Making a graph crossing-critical by multiplying its edges

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Abstract

A graph is *crossing-critical* if the removal of any of its edges decreases its crossing number. This work is motivated by the following question: to what extent is crossing-criticality a property that is inherent to the structure of a graph, and to what extent can it be induced on a noncritical graph by multiplying (all or some of) its edges? It is shown that if a nonplanar graph G is obtained by adding an edge to a cubic polyhedral graph, and G is sufficiently connected, then G can be made crossing-critical by a suitable multiplication of its edges.

1 Introduction

This work is motivated by the recent breakthrough constructions by DeVos, Mohar and Šámal [3] and Dvořák and Mohar [4], which settled two important crossing numbers questions. The graphs constructed in [3] and [4] use weighted (or "thick") edges. A graph with weighted edges can be naturally transformed into an ordinary graph by substituting weighted edges by multiedges (recall that a *multiedge* is a set of edges with the same pair of endvertices). If one wishes to avoid multigraphs, one can always substitute a

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weight t edge by a $K_{2,t}$, but still the resulting graph is homeomorphic to a multigraph. Sometimes (as in [3]) one can afford to substitute each weighted edge by a slightly richer structure (such as a graph obtained from $K_{2,t}$ by joining the degree 2 vertices with a path), but sometimes (as in [4]) one is concerned with criticality properties, and so no such superfluous edges may be added. In any case, the use of weighted edges is crucial.

After trying unsuccessfully to come up with graphs with similar crossing number properties as those presented in [3] and [4], while avoiding the use of weighted edges, we were left with a wide open question: in the realm of crossing numbers, more specifically on crossing-criticality issues, to what extent does it make a difference to allow (equivalently, to forbid) weighted edges (or, for that matter, multiedges)?

1.1 Crossing-critical graphs and multiedges

Recall that the *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of pairwise intersections of edges in a drawing of G in the plane. An edge e of G is *crossing-critical* if $\mathrm{cr}(G-e) < \mathrm{cr}(G)$. If all edges of G are crossing-critical, then G itself is *crossing-critical*. A crossing-critical graph seems naturally more interesting than a graph with some not crossing-critical edges, since a graph of the latter kind contains a proper subgraph that has all the relevant information from the crossing numbers point of view.

Earlier constructions of infinite families of crossing-critical graphs made essential use of multiedges [10]. On the other hand, constructions such the ones given by Kochol [7], Hliněný [5], and Bokal [1] deal exclusively with simple graphs.

We ask to what extent crossing-criticality is an inherent structural property of a graph, and to what extent crossing-criticality can be induced by multiplying the edges of a (noncritical) graph. Let G, H be graphs. We say that G *is obtained by multiplying edges of* H if H is a subgraph of G and, for every edge of G, there is an edge of H with the same endvertices.

Question 1. When can a graph be made crossing-critical by multiplying edges? That is, given a (noncritical) graph H, when does there exist a crossing-critical graph G that is obtained by multiplying edges of H?

Our universe of interest is, of course, the set of nonplanar graphs, since a planar graph obviously remains planar after multiplying any or all of its edges.

1.2 Main result

We show that a large, interesting family of nonplanar graphs satisfy the property in Question 1. A nonplanar graph G is *near-planar* if it has an edge e such that $G-e$ is planar. Near-planar graphs constitute a natural family of nonplanar graphs. Any thought to the effect that crossing number problems might become easy when restricted to near-planar graphs is put definitely to rest by the recent proof by Cabello and Mohar that CrossingNumber is NP-Hard for near-planar graphs [2].

A graph G is *internally* 3*-connected* if G is simple and 2-connected, and for every separation (G_1, G_2) of G of order two, either $|E(G_1)| \leqslant 2$ or $|E(G_2)| \leqslant 2$. Hence, internally 3-connected graphs are those that can be obtained from a 3-connected graph by subdividing its edges, with the condition that no edge can be subdivided more than once. We recall that a graph is *polyhedral* if it is planar and 3-connected [9].

Our main result is that any adequately connected, near-planar graph G obtained by adding an edge to a cubic polyhedral graph, belongs to the class alluded to in Question 1. Throughout this paper, if G is a graph with vertices u, v such that the edge uv is in G, then $G - uv$ is the graph that results by removing the edge uv from G, and $G - \{u, v\}$ is the graph that results by removing both u and v (as well as their incident edges) from G.

Theorem 2. *Let G be a near-planar simple graph, with an edge uv such that $G - uv$ is a cubic polyhedral graph. Suppose that $G - \{u, v\}$ is internally 3-connected. Then there exists a crossing-critical graph that is obtained by multiplying edges of G.*

We note that some connectivity assumption is needed in order to guarantee that a nonplanar graph can be made crossing-critical by multiplying edges. To see this, consider a graph G which is the union of two subgraphs G_1, G_2, where G_1 is nonplanar and G_2 is planar with at least one edge, and G_1 and G_2 have exactly one vertex in common. Since crossing number is additive on the blocks of a graph, it is easy to see that G (more specifically, the edges of G_2) cannot be made crossing-critical by multiplying edges.

1.3 Reformulating Theorem 2 in terms of weighted graphs

We recall that a *weighted graph* is a pair (G, ω), where G is a graph and ω (the *weight assignment*) is a map that assigns to each edge e of G a number $\omega(e)$, the *weight* of e. The *length* of a path in a weighted graph is the sum of the weights of the edges in the path. If u, v are vertices of G, then the *distance* $d_\omega(u, v)$ *from u to v* (*under* ω) is the length of a minimum length (also called a *shortest*) uv-path. The weight assignment ω is *positive* if $\omega(e) > 0$ for every edge e of G, and it is *integer* if each $\omega(e)$ is an integer.

In the context of Theorem 2, let G be a simple graph which we seek to make crossing-critical by multiplying (some or all of) its edges. With this in mind, let $\overline{G}$ be a multigraph (that is, a graph with multiedges allowed) whose underlying simple graph is G. Now consider the (positive integer) weight assignment ω on $E(G)$ defined as follows: for each edge uv of G, let $\omega(uv)$ be the number of edges in $\overline{G}$ whose endpoints are u and v (i.e., the *multiplicity* of uv).

If we extend the definition of crossing number to weighted graphs, with the condition that a crossing between two edges contributes to the total crossing number by the product of their weights, then, from the crossing number point of view, clearly (G, ω) captures all the relevant information from $\overline{G}$. In particular, $\mathrm{cr}(\overline{G}) = \mathrm{cr}(G, \omega)$. Moreover, by extending the definition of crossing-criticality to weighted graphs in the obvious way (which we now proceed to do), it will follow that $\overline{G}$ is crossing-critical if and only if (G, ω) is crossing-critical.

To this end, let G be a graph and ω a positive integral weight assignment on G. An edge e of (G,ω) is *crossing-critical* if $\mathrm{cr}(G,\omega_e) < \mathrm{cr}(G,\omega)$, where ω_e is the weight assignment defined by $\omega_e(f) = \omega(f)$ for $f \neq e$ and $\omega_e(e) = \omega(e) - 1$. As with ordinary graphs, (G,ω) is *crossing-critical* if all its edges are crossing-critical.

Under this definition of crossing-criticality for weighted graphs, it is now obvious that if we start with a multigraph $\overline{G}$ and derive its associated weighted graph (G,ω) as above, then $\overline{G}$ is crossing-critical if and only if (G,ω) is crossing-critical.

In view of this equivalence (for crossing number purposes) between multigraphs and weighted graphs, it follows that Theorem 2 is equivalent to the following:

Theorem 3 (**Equivalent to Theorem 2).** *Let G be a near-planar simple graph, with an edge uv such that $G-uv$ is a cubic polyhedral graph. Suppose that $G-\{u,v\}$ is internally 3-connected. Then there exists a positive integer weight assignment ω such that (G,ω) is crossing-critical.*

The next section is devoted to the proof of Theorem 3, and Section 3 contains some concluding remarks and open questions.

2 Proof of Theorem 3

Throughout this proof, G is a graph that satisfies the hypotheses of Theorem 3. That is, G is near-planar and simple, and has an edge uv such that $G-uv$ is a cubic polyhedral graph and $G_{u,v} := G-\{u,v\}$ is internally 3-connected. We let u_1, u_2, and u_3 be the vertices of G (other than v) adjacent to u. Analogously, we let v_1, v_2, and v_3 be the vertices of G (other than u) adjacent to v.

To help comprehension, we break the proof into several subsections.

2.1 Basic observations and facts about G, $G_{u,v}$, and $G-uv$

We start with an observation that follows from the connectivity properties of $G-uv$ and $G_{u,v}$.

Remark 4. Both $G-uv$ and $G_{u,v}$ admit unique (up to homeomorphism) embeddings in the plane. This allows us, for the rest of the proof, to regard these as graphs embedded in the plane. Moreover, since $G_{u,v}$ is a subgraph of $G-uv$, it follows that we may assume that the restriction of the embedding of $G-uv$ to $G_{u,v}$ is precisely the embedding of $G_{u,v}$.

Proposition 5. *$u_1, u_2, u_3, v_1, v_2, v_3$ are all distinct.*

Proof. Since G is simple, it follows that u_1, u_2, and u_3 are all distinct. Similarly, v_1, v_2, and v_3 are all distinct. Now suppose that $u_i = v_j$ for some $i,j \in \{1,2,3\}$, and consider an embedding of $G-uv$ in the plane. It is easy to see that since $u_i = v_j$, and $G-uv$ is cubic, it follows that uv can be added to the embedding of $G-uv$ without introducing any crossings, resulting in an embedding of G. This contradicts the nonplanarity of G. Thus $u_i \neq v_j$ for all $i,j \in \{1,2,3\}$. □

Proposition 6. *In $G_{u,v}$ there is a unique face F_u (respectively, F_v) incident with u_1, u_2, and u_3 (respectively, v_1, v_2, and v_3). Moreover, $F_u \neq F_v$.*

Proof. The existence of a face F_u in $G_{u,v}$ incident with u_1, u_2, and u_3 follows since $G - uv$ is planar. Since u_1, u_2, and u_3 are all distinct, and $G - uv$ is cubic, it follows that u_1, u_2, and u_3 all have degree 2 in $G_{u,v}$. The internal 3-connectivity of $G_{u,v}$ implies that no two u_is can be adjacent to each other. This implies that there cannot be two distinct faces of $G_{u,v}$ incident with more than one u_i, and, in particular, that F_u is unique. An identical argument shows that F_v exists and is unique.

To see that $F_u \neq F_v$, it suffices to note that if $F_u = F_v$ then the edge uv could be added to the embedding of $G - uv$ without introducing any crossings, resulting in an embedding of G, contradicting its nonplanarity. □

The previous three statements immediately imply the following.

Observation 7. *To obtain the embedding of $G - uv$, we start with the embedding of $G_{u,v}$, and then draw uu_1, uu_2, and uu_3 (and, of course, u) inside F_u, and vv_1, vv_2, and vv_3 (and, of course, v) inside F_v.*

2.2 Weight assignments on the dual $G^*_{u,v}$ of $G_{u,v}$

We shall make extensive use of weight assignments on the dual (embedded graph) $G^*_{u,v}$ of $G_{u,v}$. We start by noting that $G^*_{u,v}$ is well-defined (and admits a unique plane embedding) since $G_{u,v}$ admits a unique plane embedding. As with $G - uv$ and $G_{u,v}$, this allows us to unambiguously regard $G^*_{u,v}$ as an embedded graph. We shall let $\mathcal{F}$ denote the set of all faces in $G_{u,v}$ (equivalently, the set of all vertices of $G^*_{u,v}$).

A weight assignment λ on $G_{u,v}$ naturally induces a weight assignment λ^* on $G^*_{u,v}$, and vice versa: if e is an edge of $G_{u,v}$ and e^* is its dual edge in $G^*_{u,v}$, then we simply let $\lambda^*(e^*) = \lambda(e)$. Trivially, a weight assignment $\overline{\lambda}$ on the whole graph G also naturally induces a weight assignment λ^* on $G^*_{u,v}$: it suffices to consider the restriction λ of $\overline{\lambda}$ to $G_{u,v}$, and from this we obtain λ^* as we just described.

Definition 8. A weight assignment λ^* on $G^*_{u,v}$ is *balanced* if each edge e^* of $G^*_{u,v}$ belongs to a shortest F_uF_v-path in $(G^*_{u,v}, \lambda^*)$.

Now since for $i = 1, 2, 3$ the vertex u_i has degree 2 in $G_{u,v}$, it follows that u_i is incident with exactly two faces in $G_{u,v}$, one of which is F_u; let F_{u_i} denote the other face. Thus it makes sense to define the *distance* $d_{\lambda^*}(u_i, F)$ between u_i and any face $F \in \mathcal{F}$ as $\min\{d_{\lambda^*}(F_u, F), d_{\lambda^*}(F_{u_i}, F)\}$. We define F_{v_i} and $d_{\lambda^*}(v_i, F)$ analogously, for $i = 1, 2, 3$.

We note that possibly $F_{u_i} = F_v$ for some $i \in \{1, 2, 3\}$, or $F_{v_j} = F_u$ for some $j \in \{1, 2, 3\}$. On the other hand, we have the following.

Proposition 9. *F_{u_1}, F_{u_2} and F_{u_3} are all distinct. Similarly, F_{v_1}, F_{v_2} and F_{v_3} are all distinct.*

Proof. As observed in the proof of Proposition 6, the internal 3-connectivity of $G_{u,v}$ implies that no two u_is can be adjacent to each other, and so there cannot be two distinct faces of $G_{u,v}$ incident with more than one u_i. Since F_u is incident with u_1, u_2, and u_3, it follows that F_{u_1}, F_{u_2}, and F_{u_3} are all distinct. An identical argument shows that F_{v_1}, F_{v_2} and F_{v_3} are all distinct. □

2.3 Strategy of the rest of the proof

Although the core of the proof of Theorem 3 is somewhat technical, the main ideas behind it are not difficult to explain. Our aim in this subsection is to give an informal account of the strategy behind the proof. This will also give us the opportunity to motivate the introduction of five properties (namely (P1)–(P5) below) that a weight assignment ω would need to satisfy (in our strategy) for (G, ω) to be crossing-critical. We will finish this subsection by formally stating (i) that if ω satisfies (P1)–(P5), then (G, ω) is indeed crossing-critical (Lemma 10); and (ii) the existence of an ω that satisfies (P1)–(P5) (Lemma 11). Theorem 3 will obviously follow from these lemmas, which will be proved in Subsections 2.4 and 2.5, respectively.

We seek a weight assignment ω such that in every optimal drawing of (G, ω), the induced drawing of $G_{u,v}$ is its unique (see Remark 4) embedding. Moreover, we wish to adjust the weights of the edges so that if we start with $G_{u,v}$ and then draw u inside F_u and v inside F_v, and finally add uv following a shortest F_uF_v-path in $({G_{u,v}}^*, \omega^*)$, the resulting drawing is optimal. Since such a drawing will have $\omega(uv) \cdot d_{\omega^*}(F_u, F_v)$ crossings, the aim is to have $\mathrm{cr}(G, \omega) = \omega(uv) \cdot d_{\omega^*}(F_u, F_v)$.

To ensure that in every optimal drawing of (G, ω) the induced drawing of $G_{u,v}$ is an embedding, we need to discourage crossings among edges in $G_{u,v}$ in every optimal drawing of (G, ω). This is achieved by assigning large weights to the edges of $G_{u,v}$, as captured by the following property:

(P1) For every pair of edges e, e' of $G_{u,v}$, $\omega(e)\omega(e') > \omega(uv) \cdot d_{\omega^*}(F_u, F_v)$.

Now in order to guarantee that the described way of drawing G (adding uv as explained) will be best possible, we need to make sure that if we place u (respectively, v) in a face other than F_u (respectively, F_v), and then add uv in the most economical way, then the resulting drawing will not have fewer than $\omega(uv) \cdot d_{\omega^*}(F_u, F_v)$ crossings. In order to achieve this, we need to make the weights of the edges $uu_1, uu_2, uu_3, vv_1, vv_2, vv_3$ large enough, so that if we place u (respectively, v) in a face other than F_u (respectively, F_v), we introduce a large enough number of crossings with the edges incident with u. This is captured by the next property:

(P2) For each $x \in \{u, v\}$ and each $F \in \mathcal{F}$,

$$\omega(xx_1) \cdot d_{\omega^*}(x_1, F) + \omega(xx_2) \cdot d_{\omega^*}(x_2, F) + \omega(xx_3) \cdot d_{\omega^*}(x_3, F) \geqslant \omega(uv) \cdot d_{\omega^*}(F_x, F)$$

The conditions explained so far are quite easy to accomplish: it suffices to make the weights of all the edges other than uv very large compared to $\omega(uv)$. The difficulty lies on the fine-tuning of these weights in order to guarantee the criticality of every edge.

The criticality of the edges of $G - \{u, v\}$ is forced by asking ω^* to be balanced:

(P3) The dual weight assignment ω^* on $G^*_{u,v}$ induced by ω is balanced.

Indeed, this balancedness guarantees that if we choose any edge $e \in G_{u,v}$, then we can draw u on F_u and v on F_v, and then add uv in a most economical way so that uv crosses e; since e gets crossed in an optimal drawing of G, then e is obviously critical.

The criticality of the edges $uu_1, uu_2, uu_3, vv_1, vv_2, vv_3$ is a little bit harder to achieve. Consider an edge uu_i (the situation is identical for an edge vv_j). Our strategy to ensure the criticality of uu_i is to ask that there exist a face $U_i \in \mathcal{F}$ not incident with u_i, such that we can place u in U_i and join it to u_1, u_2 and u_3 with a cost (in crossings) equal to the cost of reaching U_i from F_u (with a weight $w(uv)$). The upshot is that, since u_i is not incident with U_i, we find an alternative way to join u and v by placing u in the face U_i, and in this alternative way uu_i gets crossed, thus guaranteeing its criticality. This gets captured by the following property:

(P4) For each $(x, X) \in \{(u, U), (v, V)\}$ and each $i = 1, 2, 3$, there is a face $X_i \in \mathcal{F}$ such that $d_{\omega^*}(x_i, X_i) > 0$ and

$$\omega(xx_1) \cdot d_{\omega^*}(x_1, X_i) + \omega(xx_2) \cdot d_{\omega^*}(x_2, X_i) + \omega(xx_3) \cdot d_{\omega^*}(x_3, X_i) = \omega(uv) \cdot d_{\omega^*}(F_x, X_i).$$

Now the caveat in this attempt to make uu_i critical is that since u gets drawn outside F_u, we may introduce crossings involving one edge in $\{uu_1, uu_2, uu_3\}$ and one edge in $\{vv_1, vv_2, vv_3\}$; if such crossings get introduced, then the resulting drawing is not optimal. With the aim of fixing this, we identify a property to ensure that any such crossings, if introduced, are negligible compared to the weight of any edge in $G_{u,v}$:

(P5) For all $i, j \in \{1, 2, 3\}$, $\omega(uu_i) \cdot \omega(vv_j) < (1/9) \min\{\, \omega(e) \mid e \in E(G_{u,v})\}$.

Throughout this informal discussion, we have identified the properties that we want to be satisfied by ω. Getting back to the formal setting, our task is then to establish the following two statements:

Lemma 10. *Suppose that ω is a positive integer weight assignment on G satisfying (P1)–(P5). Then (G, ω) is crossing-critical.*

Lemma 11. *There exists a positive integer weight assignment ω on G that satisfies (P1)–(P5).*

The proofs of these statements, whose combination obviously finishes the proof of Theorem 3, are given in Subsections 2.4 and 2.5, respectively.

2.4 Proof of Lemma 10

Throughout the proof, for brevity we let $t := \omega(uv)$.

To help comprehension, we break the proof into several steps.

(A) $\mathrm{cr}(G,\omega) \leqslant t \cdot d_{\omega^*}(F_u, F_v)$.

Start with the (unique) embedding of $G - uv$, and draw uv following a shortest F_uF_v-path in $(G^*_{u,v}, \omega^*)$. Then the sum of the weights of the edges crossed by uv equals the total weight of the shortest F_uF_v-path, that is, $d_{\omega^*}(F_u, F_v)$ (here we use the elementary, easy to check fact that crossings between adjacent edges can always be avoided; in this case, we may draw uv so that it crosses no edge adjacent to u or v). Since $\omega(uv) = t$, it follows that such a drawing of (G,ω) has exactly $t \cdot d_{\omega^*}(F_u, F_v)$ crossings.

(B) $\mathrm{cr}(G,\omega) = t \cdot d_{\omega^*}(F_u, F_v)$.

Consider a crossing-minimal drawing $\mathcal{D}$ of (G,ω). An immediate consequence of (P1) and (A) is that the drawing of $G_{u,v}$ induced by $\mathcal{D}$ is an embedding (that is, no two edges of $G_{u,v}$ cross each other in $\mathcal{D}$).

Now let F' (respectively, F'') denote the face of $G_{u,v}$ in which u (respectively, v) is drawn in $\mathcal{D}$. Clearly, for $i = 1,2,3$ the edge uu_i contributes at least $\omega(uu_i) \cdot d_{\omega^*}(u_i, F')$ crossings. Analogously, for $i = 1,2,3$ the edge vv_i contributes at least $\omega(vv_i) \cdot d_{\omega^*}(v_i, F'')$ crossings. Thus it follows from (P2) that the edges in $\{uu_1, uu_2, uu_3, vv_1, vv_2, vv_3\}$ contribute at least $t \cdot d_{\omega^*}(F_u, F') + t \cdot d_{\omega^*}(F_v, F'') = t \cdot (d_{\omega^*}(F_u, F') + d_{\omega^*}(F_v, F''))$ crossings. On the other hand, since the ends u, v of uv are in faces F' and F'', it follows that edge uv contributes at least $t \cdot d_{\omega^*}(F', F'')$ crossings. We conclude that $\mathcal{D}$ has at least $t \cdot \big(d_{\omega^*}(F_u, F') + d_{\omega^*}(F_v, F'') + d_{\omega^*}(F', F'')\big)$ crossings. Elementary triangle inequality arguments show that $d_{\omega^*}(F_u, F') + d_{\omega^*}(F_v, F'') + d_{\omega^*}(F', F'') \geqslant d_{\omega^*}(F_u, F_v)$, and so $\mathcal{D}$ has at least $t \cdot d_{\omega^*}(F_u, F_v)$ crossings. Thus $\mathrm{cr}(G,\omega) \geqslant t \cdot d_{\omega^*}(F_u, F_v)$. The reverse inequality is given in (A), and so (B) follows.

(C) *Crossing-criticality of the edges in $G_{u,v}$ and of the edge uv.*

Let e be any edge in $G_{u,v}$. We proceed similarly as in (A). Start with the (unique) embedding of $G - uv$, and draw uv following a shortest F_uF_v-path in $(G^*_{u,v}, \omega^*)$ that includes e^* (the existence of such a path is guaranteed by the balancedness of ω^*). This yields a drawing of (G,ω) with exactly $t \cdot d_{\omega^*}(F_u, F_v)$ crossings, in which e and uv cross each other. Since $\mathrm{cr}(G,\omega) = t \cdot d_{\omega^*}(F_u, F_v)$, it follows that e and uv are both crossed in a crossing-minimal drawing of (G,ω). Therefore both e and uv are crossing-critical in (G,ω).

(D) *Crossing-criticality of the edges $uu_1, uu_2, uu_3, vv_1, vv_2$, and vv_3.*

We prove the criticality of uu_1; the proof of the criticality of the other edges is totally analogous.

Consider the (unique) embedding of $G_{u,v}$. Put u in face U_1 (see property (P4)) and v in face F_v. Then draw uu_j, for $j = 1, 2, 3$, adding $\omega(uu_j) \cdot d_{\omega^*}(u_j, U_1)$ crossings with the edges in $G_{u,v}$. Since crossings between adjacent edges can always be avoided, it follows that uu_1, uu_2, uu_3 get drawn by adding $\omega(uu_1) \cdot d_{\omega^*}(u_1, U_1) + \omega(uu_2) \cdot d_{\omega^*}(u_2, U_1) + \omega(uu_3) \cdot d_{\omega^*}(u_3, U_1) = t \cdot d_{\omega^*}(F_u, U_1)$ crossings (using (P4)). Finally we draw vv_1, vv_2, vv_3 in face F_v. Now this last step may add crossings, but only of the edges vv_1, vv_2, vv_3 with the edges uu_1, uu_2, uu_3. In view of (P5), the last step added fewer than $9 \cdot (1/9) \min\{\ \omega(e) \mid e \in E(G_{u,v})\} = \min\{\ \omega(e) \mid e \in E(G_{u,v})\}$ crossings. We finally draw uv; since u is in face U_1 and v is in face F_v, it follows that uv can be drawn by adding $t \cdot d_{\omega^*}(U_1, F_v)$ crossings.

The described drawing $\mathcal{D}$ of G has then fewer than $t \cdot d_{\omega^*}(F_u, U_1) + t \cdot d_{\omega^*}(U_1, F_v) + \min\{\ \omega(e) \mid e \in E(G_{u,v})\} = t \cdot d_{\omega^*}(F_u, F_v) + \min\{\ \omega(e) \mid e \in E(G_{u,v})\} = \mathrm{cr}(G, \omega) + \min\{\ \omega(e) \mid e \in E(G_{u,v})\}$ crossings, where for the first equality we used the balancedness of ω^*, and for the second equality we used (B). Thus $\mathrm{cr}(\mathcal{D}) < \mathrm{cr}(G, \omega) + \min\{\ \omega(e) \mid e \in E(G_{u,v})\}$.

In $\mathcal{D}$, one weight unit of uu_1 contributes $d_{\omega^*}(u_1, U_1)$ crossings, and so uu_1 contributes $\omega(uu_1) \cdot d_{\omega^*}(u_1, U_1)$ crossings; note that (P4) implies that $\omega(uu_1) \cdot d_{\omega^*}(u_1, U_1) > 0$. Since obviously $d_{\omega^*}(u_1, U_1) \geqslant \min\{\omega(e) \mid e \in E(G_{u,v})\}$, it follows that one weight unit of uu_1 contributes at least $\min\{\omega(e) \mid e \in E(G_{u,v})\}$ crossings. Thus, if we decrease the weight of uu_1 by 1, then we obtain a drawing of $G - uu_1$ with fewer than $\mathrm{cr}(G, \omega)$ crossings. Therefore uu_1 is critical in (G, ω), as claimed. □

2.5 Proof of Lemma 11

An important ingredient in the proof of Lemma 11 is the following, somewhat curious statement for which we could not find any reference in the literature.

Proposition 12. *Let G be a 2-connected loopless graph, and let u, v be distinct vertices of G. Then there is a positive integer weight assignment μ such that every edge of (G, μ) belongs to a shortest uv-path.*

Proof. We make use of *st*-numberings [8]. We recall that if H is a graph with vertex set V and st is an edge of H, then an *st-numbering* of H is a bijection $g : V \to \{1, 2, \ldots, |V|\}$ such that $g(s) = 1, g(t) = |V|$, and for every $v \in V \setminus \{s, t\}$ there are edges xv and vy such that $g(x) < g(v) < g(y)$. It is known that if s, t is any pair of adjacent vertices in a 2-connected graph, then there is an *st*-numbering of H.

Now let G, u, v be as in the statement of the proposition, and let $n := |V(G)|$. It is easy to see that if u and v are not adjacent and the proposition holds for $G + uv$, then it also holds for G. Thus we may assume that u and v are adjacent. Let g be a uv-numbering on G, and let μ be the weight assignment defined as follows: for each edge xy of G, let $\mu(xy) = |g(x) - g(y)|$. It is straightforward to check that a μ-shortest uv-path in (G, μ) has length $n - 1$, and that every edge in G belongs to a μ-shortest uv-path. □

Proof of Lemma 11. We start with a balanced positive integer weight assignment μ^* on $G^*_{u,v}$. The existence of such a μ^* is guaranteed by Proposition 12, which applies since the connectivity assumption on $G_{u,v}$ implies that $G^*_{u,v}$ is also 3-connected. Let μ denote the (also positive integer) weight assignment naturally induced on $G_{u,v}$.

For each face $F \in \mathcal{F} \setminus \{F_u\}$, we let $H_{F,u}$ denote the halfspace defined by $d_{\mu^*}(u_1, F)x_1 + d_{\mu^*}(u_2, F)x_2 + d_{\mu^*}(u_3, F)x_3 \geqslant d_{\mu^*}(F_u, F)$, and let $\Delta_{F,u}$ denote its supporting plane. Let $\mathcal{P}_u$ denote the polyhedron defined by the (intersection of the) set of halfspaces $\{H_{F,u} \mid F \in \mathcal{F} \setminus \{F_u\}\}$ and the nonnegative octant $\{(x_1, x_2, x_3) \mid x_1 \geqslant 0, x_2 \geqslant 0, x_3 \geqslant 0\}$. It is clear that if x_1, x_2, and x_3 are all large enough then (x_1, x_2, x_3) is in $\mathcal{P}_u$, and so $\mathcal{P}_u$ is not empty.

Claim I. *There is a strictly positive rational point (x_1, x_2, x_3) in $\mathcal{P}_u$ that belongs to (not necessarily distinct) supporting planes $\Delta_{U_1,u}$, $\Delta_{U_2,u}$, $\Delta_{U_3,u}$, such that for $i = 1, 2, 3$ we have $d_{\mu^*}(u_i, U_i) > 0$.*

Proof. Suppose first that there is an $F \in \mathcal{F} \setminus \{F_u, F_{u_1}, F_{u_2}, F_{u_3}\}$ such that a facet of $\mathcal{P}_u$ is in $\Delta_{F,u}$. Since $F \in \mathcal{F} \setminus \{F_u, F_{u_1}, F_{u_2}, F_{u_3}\}$, then $d_{\mu^*}(u_i, F) > 0$ for $i = 1, 2, 3$. Thus the plane $\Delta_{F,u}$ intersects each of the coordinate axes in a positive coordinate, and so it follows that $\Delta_{F,u}$ (more specifically, the facet of $\mathcal{P}_u$ contained in $\Delta_{F,u}$) has a positive rational point (x_1, x_2, x_3) that satisfies the claim, with $U_1 = U_2 = U_3 = F$.

Suppose finally that there is no such $F \in \mathcal{F} \setminus \{F_u, F_{u_1}, F_{u_2}, F_{u_3}\}$. Since for $i, j \in \{1, 2, 3\}$ we have $d_{\mu^*}(u_i, F_{u_j}) \geqslant 0$, where equality holds if and only if $i = j$, it follows that for each $i \in \{1, 2, 3\}$ the plane $\Delta_{F_{u_i},u}$ is parallel to the i-th coordinate axis, and intersects each of the other two axes in strictly positive points. It then follows from elementary convex geometry arguments that there exists a strictly positive rational point (x_1, x_2, x_3) contained in $\Delta_{F_{u_1},u} \cap \Delta_{F_{u_2},u} \cap \Delta_{F_{u_3},u}$. Since no facet of $\mathcal{P}_u$ is in $\Delta_{F,u}$ for any $F \in \mathcal{F} \setminus \{F_u, F_{u_1}, F_{u_2}, F_{u_3}\}$, it follows that (x_1, x_2, x_3) is in $\mathcal{P}_u$. □

Analogously, for each face $F \in \mathcal{F} \setminus \{F_v\}$, we let $H_{F,v}$ denote the halfspace defined by $d_{\mu^*}(v_1, F)y_1 + d_{\mu^*}(v_2, F)y_2 + d_{\mu^*}(v_3, F)y_3 \geqslant d_{\mu^*}(F_v, F)$, and let $\Delta_{F,v}$ denote its supporting plane. Similarly, let $\mathcal{P}_v$ denote the polyhedron defined by the (intersection of the) set of halfspaces $\{H_{F,v} \mid F \in \mathcal{F} \setminus \{F_v\}\}$ and the nonnegative octant $\{(y_1, y_2, y_3) \mid y_1 \geqslant 0, y_2 \geqslant 0, y_3 \geqslant 0\}$.

The proof of the following statement is totally analogous to the proof of Claim I:

Claim II. *There is a strictly positive rational point (y_1, y_2, y_3) in $\mathcal{P}_v$ that belongs to (not necessarily distinct) supporting planes $\Delta_{V_1,v}$, $\Delta_{V_2,v}$, $\Delta_{V_3,v}$, such that for $i = 1, 2, 3$ we have $d_{\mu^*}(v_i, V_i) > 0$.* □

Let $(p_1/q_1, p_2/q_2, p_3/q_3)$ be a point as in Claim I, and let $(a_1/b_1, a_2/b_2, a_3/b_3)$ be a point as in Claim II, where all p_is, q_is, a_is, and b_is are integers. Let $M := q_1q_2q_3b_1b_2b_3$, and let $r_1 := p_1q_2q_3b_1b_2b_3, r_2 := p_2q_1q_3b_1b_2b_3$, $r_3 := p_3q_1q_2b_1b_2b_3$, $s_1 := a_1b_2b_3q_1q_2q_3, s_2 := a_2b_1b_3q_1q_2q_3$, and $s_3 := a_3b_1b_2q_1q_2q_3$.

Then (r_1, r_2, r_3) is a positive integer solution to the set of inequalities $\{d_{\mu^*}(u_1, F)r_1 + d_{\mu^*}(u_2, F)r_2 + d_{\mu^*}(u_3, F)r_3 \geqslant M \cdot d_{\mu^*}(F_u, F) : F \in \mathcal{F} \setminus \{F_u\}\}$, and for each $i = 1, 2, 3$, we

have $d_{\mu^*}(u_1, U_i)r_1 + d_{\mu^*}(u_2, U_i)r_2 + d_{\mu^*}(u_3, U_i)r_3 = M \cdot d_{\mu^*}(F_u, U_i)$.

Similarly, (s_1, s_2, s_3) is a positive integer solution to the set of inequalities $\{d_{\mu^*}(v_1, F)s_1 + d_{\mu^*}(v_2, F)s_2 + d_{\mu^*}(v_3, F)s_3 \geqslant M \cdot d_{\mu^*}(F_v, F) : F \in \mathcal{F} \setminus \{F_v\}\}$, and for each $i = 1, 2, 3$, we have $d_{\mu^*}(v_1, V_i)s_1 + d_{\mu^*}(v_2, V_i)s_2 + d_{\mu^*}(v_3, V_i)s_3 = M \cdot d_{\mu^*}(F_v, V_i)$.

Finally, let c be any integer greater than $M \cdot d_{\mu^*}(F_u, F_v)/(\min\{\mu(e) \mid e \in E(G_{u,v})\})^2$ and also greater than $9r_is_j/\min\{\mu(e) \mid e \in E(G_{u,v})\}$, for all $i, j \in \{1, 2, 3\}$.

Define the weight assignment ω on G as follows:

- $\omega(uv) = M$;
- $\omega(uu_i) = r_i$ and $\omega(vv_i) = s_i$ for $i = 1, 2, 3$;
- $\omega(e) = c \cdot \mu(e)$, for all edges e in $G_{u,v}$.

We finish the proof by showing that ω (and its induced weight assignment ω^* on $G^*_{u,v}$) satisfies (P1)–(P5).

To see that ω^* satisfies (P3), it suffices to note that ω^* inherits the balancedness (when restricted to $G^*_{u,v}$) from μ^*.

Now let e, e' be edges of $G_{u,v}$. Then $\omega(e)\omega(e') = c^2 \cdot \mu(e)\mu(e') \geqslant c^2 \cdot (\min\{\mu(f) \mid f \in E(G_{u,v})\})^2 > c \cdot M \cdot d_{\mu^*}(F_u, F_v) = \omega(uv)(c \cdot d_{\mu^*}(F_u, F_v)) = \omega(uv) \cdot d_{\omega^*}(F_u, F_v)$. This proves (P1).

Recall that (r_1, r_2, r_3) is a positive integer solution to the set of inequalities $\{d_{\mu^*}(u_1, F)r_1 + d_{\mu^*}(u_2, F)r_2 + d_{\mu^*}(u_3, F)r_3 \geqslant M \cdot d_{\mu^*}(F_u, F) : F \in \mathcal{F} \setminus \{F_u\}\}$, and for each $i = 1, 2, 3$, we have $d_{\mu^*}(u_1, U_i)r_1 + d_{\mu^*}(u_2, U_i)r_2 + d_{\mu^*}(u_3, U_i)r_3 = M \cdot d_{\mu^*}(F_u, U_i)$. Noting that for any faces $F, F' \in \mathcal{F} \setminus \{F_u\}$, $d_{\omega^*}(F, F') = c \cdot d_{\mu^*}(F, F')$, and using the definition of ω (and its induced ω^*), we immediately obtain (P2) (when $x = u$) and (P4) (when $(x, X) = (u, U)$). The proof of (P2) for the case $x = v$ and the proof of (P4) for the case $(x, X) = (v, V)$ are totally analogous.

Finally, we recall that we defined c so that $c > 9r_is_j/\min\{\mu(e) \mid e \in E(G_{u,v})\}$ for all $i, j \in \{1, 2, 3\}$. By the definition of ω, this is equivalent to $c \cdot \min\{\mu(e) \mid e \in E(G_{u,v})\} > 9\omega(uu_i)\omega(vv_j)$, that is, $\min\{\omega(e) \mid e \in E(G_{u,v})\} > 9\omega(uu_i)\omega(vv_j)$, which is in turn obviously equivalent to (P5). □

3 Concluding remarks and open questions

Let $\mathcal{G}$ be the class of graphs that can be made crossing-critical by a suitable multiplication of edges. In this work we have proved that a large family of graphs is contained in $\mathcal{G}$ (note that the cubic condition is only used around vertices $u, v, u_1, u_2, u_3, v_1, v_2$ and v_3; other vertices can have arbitrary degrees). Which other graphs belong to $\mathcal{G}$? Is there any hope of fully characterizing $\mathcal{G}$?

It is not difficult to prove that we can restrict our attention to simple graphs: if $\overline{G}$ is a graph with multiedges and G is a maximal simple graph contained in $\overline{G}$, then $\overline{G}$ is in $\mathcal{G}$ if and only if G is in $\mathcal{G}$.

One reason (but not necessarily the only one) for a graph G not to belong to $\mathcal{G}$, is the existence of an edge e of G with the following property: if $\mathcal{D}$ is a drawing of G in which e is crossed, then there is a drawing $\mathcal{D}'$ in which e is not crossed, and such that every two edges that cross each other in $\mathcal{D}'$ also cross each other in $\mathcal{D}$. It is immediately seen that such a graph is not in $\mathcal{G}$. As Jesús Leaños has pointed out, the easiest such instance is the graph $K^+_{3,3}$ obtained by adding to $K_{3,3}$ an edge (between vertices in the same chromatic class).

Following Širáň [11, 12], an edge e in a graph G is a *Kuratowski edge* if there is a subgraph H of G that contains e and is homeomorphic to a Kuratowski graph (that is, $K_{3,3}$ or K_5). It is trivial to see that the added edge in Leaños's example is not a Kuratowski edge of $K^+_{3,3}$. This observation naturally gives rise to the following.

Conjecture 13. If G is a graph all whose edges are Kuratowski edges, then G can be made crossing-critical by a suitable multiplication of its edges.

We remark that the converse of this statement is not true: Širáň [11] gave examples of graphs that contain crossing-critical edges that are not Kuratowski edges.

The only positive result we have in this direction is that Kuratowski edges can be made individually crossing-critical:

Proposition 14. *If e is a Kuratowski edge of a graph G, then e can be made crossing-critical by a suitable multiplication of the edges of G.*

Proof. Let H be a subgraph of G, homeomorphic to a Kuratowski graph, such that e is in H. Let f be another edge of H such that there is a drawing $\mathcal{D}_H$ of H with exactly one crossing, which involves e and f. Extend $\mathcal{D}_H$ to a drawing $\mathcal{D}$ of G. Let p be the number of crossings in $\mathcal{D}$. If $p = 1$ then e is already critical in G, so there is nothing to prove. Thus we may assume that $p \geqslant 2$. Add $p^2 - 1$ parallel edges to each of e and f, add $p^4 - 1$ parallel edges to all edges in $H \setminus \{e, f\}$, and do not add any parallel edge to the other edges of G. Let G' denote the resulting graph.

We claim that $\mathrm{cr}(G') \leqslant p^5$. To see this, consider the drawing $\mathcal{D}'$ of G' naturally induced by $\mathcal{D}$. It is easy to check that each crossing from $\mathcal{D}$ yields at most p^4 crossings in $\mathcal{D}'$ (here we use that e and f are the only edges in H that cross each other in $\mathcal{D}$). Thus $\mathcal{D}'$ has at most $p \cdot p^4 = p^5$ crossings, and so $\mathrm{cr}(G') \leqslant p^5$, as claimed.

On the other hand, it is clear that a drawing of G' in which e and f do not cross each other has at least p^6 crossings. Since $p^6 > p^5 \geqslant \mathrm{cr}(G')$, it follows that no such drawing can be optimal. Therefore e and f cross each other in every optimal drawing of G'. This immediately implies that e is critical in G'. □

The immediate next step towards Conjecture 13 seems already difficult enough so as to prompt us to state it:

Conjecture 15. Suppose that e, f are Kuratowski edges of a graph G. Then there exists a graph H, obtained by multiplying edges of G, such that both e and f are crossing-critical in H.

The proof technique of Proposition 14 cannot be applied to settle this conjecture. Indeed, if we were to apply the same ideas, in order to make e critical we would assign a weight to the other edges of G, and to make f critical we would also assign a weight to the other edges of G; the problem is that these edge assignments could (in principle) be irreconcilably different.

Finally, let us remark that the proof technique of Theorem 2 can be applied to other similar families of near-planar graphs. For instance, the vertices sufficiently far from u and v need not have degree 3. Moreover, neither u nor v needs to have degree 3; this requirement may be substituted by asking that there exist 3 vertices adjacent to u that are incident with faces distinct from each other and distinct from F_u (respectively, F_v). Actually, this last property is essential in our proof, and is the reason to require internal 3-connectivity instead of the weaker condition that $G - \{u, v\}$ is the subdivision of a 3-connected graph and $G - uv$ is subcubic.

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