_i's we may assume α_i/x is small for $i=1,2,\ldots,r$ and α_i/x is large for $i=r+1,\ldots,d$. Then we have

$$\operatorname*{res}_x \frac{p(x)}{D(x)} = \sum_{i=1}^{r} \frac{p(\alpha_i)}{\partial_x D(\alpha_i)}.$$

By the AFE algorithm, we can find $P_i(x) \in K[x]$ of degree at most d such that $P_i(\frac{p(\alpha_i)}{\partial_x D(\alpha_i)}) = 0$. It is clear that $P_i(x)=P_j(x)$, and therefore by finding r we can apply Remark 8 to find $\overline{P}(x)$ having $\operatorname{res}_x \frac{p(x)}{D(x)}$ as a root.

We need not compute α_i to get the value of r. Indeed we must have $r=s$. With the condition given above, we can write

$$D(x) = (-1)^{d-s} a_d x^s \alpha_{r+1}\cdots\alpha_d (1-\alpha_1/x)\cdots(1-\alpha_s/x)(1-x/\alpha_{s+1})\cdots(1-x/\alpha_d),$$

where each factor in parenthesis has 1 as the initial term. It follows that the initial term of $D(x)$ must have degree r in x, and hence $r=s$.

Finally, by the AFE2 algorithm we can find a polynomial $A(z)$ having $\operatorname{res}_x Q(x)$ as a root. By checking for all irreducible factors of $A(z)$, we can find the minimal polynomial. □

Remark 10. The following fact will be used in our algorithm. Since $\deg p(x) \leqslant d-2$, we have

$$\sum_{i=1}^{d} \frac{p(\alpha_i)}{\partial_x D(\alpha_i)} = \frac{[x^{d-1}]p(x)}{[x^d]D(x)} = 0.$$

If we take $R(x) = p(x)/\partial_x D(x)$ in the proof of Lemma 5, then we deduce that $[x^{b-1}]m(x) = 0$.

Here is an outline of the algorithm for finding the minimal polynomial, suppose the working field is $\mathbb{C}((x))((t))$.

MP algorithm

INPUT: A rational function $Q(x)$ in $\mathcal{K}$.
OUTPUT: The minimal polynomial of $\operatorname{res}_x Q(x)$.

1. By the ILSRRF algorithm, let the ILSRRF of $\operatorname{res}_x Q(x)$ with respect to x be
$$\operatorname*{res}_x Q(x) \underset{\text{res}}{\equiv} C + \sum_{i=1}^{k} \operatorname*{res}_x \frac{p_i(x)}{D_i(x)}.$$
2. Find the initial term $cM_i(x)$ of each $D_i(x)$, and let $l_i := \deg_x cM_i(x)$; then $D_i(x)$ has l_i small roots, and
$$\operatorname*{res}_x \frac{p_i(x)}{D_i(x)} = \sum_{j=1}^{l_i} \frac{p_i(\alpha_j)}{\partial_x D_i(\alpha_j)},$$
where the $\alpha_j (j = 1, \ldots, l_i)$ are the small roots of $D_i(x)$.
3. For $i = 1, \ldots, k$ do:

   let $\deg D_i(x) = d_i$; find $P_i(Z) \in K[Z]$ such that $P_i(\frac{p_i(\alpha_1)}{\partial_x D(\alpha_1)}) = 0$ by the AFE algorithm;

   if $l_i = 1$, then take $F_i(Z) := P_i(Z)$;

   else if $1 < l_i \leqslant d_i/2$, let $R(Z) := P_i(Z)$; for $j = 1, \ldots, l_i - 1$, do: let $R(Z)$ be the resultant of $R(Z - y)$ and $P_i(y)$ with respect to y; find an irreducible factor $F_i(Z)$ of $R(Z)$ such that $F_i(\operatorname{res}_x \frac{p_i(x)}{D_i(x)}) = 0$;

   else if $d_i/2 < l_i \leqslant d_i - 1$, let $R(Z) := P_i(Z)$; for $j = 1, \ldots, d_i - l_i - 1$, do: let $R(Z)$ be the resultant of $R(Z - y)$ and $P_i(y)$ with respect to y; by Remark 10 we can take $R(Z) = R(-Z)$; find an irreducible factor $F_i(Z)$ of $R(Z)$ such that $F_i(\operatorname{res}_x \frac{p_i(x)}{D_i(x)}) = 0$;

   for all the cases, $F_i(Z)$ is an algebraic functional equation for $\operatorname{res}_x \frac{p_i(x)}{D_i(x)}$.
4. Recursively apply the AFE2 algorithm to C and $F_1(Z), \ldots, F_k(Z)$ to get the minimal polynomial of $\operatorname{res}_x Q(x)$.

□

We conclude this section by giving three examples. The residues can be interpreted as certain lattice paths in the plane. The degree of the algebraic functional equations for these residues are lower than estimated. Our working field is $\mathbb{C}((x))((t))$.

Example 11. Consider the residue:
$$F(t) = \operatorname*{res}_x \frac{1}{x\left(1 - \frac{t}{x^2} - x^4 t\right)} = \operatorname*{res}_x \frac{x}{x^2 - t - x^6 t}.$$
Because the denominator $x^2 - t - x^6 t$ has initial term x^2, then it has two small roots. By Remark 8, the residue is algebraic of degree at most 15. But by the MP algorithm, we obtain the following algebraic functional equation of degree 3 for $F(t)$.
$$1 + 3\,Z + \left(27\,t^3 - 4\right) Z^3.$$

Example 12. Consider the residue:

$$F(t) = \operatorname*{res}_x \frac{1}{x\left(1 - \frac{t}{x} - xt - \frac{t}{x^2} - x^2 t\right)} = \operatorname*{res}_x \frac{x}{x^2 - xt - x^3 t - t - x^4 t}.$$

Because the denominator $x^2 - xt - x^3t - t - x^4t$ has initial term x^2, then it has two small roots. By Remark 8, the residue is algebraic of order at most 6. But by the MP algorithm, we obtain the following algebraic functional equation of degree 4 for $F(t)$.

$$(4 + 9\,t)\,(4\,t - 1)^2\, Z^4 - 2\,(4\,t - 1)\,(3\,t - 2)\, Z^2 + t.$$

Example 13. Consider the residue:

$$F(t) = \operatorname*{res}_x \frac{1}{x\left(1 - \frac{t^3}{x^2} - \frac{t^4}{x^2} - x^2 t - x^2 t^2\right)} = \operatorname*{res}_x \frac{x}{x^2 + x^4 t - x^4 t^2 - t^3 - t^4}.$$

Because the denominator $x^2 + x^4t - x^4t^2 - t^3 - t^4$ has initial term x^2, then it has two small roots. By Remark 8, the residue is algebraic of order at most 6. But by the MP algorithm, we obtain the following algebraic functional equation of degree 2 for $F(t)$.

$$1 + \left(2\,t^3 + 2\,t^2 - 1\right)\left(2\,t^3 + 2\,t^2 + 1\right) Z^2.$$

6 The archetype

Let $A(x, y, z)$ be as in Section 1, and let $B(x, y, t) = A(x, y, t/xy)$. Then we need to compute

$$\begin{aligned} F(t) &= \operatorname*{res}_{x,y} \frac{B(x,y,t)}{xy} \\ &= \operatorname*{res}_{x,y} \frac{1}{xy\,(1-x)^2\,(1-y)^2\left(1-\frac{t}{xy}\right)^2(1-y-x)\left(1-\frac{t}{xy}-y\right)\left(1-\frac{t}{xy}-x\right)}. \end{aligned}$$

Our working field is $\mathbb{C}((x))((y))((t))$.

We eliminate y first. Compute the ILSRRF with respect to y. The result is

$$\operatorname*{res}_y \frac{B(x,y,t)}{xy} = Q(x,t) - \operatorname*{res}_y \frac{x^3}{(1-x)^2\, t^2\,(-t + x^2 - x^3)\,(-t + xy - xy^2)},$$

where $Q(x, t)$ is a complicated rational function, whose ILSRRF with respect to x is given by

$$\begin{aligned} Q(x,t) \underset{\text{res}}{\equiv} 2\frac{1 - 6\,t + 10\,t^2}{(1-4\,t)\,t^4\,(x - t - x^2)} - \frac{1}{3}\frac{2 - 2\,x - 22\,t + 19\,tx + 47\,t^2 - 26\,t^2 x}{t^4\,(1-8\,t)\,(x - t - 2\,x^2 + x^3)} \\ - \frac{1}{3}\frac{2\,x - 3\,t - 19\,tx + 21\,t^2 + 26\,t^2 x}{t^4\,(1-8\,t)\,(-t + x^2 - x^3)} - \frac{2}{3}\frac{1 - 12\,t + 42\,t^2 - 95\,t^3 + 51\,t^4 - 48\,t^5 - 2\,t^6}{t^4\,(1-t)^4\,x\,(1-8\,t)}. \end{aligned}$$

The remaining residue in y is computed as follows.

$$\operatorname*{res}_y \frac{1}{(-t+xy-xy^2)} = \frac{1}{x-2xY} = \frac{1}{x\sqrt{1-4t/x}},$$

where Y is the unique small root of $-t+xy-xy^2$ for y given by

$$Y = \frac{1-\sqrt{1-4t/x}}{2} = \frac{t}{x} + \cdots .$$

Similar argument gives

$$\operatorname*{res}_x \frac{1}{(x-t-x^2)} = \frac{1}{1-2X} = \frac{1}{\sqrt{1-4t}},$$

where $X = \frac{1-\sqrt{1-4t}}{2} = t + \cdots$.

In summary, $F(t)$ is equal to $B_1+B_2+B_3+B_4$, where

$$\begin{aligned}
B_1 &= -\frac{2}{3}\frac{1-12\,t+42\,t^2-95\,t^3+51\,t^4-48\,t^5-2\,t^6}{t^4\,(1-t)^4\,(1-8\,t)} + 2\frac{1-6\,t+10\,t^2}{(1-4\,t)\,t^4\sqrt{1-4t}},\\
B_2 &= \operatorname*{res}_x -\frac{1}{3}\frac{2-2\,x-22\,t+19\,tx+47\,t^2-26\,t^2x}{t^4\,(1-8\,t)\,(x-t-2\,x^2+x^3)},\\
B_3 &= -\operatorname*{res}_x \frac{1}{3}\frac{2\,x-3\,t-19\,tx+21\,t^2+26\,t^2x}{t^4\,(1-8\,t)\,(-t+x^2-x^3)},\\
B_4 &= \operatorname*{res}_x \frac{x^3}{(1-x)^2\,t^2\,(-t+x^2-x^3)\sqrt{x^2-4tx}}.
\end{aligned}$$

Our first try is to obtain the D-finite equation. The equation for B_1, B_2, B_3 are easy. For B_4, we could use a similar idea of [9] to find the D-finite equation. Combining these together, we could obtain a D-finite equation for $F(t)$.

The D-finite equation for $F(t)$ is of order 4, but is too lengthy to be put here. This equation can be obtained by other methods, such as the theory of hypergeometric sum. Zeilberger can compute it quickly using his software.

A. Goupil believes that $F(t)$ is algebraic [1] and he is right. It is clear that B_1, B_2, B_3 are algebraic. We need to show that B_4 is algebraic, too. Note that the general theory only shows that B_4 is D-finite.

We need the well-known Jacobi's change of variable formula. See, e.g., [8].

Theorem 14 (Jacobi's Residue Formula)**.** *Let $y = f(x) \in \mathbb{C}((x))$ be a Laurent series and let b be the integer such that $f(x)/x^b$ is a formal power series with nonzero constant term. Then for any formal series $G(y)$ such that the composition $G(f(x))$ is a Laurent series, we have*

$$\operatorname*{res}_x G(f(x))\frac{\partial f}{\partial x} = b \operatorname*{res}_y G(y). \tag{2}$$

We make the change of variable $x = u^{-1}(u+t)^2$. Then

$$\sqrt{x^2 - 4tx} = u^{-1}(u+t)(u-t).$$

The advantage of this change of variable is that the square root disappears, and we are left with a residue of a rational function. This change of variable is inspired by the well-known fact if we set $x = u(1+u)^{-2}$, $\sqrt{1-4x}$ becomes a rational function in u. One more thing we need to take care is how to treat u: solving $x = u^{-1}(u+t)^2$ for u gives

$$u = -t + \frac{x + \sqrt{x^2 - 4tx}}{2} = x + \cdots, \text{ or } u = -t + \frac{x - \sqrt{x^2 - 4tx}}{2} = \frac{t^2}{x} + \cdots.$$

We choose the former representation, which means u has initial term x and we shall just treat u the same as x and hence we can replace u by x and still work in the field $\mathbb{C}((x))((t))$ of iterated Laurent series. If we choose the latter representation, then u has initial term t^2/x. Thus u is smaller than t and we need the theory of Malcev-Neumann series to continue.

To avoid producing more denominator factors, we apply this change of variable to $B_3 + B_4$. This gives the residue in u of a rational function. Now replacing u by x and applying the ILSRRF algorithm with respect to x, we obtain

$$\begin{aligned} B_3 + B_4 = &-\frac{1}{3}\frac{1 - 8\,t + 4\,t^2}{t^4\,(1 - 8\,t)} + \frac{1 - 6\,t + 10\,t^2}{(1 - 4\,t)\,t^4\sqrt{1 - 4t}} \\ &- \operatorname*{res}_x \frac{2}{3}\frac{2\,x - 22\,tx - 3\,t^2 + 47\,t^2x + 21\,t^3}{t^4\,(1 - 8\,t)\,(x^2 - x^3 - 3\,tx^2 - 3\,t^2x - t^3)}. \end{aligned}$$

Together with the formulas for B_1 and B_2, we obtain $F(t) = C_1 + C_2 + C_3 + C_4$, where

$$\begin{aligned} C_1 &= -\frac{1 - 4\,t + 10\,t^2 - 6\,t^3 + 5\,t^4}{(1-t)^4\,t^4}, \\ C_2 &= 3\frac{1 - 6\,t + 10\,t^2}{t^4\,(1 - 4\,t)\,\sqrt{1 - 4t}}, \\ C_3 &= -\operatorname*{res}_x \frac{1}{3}\frac{2 - 2\,x - 22\,t + 19\,tx + 47\,t^2 - 26\,t^2x}{t^4\,(1 - 8\,t)\,(x - t - 2\,x^2 + x^3)}, \\ C_4 &= -\operatorname*{res}_x \frac{2}{3}\frac{2\,x - 22\,tx - 3\,t^2 + 47\,t^2x + 21\,t^3}{t^4\,(1 - 8\,t)\,(x^2 - x^3 - 3\,tx^2 - 3\,t^2x - t^3)}. \end{aligned}$$

Now we deal with C_3 first. The denominator factor $-x + 2\,x^2 - x^3 + t$ has initial term $-x$, and hence has a unique small root denoted R_3. Then the residue is computed as

$$C_3 = -\frac{1}{3}\frac{2 - 2\,R_3 - 22\,t + 19\,tR_3 + 47\,t^2 - 26\,t^2R_3}{t^4\,(1 - 8\,t)\left(1 - 4\,R_3 + 3\,{R_3}^2\right)}.$$

Since R_3 is algebraic of degree 3, C_3 is algebraic of degree at most 3. By the AFE algorithm we obtain an algebraic functional equation for C_3 as follows.

$$\begin{aligned} g_3(Z) = &\, 27\,(27t - 4)\,(8t - 1)^3\,Z^3t^{12} + 9\,(8t - 1)\left(2314t^4 - 2272t^3 + 720t^2 - 91t + 4\right)Zt^4 \\ &+ 17576t^6 - 59259t^5 + 47838t^4 - 16864t^3 + 2928t^2 - 246t + 8. \end{aligned}$$

Next we deal with C_4. The denominator factor $t^3+3t^2x+3tx^2-x^2+x^3$ has initial term $-x^2$ and hence has two small roots. Thus the residue is computed as

$$C_4 = \sum_{R_4} -\frac{2}{3}\frac{2R_4 - 22tR_4 - 3t^2 + 47t^2R_4 + 21t^3}{t^4(1-8t)\left(2R_4 - 3{R_4}^2 - 6tR_4 - 3t^2\right)},$$

where the sum ranges over the two small roots of the denominator factor. By the AFE algorithm we can get an algebraic functional equation for one term, which is the following degree 3 polynomial.

$$\begin{aligned}\overline{g}_4(Z) = 27\,(27t-4)(8t-1)^3 Z^3 t^{12} + 36\,(8t-1)\left(2314t^4 - 2272t^3 + 720t^2 - 91t + 4\right) Zt^4 \\ - 64 + 134912t^3 - 23424t^2 + 1968t - 140608t^6 + 474072t^5 - 382704t^4.\end{aligned}$$

Suppose the above polynomial has $\alpha_1, \alpha_2, \alpha_3$ as the three roots. Then $C_4 = \alpha_1 + \alpha_2$ by suitably permuting the α's. Notice that $\alpha_1 + \alpha_2 + \alpha_3 = 0$. The algebraic functional equation for C_4 must be $-\overline{g}_4(-Z)$, so

$$\begin{aligned}g_4(Z) = 27\,(27t-4)(8t-1)^3 Z^3 t^{12} + 36\,(8t-1)\left(2314t^4 - 2272t^3 + 720t^2 - 91t + 4\right) Zt^4 \\ - (-64 + 134912t^3 - 23424t^2 + 1968t - 140608t^6 + 474072t^5 - 382704t^4).\end{aligned}$$

Applying the AFE2 algorithm to C_3+C_4 we obtain the minimal polynomial of C_3+C_4 as follows.

$$\begin{aligned}g_{34}(Z) = &\left(1416t^{14} + 4t^{12} - 123t^{13} + 13824t^{16} - 7232t^{15}\right) Z^3 \\ &+ \left(-12t^4 - 61470t^8 + 369t^5 + 24096t^7 - 4344t^6 + 55536t^9\right) Z \\ &+ 17576t^6 - 59259t^5 + 47838t^4 - 16864t^3 + 2928t^2 - 246t + 8.\end{aligned}$$

To obtain the minimal polynomial of $C_2+C_3+C_4$ it is easier to use Remark 7: the minimal polynomial divides $g_{234} = g_{34}(Z+C_2)g_{34}(Z-C_2)$, which is checked to be irreducible. The final step for the minimal polynomial of $C_1+C_2+C_3+C_4$ is easy, and we finally obtain the minimal polynomial of the diagonal as follows.

$$g(Z) = 7776 - 7776Z + t\cdot(\text{a lengthy polynomial}).$$

The degree of $g(Z)$ is 6 in Z and 61 in t. This shows that to guess $g(Z)$ using something like the gfun package is too expensive. Finally, we remark that $g(Z)$ has a unique power series root and this root has constant 1.

7 More examples

The following examples are chosen from [4]. Our working field for dealing with the examples is $\mathbb{C}((x))((t))$ or $\mathbb{C}((x))((y))((t))$.

Example 15. [4, 4.5,7.3,4.6] The following three diagonals are simple and similar.

$$F_1(t) = \operatorname{diag} \frac{2}{1+2x+\sqrt{1-4x}-2xy} = \operatorname*{res}_x \frac{2}{x(1+2x+\sqrt{1-4x}-2x\frac{t}{x})}, \tag{3}$$

$$F_2(t) = \operatorname{diag} \frac{2}{1+\sqrt{1-4\,x^2}-2\,xy} = \operatorname*{res}_x \frac{2}{x\left(1+\sqrt{1-4\,x^2}-2\,t\right)}, \tag{4}$$

$$\begin{aligned} F_3(t) &= \operatorname{diag} \frac{xy\,(1-x)^3}{(1-x)^4 - xy\,(1-x-x^2+x^3+x^2y)} \\ &= \operatorname*{res}_x \frac{t\,(1-x)^3}{x\big(\,(1-x)^4 - t\,(1-x-x^2+x^3+xt)\,\big)}. \end{aligned} \tag{5}$$

For $F_1(t)$ and $F_2(t)$ we need Jacobi's residue formula to deal with the square root. For F_1 we make the change of variable $x = u(1+u)^{-2}$. Then $\sqrt{1-4x} = (1-u)/(1+u)$, and

$$F_1(t) = \operatorname*{res}_u \frac{(1-u)\,(1+u)}{u\,(1-t+2\,u-2\,tu-tu^2)}.$$

Solving $x = u(1+u)^{-2}$ for u gives

$$u = \frac{-2\,x+1+\sqrt{1-4\,x}}{2x} = 1/x - 2 - x + \cdots, \quad \text{or} \quad u = -\frac{2\,x-1+\sqrt{1-4\,x}}{2x} = x + \cdots.$$

We choose the later representation and treat u the same as x.

For F_2 we make the change of variable $x = u(1+u^2)^{-1}$. Then $\sqrt{1-4x^2} = (1-u)(1+u)/(1+u^2)$, and

$$F_2(t) = \operatorname*{res}_u \frac{(1-u)\,(1+u)}{u\,(1-t-tu^2)}.$$

Solving $x = u(1+u^2)^{-1}$ for u gives

$$u = \frac{1+\sqrt{1-4\,x^2}}{2x} = \frac{1}{x} + \cdots, \quad \text{or} \quad u = \frac{1-\sqrt{1-4\,x^2}}{2x} = x + \cdots.$$

We choose the later representation and treat u the same as x.

$F_1(t), F_2(t)$ and $F_3(t)$ are all of the form

$$\operatorname*{res}_x q(t) \frac{1+x\cdot r(x)}{x(1-t+x\cdot p(x,t))},$$

where q, r, p are polynomials. It is easy to get the above residues even without the ILSRRF algorithm. Then

$$F_1(t) = F_2(t) = \frac{1}{1-t}, \quad F_3(t) = \frac{t}{1-t}.$$

Example 16. [4, 4.10] Compute the diagonal

$$F(t) = \operatorname{diag} \frac{1}{(1 - z\,(1 - x^2y^2))\,(1 - x\,(1 + y))} = \operatorname*{res}_{x,y} \frac{1}{xy\Big(1 - \frac{t}{xy}\,(1 - x^2y^2)\,\Big)\,(1 - x\,(1 + y))}. \tag{6}$$

Applying the ILSRRF algorithm with respect to y, we obtain $F(t) = B_1 + B_2$, where

$$B_1 = \operatorname*{res}_{x} \frac{1}{2x\,(1 - x - 2\,tx + tx^2)},$$
$$B_2 = -\operatorname*{res}_{x,y} \frac{2\,tx - 2\,t - 1}{2\,(1 - x - 2\,tx + tx^2)\,(xy - t + tx^2y^2)}.$$

Applying the ILSRRF algorithm to B_1 with respect to x, we get

$$B_1 = \operatorname*{res}_{x} \frac{1}{2x} = \frac{1}{2}.$$

Since the initial term of $xy - t + tx^2y^2$ in the denominator of B_2 is xy, there is a unique small root for y. Solving $xy - t + tx^2y^2 = 0$ for y gives

$$y = \frac{-1 + \sqrt{1 + 4\,t^2}}{2tx} = \frac{t}{x} + \cdots, \quad \text{or} \quad y = -\frac{1 + \sqrt{1 + 4\,t^2}}{2tx} = -\frac{x}{t} - \frac{t}{x} + \cdots.$$

The former one is small. It follows that

$$B_2 = -\operatorname*{res}_{x} \left.\frac{2\,tx - 2\,t - 1}{2\,(1 - x - 2\,tx + tx^2)\,\partial_y\,(xy - t + tx^2y^2)}\right|_{y = \frac{-1 + \sqrt{1 + 4\,t^2}}{2tx}}$$
$$= -\operatorname*{res}_{x} \frac{2\,tx - 2\,t - 1}{2\,(1 - x - 2\,tx + tx^2)\,x\sqrt{1 + 4\,t^2}}.$$

Applying the ILSRRF algorithm with respect to x, we obtain

$$B_2 = \operatorname*{res}_{x} \frac{1 + 2\,t}{2x\sqrt{1 + 4\,t^2}} = \frac{1 + 2\,t}{2\sqrt{1 + 4\,t^2}}.$$

Then

$$F(t) = \frac{1 + 2\,t}{2\sqrt{1 + 4\,t^2}} + \frac{1}{2}.$$

Example 17. [4, 5.3] Compute the diagonal

$$\begin{aligned} F(t) =& \operatorname{diag} \frac{1 + x^2y^3 + x^2y^4 + x^3y^4 - x^3y^6}{1 - x - y + x^2y^3 - x^3y^3 - x^4y^4 - x^3y^6 + x^4y^6} \\ =& \operatorname*{res}_{x} \left(1 + \frac{t^3}{x} + \frac{t^4}{x^2} + \frac{t^4}{x} - \frac{t^6}{x^3}\right) \left(x\left(1 - x - \frac{t}{x} + \frac{t^3}{x} - t^3 - t^4 - \frac{t^6}{x^3} + \frac{t^6}{x^2}\right)\right)^{-1}. \end{aligned}$$

Applying the ILSRRF algorithm with respect to x, we obtain

$$F(t) = -\frac{1}{4}\operatorname*{res}_x \frac{-t^3x^2 - t^4x^2 - 4\,t^4x - 2\,xt - 4\,t^4 + 3\,t^6 - 3\,x^2 - 2\,t^3x}{x^3 - x^4 - tx^2 + t^3x^2 - t^3x^3 - t^4x^3 - t^6 + t^6x} + \frac{1}{4}.$$

The denominator $x^3 - x^4 - tx^2 + t^3x^2 - t^3x^3 - t^4x^3 - t^6 + t^6x$ has initial term x^3, hence has three small roots. Then the residue is computed as

$$F(t) = \sum_R Q(R) + \frac{1}{4},$$

where the sum ranges over the three small roots of the denominator and

$$Q(R) = -\frac{1}{4}\,\frac{-t^3R^2 - t^4R^2 - 4\,t^4R - 2\,Rt - 4\,t^4 + 3\,t^6 - 3\,R^2 - 2\,t^3R}{3\,R^2 - 4\,R^3 - 2\,Rt + 2\,t^3R - 3\,t^3R^2 - 3\,t^4R^2 + t^6}.$$

Applying the AFE algorithm, we can get an algebraic functional equation for $Q(R)$, which is the following degree 4 polynomial.

$$\overline{g}(Z) = a_4Z^4 + a_3Z^3 + a_2Z^2 + a_1Z + a_0,$$

where

$$\begin{aligned}
a_4 &= 256\,(t+1)\left(4t^{20} + 8t^{19} - 23t^{18} - 63t^{17} - 62t^{16} - 26t^{15} + 43t^{14} + 11t^{13} + 182t^{12} + 56t^{11}\right.\\
&\quad \left.-t^{10} + 203t^9 - 66t^8 - 154t^7 + 286t^6 - 368t^5 + 233t^4 - 75t^3 - 8t^2 + 20t - 4\right),\\
a_3 &= 0,\\
a_2 &= -384t^{21} - 1152t^{20} + 1440t^{19} + 8256t^{18} + 8928t^{17} - 3840t^{16} - 20064t^{15} - 17728t^{14}\\
&\quad - 12640t^{13} - 8256t^{12} + 18016t^{11} + 15680t^{10} + 8352t^9 + 29824t^8 + 10624t^7 - 8000t^6\\
&\quad + 6816t^5 - 6464t^4 - 2528t^3 + 1408t^2 - 768t + 1152,\\
a_1 &= 128t^{21} + 384t^{20} - 480t^{19} - 2752t^{18} - 2464t^{17} + 3328t^{16} + 9504t^{15} + 6976t^{14} + 1696t^{13}\\
&\quad - 448t^{12} - 8352t^{11} - 8256t^{10} - 3552t^9 - 9344t^8 - 6656t^7 + 2240t^6 - 1248t^5 - 2112t^4\\
&\quad + 1312t^3 + 640t^2 + 128t + 512,\\
a_0 &= -12t^{21} - 36t^{20} + 45t^{19} + 258t^{18} + 183t^{17} - 504t^{16} - 1139t^{15} - 690t^{14} + 173t^{13} + 390t^{12}\\
&\quad + 907t^{11} + 882t^{10} + 229t^9 + 692t^8 + 868t^7 + 22t^6 + 21t^5 + 518t^4 - 87t^3 - 260t^2\\
&\quad + 256t + 60.
\end{aligned}$$

Suppose the above polynomial has r_1, r_2, r_3, r_4 as its four roots. Then $\sum_R Q(R) = r_1 + r_2 + r_3$ by suitably permuting the r's. Notice that $r_1 + \cdots + r_4 = 0$, then the algebraic functional equation for $\sum_R Q(R)$ must be $\overline{g}(-Z)$. Thus the algebraic functional equation for $F(t)$ is

$$g(Z) = \overline{g}(-Z + \frac{1}{4}) = a_4(Z - \frac{1}{4})^4 + a_2(Z - \frac{1}{4})^2 - a_1(Z - \frac{1}{4}) + a_0.$$

Example 18. [4, 6.3] Compute the diagonal

$$F(t) = \operatorname{diag} \frac{3xz\,(1-z)\,(3-z)}{\left(1-3\,y\,(1+z)^2\right)\left(27-xz\,(3-z)^2\right)}$$
$$= \operatorname*{res}_{x,y} \frac{3\,t\left(1-\frac{t}{xy}\right)\left(3-\frac{t}{xy}\right)}{xy^2\left(1-3\,y\left(1+\frac{t}{xy}\right)^2\right)\left(27-t\left(3-\frac{t}{xy}\right)^2 y^{-1}\right)}.$$

Applying the ILSRRF algorithm with respect to y, we obtain

$$F(t) = \operatorname*{res}_{x} Q(x,t) + \operatorname*{res}_{x,y} R(x,y,t),$$

where

$$Q(x,t) = -\frac{x\,(3\,x^2-72\,tx-2\,tx^2+378\,t^2+32\,xt^2)}{2\,(-x^4-378\,t^2x^2-32\,t^2x^3+864\,t^3x+36\,tx^3+256\,t^3x^2+729\,t^3+x^4t)},$$

and

$$R(x,y,t) = \frac{x^2\,(-3\,x^3-756\,xt^2-112\,t^2x^2+1296\,t^3+90\,tx^2+768\,t^3x+4\,tx^3)}{2\,(-x^2y+3\,x^2y^2+6\,xyt+3\,t^2)}$$
$$\cdot\left(-x^4-378\,t^2x^2-32\,t^2x^3+864\,t^3x+36\,tx^3+256\,t^3x^2+729\,t^3+x^4t\right)^{-1}.$$

Applying the ILSRRF algorithm to $Q(x,t)$ with respect to x, we get

$$\operatorname*{res}_{x} Q(x,t) = \frac{3-2t}{2(1-t)}.$$

The denominator factor $-x^2y+3\,x^2y^2+6\,xyt+3\,t^2$ has initial term $-x^2y$. Thus it has the following unique small root

$$\frac{x-6t-\sqrt{x^2-12\,tx}}{6x}.$$

It follows that

$$\operatorname{res}_y R(x,y,t)$$
$$= \frac{x\,(-x+12\,t)\,(108\,t^2+64\,xt^2-54\,tx-4\,tx^2+3\,x^2)}{2\sqrt{x\,(x-12\,t)}\,(x^4+378\,t^2x^2+32\,t^2x^3-864\,t^3x-36\,tx^3-256\,t^3x^2-729\,t^3-x^4t)}.$$

We make the change of variable $x=(u+3\,t)^2\,u^{-1}$. Then $\sqrt{x(x-12t)}=(u+3t)(u-3t)/u$. Solving $x=(u+3\,t)^2\,u^{-1}$ for u gives

$$u=\frac{x}{2}-3\,t+\frac{\sqrt{x^2-12\,tx}}{2}=x+\cdots,\quad \text{or}\quad u=\frac{x}{2}-3\,t-\frac{\sqrt{x^2-12\,tx}}{2}=9t^2/x+\cdots.$$

We choose the former representation and treat u the same as x. Therefore doing the Jacobi's change of variable and applying the ILSRRF algorithm with respect to x, we obtain

$$\operatorname*{res}_{x,y} R(x,y,t) = \frac{4\,t-3}{2\,(1-t)}.$$

Then

$$F(t) = \frac{t}{1-t}.$$

Example 19. [4, 7.1] Compute the diagonal

$$F(t) = \operatorname{diag} \frac{1}{\left(1+\sqrt{1-x}-y\right)} = \operatorname*{res}_{x} \frac{1}{\left(1+\sqrt{1-x}-\frac{t}{x}\right)x}.$$

We make the change of variable $x = \frac{4u}{(1+u)^2}$. Then $\sqrt{1-x} = \frac{1-u}{1+u}$, and

$$F(t) = \operatorname*{res}_{u} \frac{-4(1-u)}{-8\,u+t+3\,tu+3\,tu^2+tu^3}.$$

Solving $x = \frac{4u}{(1+u)^2}$ for u gives

$$u = \frac{-x+2+2\sqrt{1-x}}{x} = \frac{4}{x}+\cdots\,, \quad \text{or} \quad u = \frac{-x+2-2\sqrt{1-x}}{x} = \frac{1}{4}x+\cdots\,.$$

So we choose the later representation and treat u the same as x. The initial term of the denominator $-8u+t+3tu+3tu^2+tu^3$ is u. Then the residue is computed as

$$F(t) = \frac{-4(1-R)}{-8+3\,t+6\,tR+3\,tR^2},$$

where R is the unique small root of $-8u+t+3tu+3tu^2+tu^3$. Applying the AFE algorithm, we can get the following algebraic functional equation for $F(t)$.

$$(27\,t-32)\,tZ^3+(-12\,t+16)\,Z+8\,t-8.$$

Example 20. [4, 8.2] Compute the diagonal

$$F(t) = \operatorname{diag} \frac{1+xy+x^2y^2}{1-x-y+xy-x^2y^2} = \operatorname*{res}_{x} \frac{1+t+t^2}{(x-x^2-t+tx-t^2x)}.$$

The denominator factor $(x-x^2-t+tx-t^2x)$ has initial term x and hence has a unique small root

$$\frac{1+\,t-\,t^2-\,\sqrt{-2\,t+1-t^2-2\,t^3+t^4}}{2} = t+\cdots\,.$$

Then the residue is

$$F(t) = \sqrt{\frac{1+t+t^2}{1-3\,t+t^2}}.$$

8 Potential for the general Lipshitz theorem

It is possible to simplify residues in more variables. Suppose x and y are two variables in $\mathcal{K}$, and $Q(x, y)$ is a rational function. Then $\operatorname{res}_{x,y} Q(x, y)$ might not be algebraic. We hope to find simple conditions for $Q(x, y)$ to have an algebraic residue in x and y.

The natural condition would be that $\operatorname{res}_x Q(x, y)$ or $\operatorname{res}_y Q(x, y)$ is indeed rational. This raises the following question: Given a rational function $Q(x, y)$, determine if $\operatorname{res}_x Q(x, y)$ is rational or not. If the answer is positive, then give the explicit rational function.

Let $G = \operatorname{res}_x Q(x, y)$. Then G is rational if and only if its minimal polynomial is of degree 1. Since the minimal polynomial can be computed, we have an algorithm to determine if G is rational. Note that the D-finite approach is not appealing. If $G = p(y)/q(y)$ is rational, then G satisfies the following first order D-finite equation:

$$q(y)^2 \partial_y G(y) - q(y) \partial_y p(y) + p(y) \partial_y q(y) = 0.$$

However, it is hard to find the minimal order D-finite equation.

Jacobi's residue formula is powerful, but there is no general rule on how to make the change of variables. By studying our archetype, we find a new class of rational functions for which we can always make a change of variable to simplify further. Consider residues of the following type

$$\operatorname*{res}_{x,y} \frac{p(x, y)}{q(x)D(x, y)},$$

where p, q, D are polynomials with $\deg_y p(x, y) < \deg_y D(x, y)$, and $D(x, y)$ is irreducible. If $\deg_x D(x, y) = 1$, then we can solve $D(x, y)$ for x, giving $x = f(y)$ for rational f. We claim that $D(f(u), y)$ has $y - u$ as a factor since $D(f(u), u) = 0$. Therefore, by making the change of variable $x = f(u)$, we obtain

$$\operatorname*{res}_{x,y} \frac{p(x, y)}{q(x)D(x, y)} = \operatorname*{res}_{u,y} \frac{p(f(u), y)}{q(f(u))D(f(u), y)} \cdot \partial_u f(u).$$

Applying the ILSRRF algorithm with respect to y gives the residue of a simpler rational function. In particular, if $\deg_y D(x, y) = 2$ and $\deg_x D(x, y) = 1$, then $\operatorname{res}_{x,y} \frac{p(x,y)}{q(x)D(x,y)}$ is algebraic.

For example, when computing $\operatorname{res}_{x,y} x^{-1}y^{-1}B(x, y, t/xy)$, we met the following residue

$$\operatorname*{res}_{x,y} \frac{x^3}{(1-x)^2\, t^2\, (-t + x^2 - x^3)\, (-t + xy - xy^2)}.$$

If we make the change of variable $x = t(u - u^2)^{-1}$, then we get

$$\operatorname*{res}_{u,y} \frac{(-1 + 2\,u)\,(-1 + u)\,u}{(u - 1 + y)\,(u - y)\,(t - u^2 + u^3)\,(t - u + 2\,u^2 - u^3)\,(-u + u^2 + t)^2}.$$

Elimination of y must give a rational function, but one must be careful:

$$u = \frac{1 + \sqrt{1 - 4t/x}}{2} = 1 - t/x + \cdots \text{ or } u = \frac{1 - \sqrt{1 - 4t/x}}{2} = t/x + \cdots .$$

It follows that we can only chose the later and regard u as t/x. We omit the rest of the computation for brevity.

For residue in two or more variables, we need Lipshitz's D-finite theory. Lipshitz's idea can be reformulated as follows in its simple form. Let f be an iterated Laurent series. Suppose f is D-finite in variables x and t. Then there exist nonzero linear partial differential operators $P_1(x,t;\partial_x)$ and $P_2(x,t;\partial_t)$, called annihilating operators, such that

$$P_1(x,t;\partial_x)f=0,\ P_2(x,t;\partial_t)f=0.$$

To show that $\operatorname{res}_x f$ is D-finite in t, it is sufficient to find operators L_1 and L_2 such that

$$L_1(x,t;\partial_x,\partial_t)P_1(x,t;\partial_x)+L_2(x,t;\partial_x,\partial_t)P_2(x,t;\partial_t)=P_3(t;\partial_x,\partial_t)$$

is free of x. Then $\operatorname{res}_x f$ will be annihilated by $P_3(t;0,\partial_t)$. Lipshitz showed the existence of L_1 and L_2 by a clever dimension counting argument, but the dimension is usually too huge in practice.

It has been observed that we only need an annihilating operator of the form $P_4(t;\partial_t)+\partial_x(P_5(x,t;\partial_x,\partial_t))$. Algorithms along this line have been developed. See [5, Chapter 9] for explanation and references therein.

The idea of a reduced form can be used to simplify the computation. By using P_1 we can define a reduced form for $Q(x,t)\partial_t^i f$, where $Q(x,t)$ is a rational function. By the reduced form, it will be easy to see that the $\partial_t^k f$ have similar reduced forms, and then deduce that they lie in a finite dimensional space. Then solving a system of linear equation will give the desired D-finite equation. It is possible to use the powerful Jacobi's residue formula to reduce the dimension. This idea performs well when $f=\sqrt{S}$ for some rational function S. See [9].

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References

[1] A. Goupil, private communication, August 2010.

[2] A. Goupil and H. Cloutier. 3D Polyominoes inscribed in a rectangular prism. preprint, arXiv:1009.4859.

[3] L. Lipshitz. The diagonal of a D-finite power series is D-finite. *J. Algebra*, 113(2):373–378, 1988.

[4] R. Pemantle and M. Wilson. Twenty combinatorial examples of asymptotics derived from multivariate generating functions. *SIAM Review*, 50:199–272, 2008.

[5] M. Petkovsek, H. Wilf, and D. Zeilberger. *A=B*. A K Peters, Ltd., Wellesley, MA, 1996.

[6] R.P. Stanley. *Enumerative Combinatorics, II.* Cambridge University Press, 1999.

[7] G. Xin. A fast algorithm for MacMahon's partition analysis. *Electron. J. Combin.*, 11:R58, 2004.

[8] G. Xin. A residue theorem for Malcev-Neumann series. *Adv. in Appl. Math.*, 35:271–293, 2005.

[9] G. Xin and T.Y.J. Zhang. Enumeration of bilaterally symmetric 3-noncrossing partitions. *Discrete Math.*, 309:2497–2509, 2009.

[10] D. Zeilberger. A holonomic systems approach to special functions identities. *J. Comput. Appl. Math.*, 32:321–368, 1990.

[11] D. Zeilberger. The method of creative telescoping. *J. Symb. Computation*, 11:195–204, 1991.