A simple proof of an identity of Lacasse

Yidong Sun

Department of Mathematics,
Dalian Maritime University, 116026 Dalian, P.R. China
sydmath@yahoo.com.cn

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Abstract

In this note, using the derangement polynomials and their umbral representation, we give a simple proof of an identity conjectured by Lacasse in the study of the PAC-Bayesian machine learning theory.

Keywords: Derangement polynomial; Umbral operator.

1 Introduction

In his thesis [4], Lacasse introduced the functions $\xi(n)$ and $\xi_2(n)$ in the study of the PAC-Bayesian machine learning theory, where

$$\xi(n) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n}\right)^k \left(1-\frac{k}{n}\right)^{n-k},$$

$$\xi_2(n) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \left(\frac{k}{n}\right)^k \left(\frac{j}{n}\right)^j \left(1-\frac{k}{n}-\frac{j}{n}\right)^{n-k-j}.$$

Based on numerical verification, Lacasse presented the following conjecture.

Conjecture 1. For any integer $n \geqslant 1$, there holds

$$\xi_2(n) = \xi(n) + n. \tag{1}$$

Recently, by applying a multivariate Abel identity due to Hurwitz, Younsi [9] gave an algebraic proof of this conjecture. Later, using a decomposition of triply rooted trees into three doubly rooted trees, Chen, Peng and Yang [1] gave it a nice combinatorial interpretation. A very short proof was also obtained by Prodinger [5], based on the study of the tree function, with links to Lambert's W-function and Ramanujan's Q-function.

In this note, using the derangement polynomials and their umbral representation, we provide another simple proof of (1).

2 The derangement polynomials and the proof of (1)

Recall that the derangement polynomials $\{\mathcal{D}_n(\lambda)\}_{n\geqslant 0}$ are defined by

$$\mathcal{D}_n(\lambda) = \sum_{k=0}^{n} \binom{n}{k} D_k \lambda^{n-k}. \tag{2}$$

where $\mathcal{D}_n(1) = n!$ and $\mathcal{D}_n(0) = D_n$ is the n-th derangement number, counting permutations on $[n] = \{1, 2, \ldots, n\}$ with no fixed points. The derangement polynomials $\mathcal{D}_n(\lambda)$, also called λ-factorials of n, have been considerably investigated by Eriksen, Freij and Wästlund [2], Sun and Zhuang [8]. They have a basic recursive relation [2] and an Abel-type formula [8],

$$\mathcal{D}_n(\lambda+\mu) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_k(\lambda)\mu^{n-k}, \tag{3}$$

$$\mathcal{D}_n(\lambda+\mu) = \sum_{k=0}^{n} \binom{n}{k} (\lambda+k)^k(\mu-k-1)^{n-k}, \tag{4}$$

and obey the following property [8],

$$\sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_k(\lambda)\mathcal{D}_{n-k}(\mu+1) = (\lambda+\mu-1)^{n+1} + (n-\lambda-\mu+2)\mathcal{D}_n(\lambda+\mu). \tag{5}$$

Denote by $\mathbf{D}$ the umbral operator defined by $\mathbf{D}^n = D_n$ for $n \geqslant 0$ (See [3, 6, 7] for more information on the umbral calculus), then by (2) $\mathcal{D}_n(\lambda)$ can be represented as

$$\mathcal{D}_n(\lambda) = (\mathbf{D}+\lambda)^n.$$

Setting $\lambda = 0, \mu = n+1$ in (5), we have

$$\begin{aligned}
n^{n+1} + \mathcal{D}_n(n+1) &= \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_k(0)\mathcal{D}_{n-k}(n+2) \\
&= \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_k(0)\mu^{n-k}|_{\mu=(\mathbf{D}+n+2)} \\
&= \mathcal{D}_n(\mu)|_{\mu=(\mathbf{D}+n+2)} \qquad \text{by (3)} \\
&= \sum_{k=0}^{n} \binom{n}{k} k^k(\mu-k-1)^{n-k}|_{\mu=(\mathbf{D}+n+2)} \qquad \text{by (4)} \\
&= \sum_{k=0}^{n} \binom{n}{k} k^k \mathcal{D}_{n-k}(n-k+1),
\end{aligned}$$

which proves (1), if one notices that $\xi(n)$ and $\xi_2(n)$, by (4), can be rewritten as

$$\xi(n) = \frac{1}{n^n}\mathcal{D}_n(n+1) \text{ and } \xi_2(n) = \frac{1}{n^n}\sum_{k=0}^{n} \binom{n}{k} k^k \mathcal{D}_{n-k}(n-k+1).$$

Remark 2. By the nontrivial property of $\mathbf{D}$ [8],

$$(\mathbf{D}+\lambda)(\mathbf{D}+\lambda+n+1)^n = (n+\lambda)^{n+1},$$

one can get another expression for $\xi_2(n)$,

$$\begin{aligned}
\xi_2(n) = \xi(n) + n &= \frac{1}{n^n}\left(\mathcal{D}_n(n+1) + n^{n+1}\right) \\
&= \frac{1}{n^n}\left((\mathbf{D}+n+1)^n + \mathbf{D}(\mathbf{D}+n+1)^n\right) \\
&= \frac{1}{n^n}\left((\mathbf{D}+1)(\mathbf{D}+n+1)^n\right) = \frac{1}{n^n}\sum_{k=0}^{n}\binom{n}{k}(\mathbf{D}+1)^{k+1}n^{n-k} \\
&= \frac{1}{n^n}\sum_{k=0}^{n}\binom{n}{k}\mathcal{D}_{k+1}(1)n^{n-k} = \frac{1}{n^n}\sum_{k=0}^{n}\binom{n}{k}(k+1)!n^{n-k}.
\end{aligned}$$

This expression has been obtained by Younsi by using the Hurwitz identity on multivariate Abel polynomials and plays a critical role in his proof.

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