Non-classical hyperplanes of $DW(5, q)$

Bart De Bruyn
Department of Mathematics
Ghent University
Belgium
bdb@cage.ugent.be

Submitted: Jun 8, 2012; Accepted: Apr 18, 2013; Published: Apr 24, 2013
Mathematics Subject Classifications: 51A45, 51A50

Abstract

The hyperplanes of the symplectic dual polar space $DW(5, q)$ arising from embedding, the so-called classical hyperplanes of $DW(5, q)$, have been determined earlier in the literature. In the present paper, we classify non-classical hyperplanes of $DW(5, q)$. If $q$ is even, then we prove that every such hyperplane is the extension of a non-classical ovoid of a quad of $DW(5, q)$. If $q$ is odd, then we prove that every non-classical ovoid of $DW(5, q)$ is either a semi-singular hyperplane or the extension of a non-classical ovoid of a quad of $DW(5, q)$. If $DW(5, q)$, $q$ odd, has a semi-singular hyperplane, then $q$ is not a prime number.

Keywords: symplectic dual polar space, hyperplane, projective embedding

1 Introduction

The hyperplanes of the finite symplectic dual polar space $DW(5, q)$ that arise from some projective embedding, the so-called classical hyperplanes of $DW(5, q)$, have explicitly been determined earlier in the literature, see Cooperstein & De Bruyn [5], De Bruyn [7] and Pralle [21]. In the present paper, we give a rather complete classification for the non-classical hyperplanes of $DW(5, q)$. There are two standard constructions for such hyperplanes.

(1) Suppose $x$ is a point of $DW(5, q)$ and $O$ is a set of points of $DW(5, q)$ at distance 3 from $x$ such that every line at distance 2 from $x$ has a unique point in common with $O$. Then $x^+ \cup O$ is a non-classical hyperplane of $DW(5, q)$, the so-called semi-singular hyperplane with deepest point $x$.

(2) Suppose $Q$ is a quad of $DW(5, q)$. Then the points and lines contained in $Q$ define a generalized quadrangle $\overline{Q}$ isomorphic to $Q(4, q)$. If $O$ is a non-classical ovoid of $\overline{Q}$, then
the set of points of $DW(5, q)$ at distance at most 1 from $O$ is a non-classical hyperplane of $DW(5, q)$, the so-called extension of $O$. Several classes of non-classical ovoids of $Q(4, q)$ are known, see Section 2.2 for a discussion.

The following is our main result.

**Theorem 1.** (1) If $q$ is even, then every non-classical hyperplane of $DW(5, q)$ is the extension of a non-classical ovoid of a quad of $DW(5, q)$.

(2) If $q$ is odd, then every non-classical hyperplane of $DW(5, q)$ is either a semi-singular hyperplane or the extension of a non-classical ovoid of a quad of $DW(5, q)$.

Up to present, no semi-singular hyperplane of $DW(5, q)$ is known to exist. If a semi-singular hyperplane of $DW(5, q)$ exists, then $q$ must be odd (Theorem 19) and not a prime (Corollary 18).

The lines and quads through a given point $x$ of $DW(5, q)$ define a projective plane isomorphic to $PG(2, q)$ which we denote by $Res(x)$. If $H$ is a hyperplane of $DW(5, q)$ and $x$ is a point of $H$, then $\Lambda_H(x)$ denotes the set of lines through $x$ contained in $H$. We regard $\Lambda_H(x)$ as a set of points of $Res(x)$. If $\Lambda_H(x)$ is the whole set of points of $Res(x)$, then $x$ is called deep with respect to $H$.

The dual polar space $DW(5, q)$ has a nice full projective embedding $e$ in the projective space $PG(13, q)$, which is called the Grassmann embedding of $DW(5, q)$, see e.g. Cooperstein [4, Proposition 5.1]. A hyperplane of $DW(5, q)$ whose image under $e$ is contained in a hyperplane of of $PG(13, q)$ is said to arise from $e$. For a proof of the following proposition, we refer to Pasini [16, Theorem 9.3] or Cardinali & De Bruyn [3, Corollary 1.5].

**Proposition 2.** If $H$ is a hyperplane of $DW(5, q)$ arising from the Grassmann embedding of $DW(5, q)$, then for every point $x$ of $H$, $\Lambda_H(x)$ is one of the following sets of points of $Res(x)$: (1) a point; (2) a line; (3) the union of two distinct lines; (4) a nonsingular conic; (5) the whole set of points of $Res(x)$.

If $q \neq 2$, then the Grassmann embedding of $DW(5, q)$ is the so-called absolutely universal embedding of $DW(5, q)$ (Cooperstein [4, Theorem B], Kasikova & Shult [12, Section 4.6], Ronan [22]), implying that the classical hyperplanes of $DW(5, q)$ are precisely those hyperplanes arising from the Grassmann embedding. Combining Theorem 1 with Proposition 2, we easily find:

**Corollary 3.** If $H$ is a hyperplane of $DW(5, q)$, $q \neq 2$, then for every point $x$ of $H$, $\Lambda_H(x)$ is one of the following sets of points of $Res(x)$: (1) the empty set; (2) a point; (3) a line; (4) the union of two distinct lines; (5) a nonsingular conic; (6) the whole set of points of $Res(x)$. If $\Lambda_H(x)$ is the empty set, then $H$ is a semi-singular hyperplane whose deepest point lies at distance 3 from $x$. If $H$ is not a semi-singular hyperplane, then case (1) cannot occur.
The conclusion of Corollary 3 is false for the dual polar space $DW(5, 2)$. If $x$ is a point of $DW(5, 2)$, then for every set $Y$ of points of $Res(x) \cong PG(2, 2)$, there exists a hyperplane $H$ through $x$ such that $\Lambda_H(x) = Y$, see Pralle [21, Table 1].

If $n \geq 4$, then the symplectic dual polar space $DW(2n-1, q)$ has many full subgeometries isomorphic to $DW(5, q)$. So, Corollary 3 reveals information on the local structure of any hyperplane of any symplectic dual polar space $DW(2n-1, q)$, where $q \neq 2$ and $n \geq 4$.

Theorem 1 will be proved in Section 3. In Section 2, we give the basic definitions (including some of the notions already mentioned above) and basic properties which will play a role in the proof of Theorem 1.

## 2 Preliminaries

### 2.1 The dual polar space $DW(5, q)$

Let $S = (P, L, I)$ be a point-line geometry with nonempty point-set $P$, line set $L$ and incidence relation $I \subseteq P \times L$. A set $H \subseteq P$ is called a hyperplane of $S$ if every line of $S$ has either one or all of its points in $H$. A full projective embedding of $S$ is an injective mapping $e$ from $P$ to the point-set of a projective space $\Sigma$ satisfying (i) $\langle e(P) \rangle_\Sigma = \Sigma$; (ii) $\{e(x) \mid (x, L) \in I\}$ is a line of $\Sigma$ for every line $L$ of $S$. If $e : S \to \Sigma$ is a projective embedding of $S$ and $\Pi$ is a hyperplane of $\Sigma$, then $e^{-1}(e(P) \cap \Pi)$ is a hyperplane of $S$. A hyperplane of $S$ is said to be classical if it is of the form $e^{-1}(e(P) \cap \Pi)$, where $e$ is some full projective embedding of $S$ into a projective space $\Sigma$ and $\Pi$ is some hyperplane of $\Sigma$.

Distances $d(\cdot, \cdot)$ in $S$ will be measured in its collinearity graph. If $x$ is a point of $S$ and $i \in \mathbb{N}$, then $\Gamma_i(x)$ denotes the set of points of $S$ at distance $i$ from $x$. Similarly, if $X$ is a nonempty set of points and $i \in \mathbb{N}$, then $\Gamma_i(X)$ denotes the set of all points at distance $i$ from $X$, i.e. the set of all points $y$ for which $\min\{d(y, x) \mid x \in X\} = i$.

Let $W(5, q)$ be the polar space whose subspaces are the subspaces of $PG(5, q)$ that are totally isotropic with respect to a given symplectic polarity of $PG(5, q)$, and let $DW(5, q)$ denote the associated dual polar space. The points and lines of $DW(5, q)$ are the totally isotropic planes and lines of $PG(5, q)$, with incidence being reverse containment. The dual polar space $DW(5, q)$ belongs to the class of near polygons introduced by Shult and Yanushka in [23]. This means that for every point $x$ and every line $L$, there exists a unique point on $L$ nearest to $x$. The maximal distance between two points of $DW(5, q)$ is equal to 3. The dual polar space $DW(5, q)$ has $(q+1)(q^2+1)(q^3+1)$ points, $q+1$ points on each line and $q^2 + q + 1$ lines through each point.

If $x$ and $y$ are two points of $DW(5, q)$ at distance 2 from each other, then the smallest convex subspace $\langle x, y \rangle$ of $DW(5, q)$ containing $x$ and $y$ is called a quad. A quad $Q$ of $DW(5, q)$ consists of all totally isotropic planes of $W(5, q)$ that contain a given point $x_Q$ of $W(5, q)$. Any two lines $L$ and $M$ of $DW(5, q)$ that meet in a unique point are contained in a unique quad. We denote this quad by $\langle L, M \rangle$. Obviously, we have $\langle L, M \rangle = \langle x, y \rangle$ where $x$ and $y$ are arbitrary points of $L \setminus M$ and $M \setminus L$, respectively. The points and
lines of $DW(5, q)$ that are contained in a given quad $Q$ define a point-line geometry $\tilde{Q}$ isomorphic to the generalized quadrangle $Q(4, q)$ of the points and lines of a nonsingular parabolic quadric of $PG(4, q)$. If $Q$ is a quad of $DW(5, q)$ and $x$ is a point not contained in $Q$, then $Q$ contains a unique point $\pi_Q(x)$ collinear with $x$ and $d(x, y) = 1 + d(\pi_Q(x), y)$ for every point $y$ of $Q$. If $Q_1$ and $Q_2$ are two distinct quads of $DW(5, q)$, then $Q_1 \cap Q_2$ is either empty or a line of $DW(5, q)$. If $Q_1 \cap Q_2 = \emptyset$, then the map $Q_1 \to Q_2; x \mapsto \pi_Q(x)$ is an isomorphism between $\tilde{Q}_1$ and $\tilde{Q}_2$.

2.2 Hyperplanes of $Q(4, q)$

By Payne and Thas [18, 2.3.1], every hyperplane of the generalized quadrangle $Q(4, q)$ is either the perp $x^\perp$ of a point $x$, a $(q + 1) \times (q + 1)$-subgrid or an ovoid. An ovoid of $Q(4, q)$ is classical if it is an elliptic quadric $Q^-(3, q) \subseteq Q(4, q)$. For many values of $q$, non-classical ovoids of $Q(4, q)$ do exist: (i) $q = p^h$ with $p$ an odd prime and $h \geq 2$ [11]; (ii) $q = 2^{2h+1}$ with $h \geq 1$ [26]; (iii) $q = 3^{2h+1}$ with $h \geq 1$ [11]; (iv) $q = 3^h$ with $h \geq 3$ [24]; (v) $q = 3^5$ [19]. For several prime powers $q$, it is known that all ovoids of $Q(4, q)$ are classical:

**Proposition 4.** • ([2, 15]) Every ovoid of $Q(4, 4)$ is classical.

• ([13, 14]) Every ovoid of $Q(4, 16)$ is classical.

• ([1]) Every ovoid of $Q(4, q)$, $q$ prime, is classical.

A set $G$ of hyperplanes of $Q(4, q)$ is called a pencil of hyperplanes if every point of $Q(4, q)$ is contained in either 1 or all elements of $G$. The following lemma is precisely Lemma 3.2 and Corollary 3.3 of De Bruyn [8].

**Lemma 5.** If $G_1$ and $G_2$ are two distinct classical hyperplanes of $Q(4, q)$, then through every point $x$ of $Q(4, q)$ not contained in $G_1 \cup G_2$, there exists a unique classical hyperplane $G_x$ satisfying $G_x \cap G_1 = G_1 \cap G_2 = G_2 \cap G_x$. As a consequence, any two distinct classical hyperplanes of $Q(4, q)$ are contained in a unique pencil of classical hyperplanes of $Q(4, q)$.

2.3 Hyperplanes of $DW(5, q)$

Since $DW(5, q)$ is a near polygon, the set of points of $DW(5, q)$ at distance at most 2 from a given point $x$ is a hyperplane of $DW(5, q)$, the so-called singular hyperplane with deepest point $x$. If $O$ is a set of points of $DW(5, q)$ at distance 3 from a given point $x$ such that every line at distance 2 from $x$ has a unique point in common with $O$, then $x^\perp \cup O$ is a hyperplane of $DW(5, q)$, a so-called semi-singular hyperplane of $DW(5, q)$ with deepest point $x$. If $Q$ is a quad of $DW(5, q)$ and $G$ is a hyperplane of $\tilde{Q} \cong Q(4, q)$, then $Q \cup \{x \in \Gamma_1(Q) \mid \pi_Q(x) \in G\}$ is a hyperplane of $DW(5, q)$, the so-called extension of $G$.

If $H$ is a hyperplane of $DW(5, q)$ and $Q$ is a quad, then either $Q \subseteq H$ or $Q \cap H$ is a hyperplane of $Q \cong Q(4, q)$. If $Q \subseteq H$, then $Q$ is called a deep quad. If $Q \cap H = x^\perp \cap Q$ for some point $x \in Q$, then $Q$ is called singular with respect to $H$ and $x$ is called the deep
point of $Q$. The quad $Q$ is called ovoidal (respectively, subquadrangular) with respect to $H$ if and only if $Q \cap H$ is an ovoid (respectively, a $(q + 1) \times (q + 1)$-subgrid) of $Q$. A hyperplane $H$ of $DW(5, q)$ is called locally singular (locally subquadrangular, respectively locally ovoidal) if every non-deep quad of $DW(5, q)$ is singular (subquadrangular, respectively ovoidal) with respect to $H$. A hyperplane that is locally singular, locally ovoidal or locally subquadrangular is also called a uniform hyperplane. In the following proposition, we collect a number of known results which we will need to invoke later in the proof of the Main Theorem.

**Proposition 6.** (1) The dual polar space $DW(5, q), q \neq 2$, has no locally subquadrangular hyperplanes.

(2) The dual polar space $DW(5, q)$ has no locally ovoidal hyperplanes.

(3) Every nonuniform hyperplane of $DW(5, q)$ admits a singular quad.

Proposition 6(1) is due to Pasini & Shpectorov [17]. Locally ovoidal hyperplanes of $DW(5, q)$ are just ovoids and cannot exist by Thomas [25, Theorem 3.2], see also Cooperstein and Pasini [6]. Proposition 6(3) is due to Pralle [20].

The classical hyperplanes of the dual polar space $DW(5, q)$ have already been classified in the literature. The dual polar space $DW(5, q), q \neq 2$, has six isomorphism classes of classical hyperplanes by Cooperstein & De Bruyn [5] and De Bruyn [7]. This fact is not true if $q = 2$. The dual polar space $DW(5, 2)$ has twelve isomorphism classes of hyperplanes by Pralle [21], see also De Bruyn [7, Section 9]. Observe that all these hyperplanes are classical by Ronan [22, Corollary 2]. By De Bruyn [8], the classical hyperplanes of $DW(5, q)$ can be characterized as follows.

**Proposition 7.** The classical hyperplanes of $DW(5, q)$ are precisely those hyperplanes $H$ of $DW(5, q)$ that satisfy the following property: if $Q$ is an ovoidal quad, then $Q \cap H$ is a classical ovoid of $Q$.

### 2.4 Hyperbolic sets of quads of $DW(5, q)$

As in Section 2.1, let $W(5, q)$ be the polar space associated with a symplectic polarity of $PG(5, q)$. If $L$ is a hyperbolic line of $PG(5, q)$ (i.e. a line of $PG(5, q)$ that is not a line of $W(5, q)$), then the set of the $q + 1$ (mutually disjoint) quads of $DW(5, q)$ corresponding to the points of $L$ satisfy the property that every line that meets at least two of its members meets each of its members in a unique point. Any set of $q + 1$ quads that is obtained in this way will be called a hyperbolic set of quads of $DW(5, q)$. Every two disjoint quads $Q_1$ and $Q_2$ of $DW(5, q)$ are contained in a unique hyperbolic set of quads of $DW(5, q)$. We will denote this hyperbolic set of quads by $\mathcal{H}(Q_1, Q_2)$. Considering all the lines meeting $Q_1$ and $Q_2$, we easily see that the following holds.

**Lemma 8.** Let $\{Q_1, Q_2, \ldots, Q_{q+1}\}$ be a hyperbolic set of quads of $DW(5, q)$ and let $H$ be a hyperplane of $DW(5, q)$ such that $H \cap Q_1$ and $\pi_{Q_1}(H \cap Q_2)$ are distinct hyperplanes of $\overline{Q}_1$. Then $\{\pi_{Q_i}(H \cap Q_i) \mid 1 \leq i \leq q + 1\}$ is a pencil of hyperplanes of $\overline{Q}_1$.  

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3 Proof of Theorem 1

Throughout this section, we suppose that $H$ is an arbitrary hyperplane of $DW(5, q)$. In De Bruyn [9], we classified for every field $\mathbb{K}$ of size at least three the hyperplanes of $DW(5, \mathbb{K})$ containing a quad. The main theorem of [9] implies the following:

**Proposition 9.** Every non-classical hyperplane of $DW(5, q)$, $q \neq 2$, containing a quad is the extension of a non-classical ovoid of a quad.

We have already mentioned above that every hyperplane of $DW(5, 2)$ is classical by Ronan [22, Corollary 2]. Since we are interested in the classification of all non-classical hyperplanes of $DW(5, q)$, we may by the above assume that the following holds:

**Assumption:** We have $q \geq 3$ and the hyperplane $H$ does not contain quads.

We denote by $v$ the total number of points of $H$ and by $l$ the total number of lines of $DW(5, q)$ contained in $H$. In Section 3.1, we prove that there are only three possible values for $v$, namely $q^5 + q^3 + q^2 + q + 1$, $q^5 + q^4 + q^3 + q^2 + 2q + 1$ or $q^5 + q^4 + q^3 + q^2 + q + 1$. In Section 3.2, we prove that if $v = q^3 + q^2 + q + 1$, then $H$ is a semi-singular hyperplane. We also prove there that semi-singular hyperplanes cannot exist if $q$ is even. In [10] (see also Corollary 18), the nonexistence of semi-singular hyperplanes was already shown for prime values of $q$. In Section 3.3, we prove that the case $v = q^5 + q^4 + q^3 + q^2 + 2q + 1$ cannot occur and in Section 3.4, we prove that $H$ must be classical if $v = q^5 + q^4 + q^3 + q^2 + q + 1$. All these results together imply that Theorem 1 must hold.

3.1 The possible values of $v$

The following lemma is an immediate consequence of Proposition 6.

**Lemma 10.** The hyperplane admits singular quads.

**Lemma 11.** We have $l = \frac{v(q^2 + q + 1) - (q^2 + 1)(q^3 + 1)(q^2 + q + 1)}{q}$.

**Proof.** We count the number of lines not contained in $H$. There are $(q + 1)(q^2 + 1)(q^3 + 1) - v$ points outside $H$ and each of these points is contained in $q^2 + q + 1$ lines which contain a unique point of $H$. Hence, the total number of lines not contained in $H$ is equal to \( \frac{(q^2 + 1)(q^3 + 1)(q^2 + q + 1)}{q} \). Since the total number of lines of $DW(5, q)$ equals $(q^2 + 1)(q^3 + 1)(q^2 + q + 1)$, we have $l = (q^2 + 1)(q^3 + 1)(q^2 + q + 1) - \frac{(q^2 + 1)(q^3 + 1)(q^2 + q + 1)}{q} = \frac{v(q^2 + q + 1) - (q^2 + 1)(q^3 + 1)(q^2 + q + 1)}{q}$. \hfill \Box

**Lemma 12.** If $Q$ is a singular quad with deep point $x$, then one of the following cases occurs:

1. $x^1 \cap H = x^1 \cap Q$;
2. there exists a line $L$ through $x$ not contained in $Q$ such that $x^1 \cap H = (x^1 \cap Q) \cup L$;
3. there exists a quad $R$ through $x$ distinct from $Q$ such that $x^1 \cap H = (x^1 \cap Q) \cup (x^1 \cap R)$;
4. $x^1 \subseteq H$.
Proof. Since \( x^+ \cap Q \subseteq x^+ \cap H \), \(|\Lambda_H(x)|\geq q+1\). If \(|\Lambda_H(x)|\in\{q+1,q+2\}\), then either case (1) or (2) of the lemma occurs. Suppose therefore that \(|\Lambda_H(x)|\geq q+3\) and let \(L_1\) and \(L_2\) be two distinct lines through \(x\) that are contained in \(H\), but not in \(Q\). Put \(R := \langle L_1, L_2 \rangle\). Since \(L_1 \subseteq R \cap H\), \(L_2 \subseteq R \cap H\) and \(R \cap Q \subseteq R \cap H\), \(R\) is singular with deep point \(x\) and hence every line of \(R\) through \(x\) is contained in \(H\). So, \(|\Lambda_H(x)|\geq 2q+1\).

If \(|\Lambda_H(x)| = 2q+1\), then obviously case (3) of the lemma occurs. Suppose therefore that \(|\Lambda_H(x)|\geq 2q+2\). Then there exists a line \(L_3 \subseteq H\) through \(x\) not contained in \(Q \cup R\). If \(Q'\) is a quad through \(L_3\) distinct from \(\langle L_3, Q \cap R \rangle\), then since \(Q' \cap Q \subseteq H\), \(Q' \cap R \subseteq H\) and \(L_3 \subseteq H\), \(Q'\) is singular with deep point \(x\) and hence every line of \(Q'\) through \(x\) is contained in \(H\). It follows that all lines of \(DW(5,q)\) through \(x\) are contained in \(H\), except maybe for the \(q - 1\) lines through \(x\) contained in \(\langle L_3, Q \cap R \rangle\) and distinct from \(L_3\) and \(Q \cap R\). Let \(L'\) be one of these \(q - 1\) lines and let \(Q''\) be a quad through \(L'\) distinct from \(\langle L_3, Q \cap R \rangle\). Since \(q \geq 3\) lines of \(Q''\) through \(x\) are contained in \(H\), \(Q''\) is singular with deep point \(x\) and hence also \(L'\) is contained in \(H\). So, \(x^+ \subseteq H\) and case (4) of the lemma occurs.

Lemma 13. If \(Q\) is a singular quad with deep point \(x\), then \(|\Gamma_3(x) \cap H| = q^5\).

Proof. Every point of \(\Gamma_3(x) \cap H\) is collinear with a unique point of \(\Gamma_2(x) \cap Q\). Conversely, every point \(u\) of \(\Gamma_2(x) \cap Q\) is collinear with precisely \(q^2\) points of \(\Gamma_3(x) \cap H\). (One on each line through \(u\) not contained in \(Q\).) Hence, \(|\Gamma_3(x) \cap H| = |\Gamma_2(x) \cap Q| \cdot q^2 = q^5\).

Lemma 14. Suppose \(Q\) is a singular quad with deep point \(x\).

- If case (1) of Lemma 12 occurs, then \(v = q^5 + q^4 + q^3 + q^2 + q + 1\) and \(l = q^5 + q^4 + q^3 + q^2 + q + 1\).
- If case (2) of Lemma 12 occurs, then \(v = q^5 + q^4 + q^3 + q^2 + 2q + 1\) and \(l = (q^2 + q + 1)(q^3 + 2)\).
- If case (3) of Lemma 12 occurs, then \(v = q^5 + q^4 + q^3 + q^2 + q + 1\) and \(l = q^5 + q^4 + q^3 + q^2 + q + 1\).
- If case (4) of Lemma 12 occurs, then \(v = q^5 + q^3 + q^2 + q + 1\) and \(l = q^2 + q + 1\).

Proof. Suppose case (1) of Lemma 12 occurs. Then \(x\) is contained in 1 singular quad that has \(x\) as deep point (namely \(Q\)) and \(q^2 + q\) singular quads that do not have \(x\) as deep point. In this case, \(|\Gamma_0(x) \cap H| = 1\), \(|\Gamma_1(x) \cap H| = q^2 + q\), \(|\Gamma_2(x) \cap H| = (q + 2)q = q^2 + 2q\), \(|\Gamma_3(x) \cap H| = q^5\).

Suppose case (2) of Lemma 12 occurs. Then \(x\) is contained in 1 singular quad with deep point equal to \(x\), \(q + 1\) subquadrangular quads and \(q^2 - 1\) singular quads with deep point different from \(x\). In this case, \(|\Gamma_0(x) \cap H| = 1\), \(|\Gamma_1(x) \cap H| = (q + 2)q = q^2 + 2q\), \(|\Gamma_2(x) \cap H| = 1 \cdot 0 + (q + 1) \cdot q^2 + (q^2 - 1) \cdot q^2 = q^4 + q^3\) and \(|\Gamma_3(x) \cap H| = q^5\).

Suppose case (3) of Lemma 12 occurs. Then \(x\) is contained in 2 singular quads with deep point \(x\), \(q - 1\) singular quads with deep point different from \(x\) and \(q^2\) subquadrangular
quads. In this case, $|\Gamma_0(x) \cap H| = 1$, $|\Gamma_1(x) \cap H| = (2q + 1)q = 2q^2 + q$, $|\Gamma_2(x) \cap H| = 2 \cdot 0 + (q - 1) \cdot q^2 + q^3 \cdot q^2 = q^4 + q^3 - q^2$ and $|\Gamma_3(x) \cap H| = q^5$. Hence, $v = 1 + (2q^2 + q) + (q^4 + q^3 - q^2) + q^5 = q^3 + q^4 + q^2 + q^2 + q + 1$.

Suppose case (4) of Lemma 12 occurs. Then $x$ is contained in $q^2 + q + 1$ singular quads that have $x$ as deep point. Hence, $v = |\Gamma_0(x) \cap H| + |\Gamma_1(x) \cap H| + |\Gamma_2(x) \cap H| + |\Gamma_3(x) \cap H| = 1 + q(q^2 + q + 1) + 0 + q^5 = q^5 + q^3 + q^2 + q + 1$.

In each of the four cases, the value of $l$ can be derived from Lemma 11. □

By Lemmas 10, 12 and 14, we have:

**Corollary 15.** $v \in \{q^5 + q^3 + q^2 + q + 1, q^5 + q^4 + q^3 + q^2 + q + 1, q^5 + q^4 + q^3 + q^2 + q + 1\}$.

We see that if case (2) of Lemma 12 occurs for one singular quad $Q$, then case (2) occurs for all singular quads $Q$. A similar remark holds applies to case (4) of Lemma 12.

### 3.2 The case $v = q^5 + q^3 + q^2 + q + 1$

Let $Q^*$ denote a singular quad and $x^*$ its deep point.

**Lemma 16.** If $v = q^5 + q^3 + q^2 + q + 1$, then $H$ is a semi-singular hyperplane of $DW(5, q)$ with deepest point $x^*$.

**Proof.** If $v = q^5 + q^3 + q^2 + q + 1$, then case (4) of Lemma 12 occurs for the pair $(Q^*, x^*)$. So, we have that $x^* \perp H$ and $\Gamma_2(x^*) \cap H = \emptyset$ (no deep quad through $x^*$). Since $\Gamma_2(x^*) \cap H = \emptyset$, every line at distance 2 from $x^*$ contains a unique point of $\Gamma_3(x^*) \cap H$. It follows that $H$ is a semi-singular hyperplane of $DW(5, q)$ with deepest point $x^*$.

The following proposition was proved in De Bruyn and Vandecasteele [10, Corollary 6.3].

**Proposition 17.** If $q$ is a prime power such that every ovoid of $Q(4, q)$ is classical, then $DW(5, q)$ does not have semi-singular hyperplanes.

By Propositions 4 and 17, we have

**Corollary 18.** If $q$ is prime, then $DW(5, q)$ has no semi-singular hyperplanes.

We will now use hyperbolic sets of quads of $DW(5, q)$ to prove the nonexistence of semi-singular hyperplanes of $DW(5, q)$, $q$ even.

**Theorem 19.** The dual polar space $DW(5, q)$, $q$ even, has no semi-singular hyperplanes.

**Proof.** Suppose $H$ is a semi-singular hyperplane of $DW(5, q)$, $q$ even, and as before let $x^*$ denote the deepest point of $H$. Let $Q$ be a quad through $x^*$, let $G$ be a $(q + 1) \times (q + 1)$-subgrid of $\overline{Q}$ not containing $x^*$, let $L_1$ and $L_2$ be two disjoint lines of $G$ and let $Q_i$, $i \in \{1, 2\}$, be a quad through $L_i$ distinct from $Q$. Then $Q_1$ and $Q_2$ are disjoint.
\( \mathcal{H} = \mathcal{H}(Q_1, Q_2) \). Every \( Q_3 \in \mathcal{H} \) intersects \( Q \) in a line of \( G \) and hence \( x^* \not\in Q_3 \). It follows that every \( Q_3 \in \mathcal{H} \) is ovoidal with respect to \( H \). Suppose \( Q_3 \in \mathcal{H} \setminus \{Q_1\} \) and \( x_3 \in Q_3 \cap H \) such that \( x_1 = \pi_{Q_1}(x_3) \in Q_1 \cap H \). Then the line \( x_1x_3 \) is contained in \( H \) and hence \( x^* \in x_1x_3 \). But this is impossible, since no quad of \( \mathcal{H} \) contains \( x^* \). Hence, \( \pi_{Q_1}(Q_3 \cap H) \) is disjoint from \( Q_1 \cap H \). By Lemma 8, the set \( \{\pi_{Q_1}(Q_3 \cap H) \mid Q_3 \in \mathcal{H}\} \) is a partition of \( Q_1 \) into ovoids. This is however impossible since the generalized quadrangle \( Q(4, q) \), \( q \) even, has no partition in ovoids by Payne and Thas [18, Theorem 1.8.5].

\[ \Box \]

### 3.3 The case \( v = q^5 + q^4 + q^3 + q^2 + 2q + 1 \)

We suppose that \( v = q^5 + q^4 + q^3 + q^2 + 2q + 1 \) and \( l = (q^2 + q + 1)(q^3 + 2) \). Recall that if \( Q \) is a singular quad and \( x \) is the deep point of \( Q \), then case (2) of Lemma 12 occurs for the pair \((Q, x)\).

**Lemma 20.** Let \( Q \) be a singular quad, let \( x \) be the deep point of \( Q \), let \( L \) be the line through \( x \) not contained in \( Q \) such that \( x^\perp \cap H = (x^\perp \cap Q) \cup L \) and let \( y \) be a point of \( L \setminus \{x\} \). Then there are \( q + 1 \) lines \( L_1, L_2, \ldots, L_{q+1} \) through \( y \) different from \( L \) that are contained in \( H \). The \( q + 2 \) lines \( L, L_1, L_2, \ldots, L_{q+1} \) form a hyperoval of the projective plane \( \text{Res}(y) \cong \text{PG}(2, q) \). (Hence, \( q \) must be even.)

**Proof.** The \( q + 1 \) quads \( R_1, \ldots, R_{q+1} \) through \( L \) determine a partition of the set of lines through \( y \) different from \( L \). Each of these quads is subquadrangular. Hence, \( R_i, i \in \{1, 2, \ldots, q + 1\} \), contains a unique line \( L_i \neq L \) through \( y \) that is contained in \( H \).

For all \( i, j \in \{1, 2, \ldots, q + 1\} \) with \( i \neq j \), the lines \( L, L_i \) and \( L_j \) are not contained in a quad since the quad \( \langle L, L_i \rangle \) is subquadrangular. Suppose there exist mutually distinct \( i, j, k \in \{1, 2, \ldots, q + 1\} \) such that \( L_i, L_j \) and \( L_k \) are contained in a quad \( Q' \). Then \( L \) is not contained in \( Q' \) and hence \( Q \cap Q' = \emptyset \). Since \( L_i, L_j \) and \( L_k \) are contained in \( H \), \( Q' \) is singular with deep point \( y \). Let \( z' \in Q' \setminus y^\perp \) and \( z := \pi_Q(z') \). Since \( z \) and \( z' \) are not contained in \( H \), the line \( zz' \) contains a unique point \( z'' \in H \). Let \( Q'' \) denote the unique quad through \( z'' \) intersecting \( L \) in a point \( u \). Then \( Q'' \in \mathcal{H}(Q, Q') \). So, every point of \( u^\perp \cap Q'' \) is contained in a line joining a point of \( y^\perp \cap Q' \) with a point of \( x^\perp \cap Q \) and hence is contained in \( H \). Since also \( z'' \in H \), \( Q'' \subseteq H \), contradicting the fact that there are no deep quads.

\[ \Box \]

**Lemma 21.** There are four possible types of points in \( H \):

(A) points \( x \) for which \( \Lambda_H(x) \) is the union of a line of \( \text{Res}(x) \) and a point of \( \text{Res}(x) \) not belonging to that line;

(B) points \( x \) for which \( \Lambda_H(x) \) is a hyperoval of \( \text{Res}(x) \);

(C) points \( x \) for which \( |\Lambda_H(x)| = 2 \);

(D) points \( x \) for which \( \Lambda_H(x) \) is empty.

Moreover, we have:

(i) Every point of Type (A) has distance 1 from precisely \( q^2 - 1 \) points of Type (A), \( q \) points of Type (B) and \( q + 1 \) points of Type (C).

(ii) Every point of Type (B) has distance 1 from precisely \( q + 2 \) points of Type (A), \((q + 2)(q - 1) \) points of Type (B) and 0 points of Type (C).
(iii) Every point of Type (C) has distance 1 from precisely 2q points of Type (A), 0 points of Type (B) and 0 points of Type (C).

**Proof.** Suppose $Q^*$ is a singular quad and $x^*$ is its deep point. Consider the collinearity graph $\Gamma$ of $DW(5,q)$ and let $\Gamma_H$ denote the subgraph of $\Gamma$ induced on the vertex set $H$. Suppose $x$ is a point of $H$ such that $x$ and $x^*$ belong to different connected components of $\Gamma_H$. We prove that $\Lambda_H(x)$ is empty. Suppose to the contrary that there exists a line $L$ through $x$ contained in $H$. If $L$ meets $Q^*$, then $L \cap Q^*$ must be contained in $x^\perp$, contradicting the fact that $x^*$ and $x$ belong to different connected components of $\Gamma_H$. So, $L$ is disjoint from $Q^*$. Then $\pi_{Q^*}(L)$ meets $x^\perp$ and hence $x^*$ and $x$ are connected by a path of $\Gamma_H$, again a contradiction.

Notice that by Lemma 14 and the fact that $v = q^5 + q^4 + q^3 + q^2 + 2q + 1$, $x^*$ is a point of Type (A). So, in order to prove the first part of the lemma, it suffices to verify that every vertex $x \in \{A,B,C\}$, of $\Gamma_H$ is adjacent with only vertices of Type (A), (B) or (C). As a by-product of our verification, also the conclusions of the second part of the lemma will be obtained.

First, suppose that $x$ is a point of Type (A). Without loss of generality, we may suppose that $x = x^*$. Let $L^*$ denote the unique line through $x^*$ such that $x^\perp \cap H = (x^\perp \cap Q^*) \cup L^*$. By Lemma 20, every point of $L^* \setminus \{x^*\}$ has Type (B). Now, let $L$ be a line through $x^*$ contained in $Q^*$. Then $(L, L^*)$ is a subquadrangular quad. Any quad through $L$ different from $(L, L^*)$ and $Q^*$ is singular with deep point contained in $L \setminus \{x^*\}$. By Lemmas 12 and 14 and the fact that $v = q^5 + q^4 + q^3 + q^2 + 2q + 1$, every point of $L \setminus \{x^*\}$ is the deep point of at most 1 such singular quad. Hence, $q-1$ points of $L \setminus \{x^*\}$ have Type (A) and the remaining point of $L \setminus \{x^*\}$ has type (C).

Suppose $x$ is a point of Type (C). Let $L_1$ and $L_2$ denote the two lines through $x$ that are contained in $H$. Then $(L_1, L_2)$ is a subquadrangular quad. If $Q$ is a quad through $L_1$ distinct from $(L_1, L_2)$, then $Q$ is singular with deep point on $L_1 \setminus \{x\}$. By Lemmas 12 and 14 and the fact that $v = q^5 + q^4 + q^3 + q^2 + 2q + 1$, every point of $L_1 \setminus \{x\}$ is the deep point of at most 1 such singular quad. It follows that every point of $L_1 \setminus \{x\}$ has Type (A). In a similar way, one shows that every point of $L_2 \setminus \{x\}$ has Type (A).

Suppose $x$ is a point of Type (B). Let $L$ be an arbitrary line through $x$ contained in $H$. Every quad through $L$ is subquadrangular. It follows that through every point $u \in L$ there are precisely $q+2$ lines that are contained in $H$. If at least three of these lines are contained in a certain quad $R$, then $R$ is singular with deep point $u$ and hence $u$ is of type (A). Otherwise, $u$ is of type (B). By Lemma 20, there are two possibilities.

(1) $L$ contains a unique point of Type (A) and $q$ points of Type (B).

(2) $L$ contains $q+1$ points of Type (B).

We show that case (2) cannot occur. Suppose it does occur. Then $|\Gamma_0(L) \cap H| = q+1$ and $|\Gamma_1(L) \cap H| = (q+1)^2q$. Each quad intersecting $L$ in a unique point is either ovoidal or subquadrangular and contributes $q^2$ to the value of $|\Gamma_2(L) \cap H|$. Since every point of $\Gamma_2(L)$ is contained in a unique quad that intersects $L$ in a unique point, $|\Gamma_2(L) \cap H| = (q+1)q^2 \cdot q^2$. 

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It follows that \(|H| = |\Gamma_0(L) \cap H| + |\Gamma_1(L) \cap H| + |\Gamma_2(L) \cap H| = (q+1)+(q+1)^2+q(q+1)q^4 = q^5 + q^4 + q^3 + 2q^2 + 2q + 1\), contradicting the fact that \(|H| = q^5 + q^4 + q^3 + q^2 + 2q + 1\). □

Now, let \(n_A, n_B, n_C\) respectively \(n_D\), denote the total number of points of \(H\) of Type (A), (B), (C), respectively (D). Then by Lemma 21, we have \(n_A \cdot q = n_B \cdot (q + 2)\) and \(n_A \cdot (q + 1) = n_C \cdot 2q\). Hence,

\[
\begin{align*}
n_B &= \frac{n_A \cdot q}{q + 2}, \quad (1) \\
n_C &= \frac{n_A \cdot (q + 1)}{2q}. \quad (2)
\end{align*}
\]

Now, counting in two different ways the number of pairs \((x, L)\), with \(x \in H\) and \(L\) a line through \(x\) contained in \(H\), we obtain

\[
n_A \cdot (q + 2) + n_B \cdot (q + 2) + n_C \cdot 2 = l \cdot (q + 1) = (q^2 + q + 1)(q + 1)(q^3 + 2). \quad (3)
\]

From equations (1), (2) and (3), we find \(n_A = \frac{(q^2+q+1)(q^3+2)q}{2(q+1)}\), \(n_B = \frac{(q+2)(q^2+1)(q^3+2)}{(q+2)(q+1)}\) and \(n_C = \frac{(q^2+q+1)(q^3+2)}{2(q+1)}\). If \(q = 3\), then \(n_A \notin \mathbb{N}\). If \(q \geq 4\), then

\[
n_A + n_B + n_C = (q^2 + q + 1)(q^3 + 2) \cdot \frac{5q^2 + 7q + 2}{2(q + 2)(2q + 1)} > (q^5 + q^4 + q^3 + q^2 + 2q + 1) \cdot 1 = v,
\]

a contradiction. Hence, the case \(v = q^5 + q^4 + q^3 + q^2 + 2q + 1\) cannot occur.

### 3.4 The case \(v = q^5 + q^4 + q^3 + q^2 + q + 1\)

Suppose \(v = q^5 + q^4 + q^3 + q^2 + q + 1\).

**Lemma 22.** There are five possible types of points in \(H\):

(A) points \(x\) for which \(|\Lambda_H(x)| = 1\);
(B) points \(x\) for which \(\Lambda_H(x)\) is a line of \(\text{Res}(x)\);
(C) points \(x\) for which \(\Lambda_H(x)\) is the union of two distinct lines of \(\text{Res}(x)\);
(D) points \(x\) for which \(\Lambda_H(x)\) is an oval of \(\text{Res}(x)\);
(E) points \(x\) for which \(\Lambda_H(x)\) is empty.

**Proof.** Suppose \(Q^*\) is a singular quad and \(x^*\) is its deep point. Consider the collinearity graph \(\Gamma\) of \(DW(5, q)\) and let \(\Gamma_H\) denote the subgraph of \(\Gamma\) induced on the vertex set \(H\). Suppose \(x\) is a point of \(H\) such that \(x\) and \(x^*\) belong to different connected components of \(\Gamma_H\). Then we prove that \(\Lambda_H(x)\) is empty. Suppose to the contrary that there exists a line \(L\) through \(x\) contained in \(H\). If \(L\) meets \(Q^*\), then \(L \cap Q^*\) must be contained in \(x^*\), contradicting the fact that \(x^*\) and \(x\) belong to different connected components of \(\Gamma_H\). So,
L is disjoint from $Q^*$. Then $\pi_{Q^*}(L)$ meets $x^* \perp$ and hence $x^*$ and $x$ are connected by a path of $\Gamma_H$, again a contradiction.

By Lemmas 12 and 14 applied to the pair $(Q^*, x^*)$, $x^*$ is a point of Type (B) or (C). So, in order to prove the lemma, it suffices to prove that if $x$ is a point of Type $(X) \in \{(A), (B), (C), (D)\}$ and $y$ is a point of $H \setminus \{x\}$ collinear with $x$, then $y$ is of Type (A), (B), (C) or (D). Put $L := xy$. Since $x$ is of Type (A), (B), (C) or (D), one of the following two possibilities occurs:

1. $L$ is contained in $q + 1$ singular quads with deep point on $L$.
2. $L$ is contained in a unique singular quad with deep point on $L$ and $q$ subquadrangular quads.

Observe that case (1) can only occur if $x$ has Type (A), (B) or (C), while case (2) can only occur if $x$ has Type (C) or (D).

Suppose case (1) occurs. Then $\Lambda_H(y)$ is the union of a number of lines of $\text{Res}(y)$ through a given point of $\text{Res}(y)$, union this point. Since every quad through $y$ is singular, subquadrangular or ovoidal, every line of $\text{Res}(y)$ intersects $\Lambda_H(y)$ in either 0, 1, 2 or $q + 1$ points. Notice also that the point $y$ cannot be deep with respect to $H$, since otherwise Lemmas 12 and 14 applied to any singular quad through $y$ would yield that $v = q^5 + q^3 + q^2 + q + 1$, which is impossible. It follows that $y$ is of Type (A), (B) or (C).

If case (2) occurs, then there are two possibilities:

1a. $\Lambda_H(y)$ is a line of $\text{Res}(y)$ + $q$ extra points. By Lemma 12, $y$ necessarily is a point of Type (C).
1b. $|\Lambda_H(y)| = q + 1$. If at least three of the points of $\Lambda_H(y)$ are collinear, then $\Lambda_H(y)$ is necessarily a line of $\text{Res}(y)$. But this is impossible since $y$ is not the deep point of a singular quad through $L$. So, no three points of $\Lambda_H(y)$ are collinear. This implies that $\Lambda_H(y)$ is an oval of $\text{Res}(y)$, i.e. $y$ is a point of Type (D).

Definition. As we have already noticed in the proof of Lemma 22, every line $L \subset H$ must be contained in either $q + 1$ singular quads or one singular quad and $q$ subquadrangular quads. If all quads on $L$ are singular, then $L$ is said to be special.

Lemma 23. If $L$ is a special line, then $L$ contains only points of Type (A), (B) and (C). Moreover, the number of points of Type (A) on $L$ equals the number of points of Type (C) on $L$.

Proof. Since every quad through $L$ is singular, there are $(q + 1)q$ lines contained in $H$ that meet $L$ in a unique point. Moreover, for every $y \in L$, $\Lambda_H(y)$ is the union of a number of lines of $\text{Res}(y)$, union the point of $\text{Res}(y)$ corresponding to $L$. It follows that every point of $L$ is of Type (A), (B) or (C). Let $n_1, n_2$, respectively $n_3$, denote the number of points of Type (A), (B), respectively (C), contained in $L$. Then $n_1 + n_2 + n_3 = q + 1$ and $n_1 \cdot 0 + n_2 \cdot q + n_3 \cdot 2q = q(q + 1)$. It follows that $n_1 = n_3$. □

The proof of the following lemma is straightforward.
**Lemma 24.** Every point of Type (A) is contained in a unique special line. Every point of Type (C) is contained in a unique special line.

Let \( n_A, n_B, n_C, n_D \), respectively \( n_E \), denote the total number of points of \( H \) of Type (A), (B), (C), (D), respectively (E). The following is an immediate corollary of Lemmas 23 and 24.

**Corollary 25.** We have \( n_C = n_A \).

**Lemma 26.** We have \( n_E = 0 \).

**Proof.** We count in two different ways the number of pairs \((x, L)\) with \( x \in H \) and \( L \) a line of \( H \) through \( x \). We find

\[
n_A \cdot 1 + n_B \cdot (q + 1) + n_C \cdot (2q + 1) + n_D \cdot (q + 1) + n_E \cdot 0 = l(q + 1).
\]

Using the facts that \( n_A = n_C \) and \( l = (q^2+q+1)(q^3+1) = v \), we find \( n_A + n_B + n_C + n_D = v \). Hence, \( n_E = 0 \). \( \square \)

**Lemma 27.** We have \( n_D = \frac{2q^2}{q+1} n_A \).

**Proof.** We count in two different ways the number of pairs \((x, Q)\) where \( Q \) is a singular quad and \( x \) is its deep point. We find

\[
S_i = n_B + 2 \cdot n_C, \tag{4}
\]

where \( S_i \) denotes the total number of singular quads. We count in two different ways the number of pairs \((x, Q)\) where \( Q \) is a singular quad and \( x \) is a point of \( Q \cap H \) distinct from the deep point of \( Q \). We find

\[
(q + 1)q \cdot S_i = (q + 1)n_A + q(q + 1)n_B + (q - 1)n_C + (q + 1)n_D. \tag{5}
\]

From (4) and (5) and the fact that \( n_A = n_C \), it readily follows that \( n_D = \frac{2q^2}{q+1} n_A \). \( \square \)

Now, put \( \delta := n_A \). Then we have \( n_A = n_C = \delta \), \( n_D = \frac{2q^2}{q+1} \cdot \delta \) and \( n_B = (q^2 + q + 1)(q^3 + 1) - \frac{2(q^2+q+1)}{q+1} \cdot \delta \).

**Lemma 28.** We have \( 0 \leq \delta \leq \lfloor \frac{1}{2}(q+1)(q^3+1) \rfloor \).

**Proof.** This follows from the fact that \( n_B \geq 0 \). \( \square \)

**Remark.** If \( q \geq 4 \) is even, then by De Bruyn [7], the dual polar space \( DW(5, q) \) has up to isomorphism two hyperplanes not containing quads. The values of \( \delta \) corresponding to these two hyperplanes are respectively equal to 0 and \( \frac{q^2(q+1)}{2} \). If \( q \) is odd, then by Cooperstein and De Bruyn [5], the dual polar space \( DW(5, q) \) has up to isomorphism two hyperplanes not containing quads. The values of \( \delta \) corresponding to these two hyperplanes
are respectively equal to $\frac{1}{2}(q+1)(q^3 - 1)$ and $\frac{1}{2}(q+1)(q^3 + 1)$. So, the lower and upper bounds in Lemma 28 can be tight.

**Definition.** Recall that if $Q$ is a quad of $\text{DW}(5, q)$ then the points and lines of $\text{DW}(5, q)$ contained in $Q$ bijectively correspond to the points and lines of $\text{PG}(4, q)$ that are contained in a given nonsingular parabolic quadric $Q(4, q)$ of $\text{PG}(4, q)$. A conic of $Q$ is a set of $q+1$ points of $Q$ that corresponds to a nonsingular conic of $Q(4, q)$, i.e. with a set of $q+1$ points of $Q(4, q)$ contained in a plane $\pi$ of $\text{PG}(4, q)$ intersecting $Q(4, q)$ in a nonsingular conic of $\pi$.

**Lemma 29.** Let $\{Q_1, Q_2, \ldots, Q_{q+1}\}$ be a hyperbolic set of quads of $\text{DW}(5, q)$ such that $Q_1$ is ovoidal with respect to $H$ and $|\pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)| \geq 2$. Then:

1. $\pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)$ is a conic of $Q_1$.
2. The number of ovoidal quads of $\{Q_1, \ldots, Q_{q+1}\}$ is bounded above by $\frac{q+1}{2}$. If the number of these ovoidal quads is precisely $\frac{q+1}{2}$, then the remaining $\frac{q+1}{2}$ quads of $\{Q_1, \ldots, Q_{q+1}\}$ are subquadrangular with respect to $H$.

**Proof.** We first prove that $\pi_{Q_1}(Q_2 \cap H) \neq Q_1 \cap H$. Suppose to the contrary that $\pi_{Q_1}(Q_2 \cap H) = Q_1 \cap H$. Let $u$ be a point of $Q_1 \setminus H$, let $L$ be the unique line through $u$ meeting each quad of $\{Q_1, Q_2, \ldots, Q_{q+1}\}$, let $v$ denote the unique point of $L$ contained in $H$, and let $i$ be the unique element of $\{3, \ldots, q+1\}$ such that $v \in Q_i$. Now, since $Q_i \cap H$ contains $\pi_{Q_1}(Q_2 \cap H)$ and the point $v \in Q_i \setminus \pi_{Q_1}(Q_2 \cap H)$, we must have $Q_i \subseteq H$. This is however impossible since no quad is contained in $H$.

So, $\pi_{Q_1}(Q_2 \cap H) \neq Q_1 \cap H$. By Lemma 8, $\{\pi_{Q_i}(Q_i \cap H) \mid 1 \leq i \leq q+1\}$ is a pencil of hyperplanes of $Q_1$. Let $\alpha_1$, $\alpha_2$, respectively $\alpha_3$, denote the number of quads of $\{Q_1, Q_2, \ldots, Q_{q+1}\}$ that are ovoidal, singular, respectively subquadrangular, with respect to $H$. Put $\beta := |\pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)| \geq 2$. We prove that $\beta = q+1$.

If $\alpha_1 = q+1$ and $\alpha_2 = \alpha_3 = 0$, then $(q+1)(q^2+1) = |Q_1| = \beta + (q+1)(q^2+1 - \beta) = (q+1)(q^2+1) - q\beta < (q+1)(q^2+1)$, a contradiction. So, without loss of generality, we may suppose that $Q_2$ is not ovoidal with respect to $H$. If $Q_2$ is subquadrangular with respect to $H$, then $\beta = |\pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)| = q+1$. If $Q_2$ is singular with respect to $H$ with deep point $u$ such that $\pi_{Q_1}(u) \notin Q_1 \cap H$, then also $\beta = |\pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)| = q+1$. If $Q_1$ were singular with respect to $H$ with deep point $u$ such that $\pi_{Q_1}(u) \in Q_1 \cap H$, then $\beta = |\pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)| = 1$, a contradiction. Hence, $\beta = q+1$ as claimed.

Now, we have $\alpha_1 + \alpha_2 + \alpha_3 = q+1$ and $(q+1)(q^2+1) = |Q_1| = \beta + (q+1)(q^2+q) + \alpha_2 q^2 + \alpha_3 (q^2 + q) = (q+1)(q+1)q^2 + q(\alpha_3 - \alpha_1)$, i.e. $\alpha_1 + \alpha_2 + \alpha_3 = q+1$ and $\alpha_1 = \alpha_3$. Hence, $\alpha_1 = \alpha_3 \leq \frac{q+1}{2}$. Moreover, if $\alpha_1 = \alpha_3 = \frac{q+1}{2}$, then $\alpha_2 = 0$. This proves claim (2).

Now, $\alpha_2 + \alpha_3 \geq \frac{q+1}{2}$. So, $\alpha_2 + \alpha_3 \geq 2$. Without loss of generality, we may suppose that the quads $Q_2$ and $Q_3$ are singular or subquadrangular with respect to $H$.

The points and lines contained in $Q_1$ can be identified (in a natural way) with the points and lines lying on a given nonsingular parabolic quadric $Q(4, q)$ of $\text{PG}(4, q)$. Now, each of $\pi_{Q_1}(Q_2 \cap H)$ and $\pi_{Q_1}(Q_3 \cap H)$ is either a singular hyperplane or a subgrid of $\tilde{Q}_1$ and hence arises by intersecting $Q(4, q)$ with a hyperplane of $\text{PG}(4, q)$. Since $\pi_{Q_1}(Q_2 \cap$
Without loss of generality, we may suppose that Lemma 30. If $Q_1$ is an ovoidal quad, then through every two points of $Q_1 \cap H$, there is a conic of $Q_1$ that is completely contained in $Q_1 \cap H$.

**Proof.** Let $x_1$ and $x_2$ be two distinct points of $Q_1 \cap H$. By Lemmas 22 and 26, there exists a line $L_i$, $i \in \{1,2\}$ through $x_i$ that is contained in $H$. Let $Q_2$ be a quad distinct from $Q_1$ that meets $L_1$ and $L_2$, and let $\{Q_1,Q_2,\ldots,Q_{q+1}\}$ be the unique hyperbolic set of quads of $DW(5,q)$ containing $Q_1$ and $Q_2$. Since $\{x_1,x_2\} \subseteq \pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)$, Lemma 29 applies. We conclude that $\pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)$ is a conic containing $x_1$ and $x_2$.

**Lemma 31.** For every quad $Q_1$ that is ovoidal with respect to $H$, there is a quad $Q_2$ disjoint from $Q_1$ that is singular with respect to $H$ such that $\pi_{Q_1}(u) \not\subseteq Q_1 \cap H$ where $u$ is the deepest point of the singular hyperplane $Q_2 \cap H$ of $\tilde{Q}_2$.

**Proof.** The number of points $x \in \Gamma_1(Q_1) \cap H$ for which $\pi_{Q_1}(x) \not\subseteq Q_1 \cap H$ is equal to $\left(\frac{|Q_1| - (Q_1 \cap H)}{|Q_1|}\right) \cdot q^2 = q^3(q^2 + 1)$. Now, since $n_D = \frac{2q^2}{q^2+1} \leq \frac{2q^2}{q^2+1} \cdot \frac{1}{2}(q+1)(q^3+1) = q^3(q^3+1) < q^3(q^2+1)$, there exists a point $y \in \Gamma_1(Q_1) \cap H$ not of type (D) for which $\pi_{Q_1}(y) \not\subseteq Q_1 \cap H$. Let $L \subseteq H$ be a special line through $y$ and let $z$ denote the unique point of $L$ for which $\pi_{Q_1}(z) \in Q_1 \cap H$. By Lemma 22, there are at most two quads $R$ through $L$ for which $z$ is the deep point of the singular hyperplane $R \cap H$ of $\tilde{R}$. Hence, there exists a quad $Q_2$ through $L$ for which the deep point $u$ of the singular hyperplane $Q_2 \cap H$ of $\tilde{Q}_2$ is distinct from $z$. Since $u$ is not collinear with a point of $Q_1 \cap H$, $Q_1$ and $Q_2$ are disjoint.

**Lemma 32.** If $Q_1$ is ovoidal with respect to $H$, then $Q_1 \cap H$ is a classical ovoid of $\tilde{Q}_1$.

**Proof.** By Lemma 31, there exists a quad $Q_{q+1}$ disjoint from $Q_1$ that is singular with respect to $H$ such that $\pi_{Q_1}(u) \not\subseteq Q_1 \cap H$ where $u$ is the deep point of the singular hyperplane $Q_{q+1} \cap H$ of $\tilde{Q}_{q+1}$. Let $\{Q_1,Q_2,\ldots,Q_{q+1}\}$ denote the unique hyperbolic set of quads of $DW(5,q)$ containing $Q_1$ and $Q_{q+1}$. By Lemma 29, we then have:

1. $X := \pi_{Q_1}(Q_{q+1} \cap H) \cap (Q_1 \cap H)$ is a conic of $Q_1$;
2. the number $k$ of ovoidal quads of the set $\{Q_1,Q_2,\ldots,Q_{q+1}\}$ is at most $\frac{q}{2}$.

Without loss of generality, we may suppose that $Q_1,\ldots,Q_k$ are the quads of $\{Q_1,Q_2,\ldots,Q_{q+1}\}$ that are ovoidal with respect to $H$. Since $(q+1) - \frac{q}{2} \geq 2$, $Q_q$ and $Q_{q+1}$ are not ovoidal with respect to $H$. By Lemmas 5 and 8, $\pi_{Q_1}(Q_q \cap H)$ and $\pi_{Q_1}(Q_{q+1} \cap H)$ are contained in a unique pencil of classical hyperplanes of $Q_1$. Moreover, this pencil contains the hyperplanes $\pi_{Q_1}(Q_i \cap H)$, $i \in \{k+1,\ldots,q+1\}$. Let $A_1,\ldots,A_k$ denote the remaining elements of this pencil. Then $X \subseteq A_1 \cap \cdots \cap A_k$ and $A_1 \cup \cdots \cup A_k = \pi_{Q_1}(Q_1 \cap H) \cup \cdots \cup \pi_{Q_1}(Q_k \cap H)$. Now, $|A_1 \cup \cdots \cup A_k| = |X| + k(q^2 + 1 - |X|) = (q+1) + k(q^2-q)$ and equality holds if and only if every $A_j$, $j \in \{1,\ldots,k\}$, is a classical ovoid of $Q_1$. Now, since $|\pi_{Q_1}(Q_1 \cap H) \cup \cdots \cup \pi_{Q_1}(Q_k \cap H)| = |X| + k(q^2 + 1 - |X|) = (q+1) + k(q^2-q)$, we can conclude that every $A_j$, $j \in \{1,\ldots,k\}$, is a classical ovoid of $\tilde{Q}_1$. 

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Now, let \( i \in \{1, \ldots, k\} \) and suppose there exists no \( j \in \{1, \ldots, k\} \) such that \( \pi_{Q_1}(Q_i \cap H) = A_j \). Then \( X \subseteq \pi_{Q_1}(Q_i \cap H) \subseteq A_1 \cup \cdots \cup A_k \) and there exist two distinct \( j_1, j_2 \in \{1, \ldots, k\} \) such that \( \pi_{Q_1}(Q_i \cap H) \cap (A_{j_1} \setminus X) \neq \emptyset \) and \( \pi_{Q_1}(Q_i \cap H) \cap (A_{j_2} \setminus X) \neq \emptyset \). Let \( y_1 \) be an arbitrary point of \( \pi_{Q_1}(Q_i \cap H) \cap (A_{j_1} \setminus X) \) and let \( y_2 \) be an arbitrary point of \( \pi_{Q_1}(Q_i \cap H) \cap (A_{j_2} \setminus X) \). By Lemma 30, there exists a conic \( C \) through \( y_1 \) and \( y_2 \) that is completely contained in \( \pi_{Q_1}(Q_i \cap H) \) and hence also in \( A_1 \cup \cdots \cup A_k \). Since \( |C| = q+1 \) and \( k \leq \frac{q}{2} \), there exists a \( j_3 \in \{1, \ldots, k\} \) such that \( |C \cap A_{j_3}| \geq 3 \). Since \( A_{j_3} \) is a classical ovoid of \( Q_1 \), this necessarily implies that \( C \subseteq A_{j_3} \), contradicting the fact that \( y_1 \in A_{j_1} \setminus X \), \( y_2 \in A_{j_2} \setminus X \) and \( j_1 \neq j_2 \). Hence, there exists a \( j \in \{1, \ldots, k\} \) such that \( \pi_{Q_1}(Q_i \cap H) = A_j \). This implies that the ovoid \( Q_i \cap H \) of \( \tilde{Q}_i \) is classical.

\[ \text{} \]

**Corollary 33.** The hyperplane \( H \) is classical.

**Proof.** This is an immediate corollary of Proposition 7 and Lemma 32.

**Remark.** With the terminology of Cooperstein & De Bruyn [5] and De Bruyn [7], the hyperplane \( H \) is either a hyperplane of Type V or a hyperplane of Type VI.

**Acknowledgment**

At the time of the writing of this paper, the author was a Postdoctoral Fellow of the Research Foundation - Flanders (Belgium).

**References**


