An Extension Theorem for Terraces

M. A. Ollis* and Devin T. Willmott

Marlboro College, P.O. Box A, Marlboro
Vermont 05344, USA

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Abstract

We generalise an extension theorem for terraces for abelian groups to apply to non-abelian groups with a central subgroup isomorphic to the Klein 4-group V. We also give terraces for three of the non-abelian groups of order a multiple of 8 that have a cyclic subgroup of index 2 that may be used in the extension theorem. These results imply the existence of terraces for many groups that were not previously known to be terraced, including 27 non-abelian groups of order 64 and all groups of the form $V^s \times D_{8t}$ for all s and all $t > 1$ where D_{8t} is the dihedral group of order $8t$.

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1 Introduction

Let G be a group of order n and let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be an arrangement of all of the elements of G. Define $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1})$ by $b_i = a_i^{-1}a_{i+1}$. If each involution of G appears once in $\mathbf{b}$ and there are two appearances from each set $\{g, g^{-1} : g^2 \neq e\}$ in $\mathbf{b}$ then $\mathbf{a}$ is a *terrace* for G and $\mathbf{b}$ is its associated *2-sequencing*. If a group has a terrace then it is *terraced*. Left-multiplying each element of a terrace by any element of the group produces another terrace for the group; choosing a_1^{-1} gives a terrace with the identity, e, as the first element—such a terrace is called *basic*.

Terraces for cyclic groups were implicitly used by Williams in [13] and the concept was formally defined and extended to arbitrary groups by Bailey [3]. They were originally of interest because the Cayley table of a group may be presented as a quasi-complete Latin square if and only if the group is terraced [3] but have since been used for other applications and studied as objects of interest in their own right. The purpose of this paper is to move closer to a proof of Bailey's Conjecture:

*Corresponding author, email address: matt@marlboro.edu.

Conjecture 1 [3] *All groups, except the non-cyclic elementary abelian 2-groups, are terraced.*

It is known that the non-cyclic elementary abelian 2-groups are not terraced [3].

Example 1 *Let $\mathbb{Z}_n$ be the additively-written cyclic group of order n. The Lucas-Walecki-Williams terrace (so-called because it was implicitly used by Lucas and Walecki [7] for even n and by Williams [13] for all n) for $\mathbb{Z}_n$ is* $(0, 1, n-1, 2, n-2, \ldots)$ *and has associated 2-sequencing* $(1, n-2, 3, n-4, \ldots)$.

There have been two main lines of attack on Bailey's Conjecture. First, one may directly construct terraces for a particular family of groups. Second, one can produce theorems that build a terrace for a group out of terraces for smaller groups. The most powerful example of the second approach is the following result:

Theorem 1 [1, 2] *Let G be a group with normal subgroup N. If N has odd index and is terraced, then G is terraced. If N has odd order and G/N is terraced then G is terraced.*

In [9] a theorem that constructed a terrace for an abelian group G that has a subgroup of order 4 and a particular type of terrace for the quotient group was presented. This theorem is not fully correct in the case when the subgroup of order 4 is cyclic; see [11] in which the error is corrected and it is shown that all of the groups claimed to be terraced are indeed terraced. In the next section we present a more general version of the correct case (when the subgroup is isomorphic to V, the Klein 4-group) that applies to many non-abelian groups.

The extension result in the next section allows us to find terraces for a considerable array of previously unterraced groups. As input the theorem requires terraces with particular properties; some such terraces are catalogued in Section 3.

2 The extension theorem

We first define the properties we require of a terrace to be used in the theorem. Let K be a group of order $m \geqslant 6$ and let $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ be a basic terrace for K. If $a_m = a_2^2$ and $a_{j-1}a_{j+1} = a_j = a_{j+1}a_{j-1}$ for some $5 \leqslant j < m$ then $\mathbf{a}$ is *extendable*.

An important intermediary object is an R-terrace, or rotational terrace. Following the convention of earlier papers we write circular lists in square brackets and consider the subscripts to be calculated modulo the length of the list. Let K be a group of order m and let $\mathbf{a} = [a_1, a_2, \ldots, a_{m-1}]$ be a circular arrangement of the non-identity elements of K. Define $\mathbf{b} = [b_1, b_2, \ldots, b_{m-1}]$ by $b_i = a_i^{-1}a_{i+1}$ for $1 \leqslant i \leqslant m-1$. If $\mathbf{b}$ contains exactly one occurrence of each involution of K and exactly two occurrences from each set $\{k, k^{-1} : k^2 \neq e\}$ then $\mathbf{a}$ is a *rotational terrace* or *R-terrace* for K and $\mathbf{b}$ is the associated *rotational 2-sequencing* or *R-2-sequencing* of K. If there are no repeats among the elements of $\mathbf{b}$ then the R-terrace is *directed* and the R-2-sequencing is an *R-sequencing*. Further, if $a_1 = a_{m-1}a_2 = a_2a_{m-1}$ then $\mathbf{a}$ is a *standard R^*-terrace* for K and if $b_r = a_{r+1}^{-1}$ for some r then r is a *right match-point* of **b**. Note that a standard R*-terrace cannot have an R-2-sequencing with 1 as a right match-point.

Standard R*-terraces whose R-2-sequencings have particular right match-points and extendable terraces are equivalent: The circular list $[a_1, a_2, \dots, a_{m-1}]$ is a standard R*-terrace whose R-2-sequencing has a right match-point r for some $2 \leqslant r \leqslant m-3$ if and only if

$$(e, a_{r+1}, a_{r+2}, \dots, a_{m-1}, a_1, a_2, \dots, a_r)$$

is an extendable terrace.

The following lemma restricts which groups may have an extendable terrace.

Lemma 1 [9] *If the order of G is congruent to 2 modulo 4 then G does not have a rotational terrace.*

We can now prove our main result. The *Klein 4-group* is the non-cyclic group of order 4 and a subgroup is *central* if each of its elements commutes with every element of the group (that is, it is contained in the centre of the group).

Theorem 2 *Let G be a group with a central subgroup V of index $m \geqslant 7$, where V is isomorphic to the Klein 4-group. Suppose G/V has a standard R*-terrace $[K_1, K_2, \dots, K_{m-1}]$ whose R-2-sequencing has a match-point r for some $2 \leqslant r \leqslant m-3$ and such that there is a pair of elements, one in K_2 and one in K_{m-1}, that commute. Then G has an extendable terrace.*

Proof. Choose coset representatives k_i, for $1 \leqslant i \leqslant m-1$, such that $k_i \in K_i$ and that both $k_{m-1}k_2 = k_1 = k_2k_{m-1}$ and $k_r^{-1}k_{r+1} = k_{r+1}^{-1}$. These two criteria potentially interact if $r = 2$. In this case, choose any $k_3 \in K_3$, set $k_2 = k_3^2$ and then there is a $k_{m-1} \in K_{m-1}$ that commutes with k_2: if $\ell_2 \in K_2$ and $\ell_{m-1} \in K_{m-1}$ are the commuting elements we know to exist then there is a $v_0 \in V$ with $k_2 = v_0\ell_2$ and this commutes with ℓ_{m-1}, which we may set to be k_{m-1}, by the centrality of V.

Note that each element of G is uniquely expressible in the form vk for $v \in V$ and $k \in \{e, k_1, k_2, \dots, k_{m-1}\}$.

We build the standard R*-terrace by showing the lists of the v components and k components separately. Let $V = \{e, v_1, v_2, v_3\}$, then $[v_1, v_2, v_3]$ is an R-terrace for V (that is, any circular list of the non-identity elements of V is an R-terrace). We list the v components as the rows of a $4 \times m$ matrix. Let $(v_1, v_2, v_3)_{t-1}$ denote $t-1$ repetitions of the sequence (v_1, v_2, v_3), and similarly for other subscripted sequences. There are three slightly different matrices for the v components as m varies modulo 3.

Case 1: $m = 3t$ for $t \geqslant 3$. Take

$$\begin{bmatrix} e & e & \dots & & e & v_2 & v_1 \\ v_3 & (v_1, v_2, v_3)_{t-2} & v_1 & v_3 & v_3 & v_2 & v_1 \\ v_3 & v_2 & (v_3, v_1, v_2)_{t-2} & v_3 & v_2 & v_1 & v_3 \\ v_2 & v_1 & (v_2, v_3, v_1)_{t-2} & v_2 & v_1 & e & \end{bmatrix}$$

to be the v component matrix.

Case 2: $m = 3t + 1$ for $t \geqslant 2$. Take

$$\begin{bmatrix} e & e & \dots & v_2 & v_1 \\ v_3 & (v_2, v_1, v_3)_{t-1} & v_3 & v_2 & v_1 \\ v_3 & v_2 & (v_1, v_3, v_2)_{t-1} & v_1 & v_3 \\ v_2 & v_1 & (v_3, v_2, v_1)_{t-1} & e & \end{bmatrix}$$

to be the v component matrix.

Case 3: $m = 3t + 2$ for $t \geqslant 2$. Take

$$\begin{bmatrix} e & e & \dots & e & v_2 & v_1 \\ v_3 & (v_2, v_1, v_3)_{t-1} & v_2 & v_3 & v_2 & v_1 \\ v_3 & v_2 & (v_1, v_3, v_2)_{t-1} & v_1 & v_1 & v_3 \\ v_2 & v_1 & (v_3, v_2, v_1)_{t-1} & v_3 & e & \end{bmatrix}$$

to be the h component matrix.

For each of the above cases the k component matrix is

$$\begin{bmatrix} k_1 & k_2 & \dots & & k_{m-1} & e \\ k_2 & k_3 & \dots & k_{m-1} & k_1 & k_1 \\ k_1 & k_2 & \dots & & k_{m-1} & e \\ e & k_2 & k_3 & \dots & k_{m-1} & \end{bmatrix}$$

As V is central in G and $k_{m-1}k_2 = k_1 = k_2k_{m-1}$ we get the following matrix of quotients in the k component:

$$\begin{bmatrix} k_1^{-1}k_2 & k_2^{-1}k_3 & \dots & \dots & k_{m-2}^{-1}k_{m-1} & k_1^{-1}k_2 & k_{m-1}^{-1}k_1 \\ k_2^{-1}k_3 & k_3^{-1}k_4 & \dots & k_{m-2}^{-1}k_{m-1} & k_{m-1}^{-1}k_1 & e & e \\ k_1^{-1}k_2 & k_2^{-1}k_3 & \dots & \dots & k_{m-2}^{-1}k_{m-1} & k_1^{-1}k_2 & e \\ k_{m-1}^{-1}k_1 & k_2^{-1}k_3 & k_3^{-1}k_4 & \dots & k_{m-2}^{-1}k_{m-1} & k_{m-1}^{-1}k_1 & \end{bmatrix}$$

Each repeated sequence in the v component matrix is a directed R-terrace and so when the quotient matrices are combined we get a sequence that obeys the conditions of an R-2-sequencing.

Further, as the first two entries and the last entry of every v component matrix is e, it follows from our choices of k_1, k_2, and k_{m-1} that the R-terrace is a standard R*-terrace. As the first $m-2$ entries of every v component matrix are all e, our choices of k_r and k_{r+1} give us the match-point we require in position r of the R-2-sequencing. $\square$

The awkward condition in Theorem 2 regarding commuting elements in commuting cosets is automatically satisfied in the cases where we have appropriate direct products or abelian (sub)groups. Hence an immediate consequence of the theorem is:

Corollary 1 *Let A be an abelian 2-group that has a normal series with all factors isomorphic to the Klein 4-group and let K be a group with an extendable terrace. Then $A \times K$ has an extendable terrace. In particular, $\mathbb{Z}_2^{2s} \times K$ has an extendable terrace for all s.*

Our goal now is to construct extendable terraces for as many groups as possible.

3 Extendable terraces

The following results for abelian groups are established in [9, 10, 11]:

- The cyclic group $\mathbb{Z}_n$ has an extendable terrace if and only if $n \geqslant 7$ and n is not twice an odd number.
- All abelian 2-groups of order at least 8, except the elementary abelian 2-groups, have an extendable terrace.
- Let p be an odd prime. The group $\mathbb{Z}_2^{2t} \times \mathbb{Z}_p$ has an extendable terrace unless $t = 0$ and $p \leqslant 5$.
- The groups $\mathbb{Z}_2^{2t+1} \times \mathbb{Z}_3$ and $\mathbb{Z}_2^{2t+1} \times \mathbb{Z}_5$ have an extendable terrace for all $t \geqslant 1$.

Other than the unterraceable elementary abelian 2-groups, these results and Theorem 1 now give terraces for all abelian groups except for those of order coprime to 15 with elementary abelian Sylow 2-subgroup of order 2^{2t+1} for $t \geqslant 1$ [9, 10, 11]. When $t \geqslant 2$ it is known that these groups are terraced [10].

In this section we present extendable terraces for each of three non-abelian groups of order $8t$ with $t \geqslant 2$: the dihedral group D_{8t}, the semidihedral group S_{8t} and a third group that also has a cyclic subgroup of index 2 but does not appear to have a common name in the literature—we denote it M_{8t} following Gorenstein's use, reported in [6], of the letter M (but with a different subscript convention) for this group when it has order a power of 2. For even t, other than finitely many small cases, the terraces given for S_{8t} and M_{8t} are the first known. Here are presentations for these groups:

$$\begin{aligned} D_{8t} &= \langle u, v : u^{4t} = e = v^2,\ vu = u^{4t-1}v \rangle \\ S_{8t} &= \langle u, v : u^{4t} = e = v^2,\ vu = u^{2t-1}v \rangle \\ M_{8t} &= \langle u, v : u^{4t} = e = v^2,\ vu = u^{2t+1}v \rangle \end{aligned}$$

Before constructing the desired terraces we introduce a related concept and prove a lemma that is crucial to the construction.

An arrangement $\mathbf{g} = (g_1, g_2, \ldots, g_n)$ of the integers $\{0, 1, \ldots, n-1\}$ is a *graceful sequence of length n* if each element of the set $\{1, 2, \ldots, n-1\}$ can be written $|g_{i+1} - g_i|$ for some i. This is equivalent to the notion of a graceful labelling of a path in graph theory [4]. If $\mathbf{g}$ is a graceful sequence then so are its *reverse* $(g_n, g_{n-1}, \ldots, g_1)$ and its *complement* $((n-1) - g_1, (n-1) - g_2, \ldots, (n-1) - g_n)$. Considered to be a sequence in $\mathbb{Z}_n$ rather than $\mathbb{Z}$ a graceful sequence is a terrace, called a *graceful terrace*.

Example 2 *The* negated LWW terrace *for $\mathbb{Z}_n$, obtained by negating each element of the LWW terrace of Example 1, is a graceful terrace.*

Lemma 2 *For all $t \geqslant 2$ there is a graceful sequence of length $2t-1$ with endpoints $t-2$ and $2t-3$.*

Proof. When $t \equiv 5 \pmod 6$ we use the complement of the "3-twizzler" graceful terrace described in [12]. The 3-twizzler terrace is obtained from the negated LWW terrace for $\mathbb{Z}_{2t-1}$ by dividing the terrace into subsequences of length 3 and reversing ("twizzling") each of them. After taking the complement we have:

$$\underbrace{2t-3,\ 0,\ 2t-2}, \underbrace{2,\ 2t-4,\ 1}, \ldots, \underbrace{t-1,\ t,\ t-2}.$$

When $t \equiv 2 \pmod 6$ the complement of 3-twizzler terrace begins the same way but ends $t-1, t-2, t$. Switching the last two elements preserves the gracefulness of the sequence and gives us the $t-2$ that we need as an endpoint.

When $t \equiv 0 \pmod 3$ we can use the complement of the "imperfect 3-twizzler" graceful terrace of [12]. In Preece's imperfect 3-twizzler terrace all but the final two elements are obtained by 3-twizzling as above. Here is its complement:

$$\underbrace{2t-3,\ 0,\ 2t-2}, \underbrace{2,\ 2t-4,\ 1}, \ldots, \underbrace{t,\ t-3,\ t+1},\ t-1,\ t-2.$$

Finally, when $t \equiv 1 \pmod 3$ we give a new graceful terrace using similar ideas. We begin as in the previous cases by twizzling subsequences of length 3 from the negated LWW graceful terrace, however this time we stop with 7 elements remaining and rearrange those to give a final element of $t-2$ while preserving the gracefulness of the sequence:

$$\underbrace{2t-3,\ 0,\ 2t-2}, \underbrace{2,\ 2t-4,\ 1}, \ldots, \underbrace{t-5,\ t+3,\ t-6},$$

$$t+1,\ t-3,\ t+2,\ t-4,\ t-1,\ t,\ t-2.$$

This completes the proof. □

Theorem 3 *The groups D_{8t}, S_{8t} and M_{8t} have an extendable terrace for all $t \geqslant 2$.*

Proof. The similar structure of the three groups allows us to use a slightly unusual approach. We give a sequence of elements of the form $u^x v^y$ with $0 \leqslant x \leqslant 4t-1$ and $y \in \{0,1\}$ and this sequence is a terrace regardless of to which group we interpret the elements belonging.

The terrace takes the form $\mathbf{a} = (e, \alpha, \beta, u^{2t}, v, \gamma, u^t v, \delta)$, where each Greek letter represents a sequence of elements. With the exception of δ, each of these sequences can be expressed in a "zigzag" pattern. The partial terrace up to $u^t v$ is given in Table 1.

Table 1: Partial extendable terrace for D_{8t}, S_{8t} and M_{8t}

e	e
α	$u^{2t-1}, uv, u^{2t-2}, u^2v, u^{2t-3}, \dots, u^{t+1}, u^{t-1}v, u^t$
β	$u^{t+1}v, u^{t-1}, u^{t+2}v, u^{t-2}, \dots, u^2, u^{2t-1}v, u, u^{2t}v$
u^{2t}	u^{2t}
v	v
γ	$u^{2t+1}v, u^{4t-1}v, u^{2t+2}v, u^{4t-2}v, \dots, u^{3t-1}v, u^{3t+1}v, u^{3t}v$
u^tv	u^tv

Table 2: Partial 2-sequencing for D_{8t}

α	$u^{2t-1}, u^{2t+2}v, u^{2t+3}v, u^{2t+4}v, \dots, u^{4t-2}v, u^{4t-1}v$
β	$uv, u^2v, u^3v, \dots, u^{2t-2}v, u^{2t-1}v$
u^{2t}	v
v	$u^{2t}v$
γ	$u^{2t-1}, u^{2t+2}, u^{2t-3}, u^{2t+4}, \dots, u^3, u^{4t-2}, u$
u^tv	u^{2t}

The associated partial 2-sequencings arising from the partial terrace for D_{8t}, S_{8t} and M_{8t} are given in Tables 2, 3 and 4 respectively with each row starting with the difference created by joining the subsequence with the previous one.

In each case, to complete the sequence to a terrace δ needs to satisfy three conditions. First, it must generate the final quotient of the form u^xv, which it can do by starting with u^{3t-1}. Second, it must contain the elements of the form u^x for $2t+1 \leqslant x \leqslant 4t-1$. Third, it must generate one from each inverse pair within $\langle u \rangle$ except for u^{2t} and $u^{\pm(2t-1)}$. Further, for the terrace to be extendable, the last element of δ must be u^{4t-2}.

These conditions can be met by taking a graceful sequence $(g_1, g_2, \dots, g_{2t-1})$ that starts with $t-2$ and ends with $2t-3$ and defining the ith element of δ to be u^{2t+1+g_i}. Such a sequence exists by Lemma 2.

Finally, we need to check the other condition to be extendable; that $a_{j-1}a_{j+1} = a_j = a_{j+1}a_{j-1}$ for some $j \geqslant 5$. Setting $j = 4t-1$, we find that $a_{j-1} = u^{2t}v$, $a_j = u^{2t}$ and $a_{j+1} = v$; a valid choice in each of the three groups. □

Example 3 *The terrace for D_{32}, S_{32} and M_{32} given by Theorem 3 is*

$$e, u^7, uv, u^6, u^2v, u^5, u^3v, u^4, u^5v, u^3, u^6v, u^2, u^7v, u, u^8v, u^8,$$
$$v, u^9v, u^{15}v, u^{10}v, u^{14}v, u^{11}v, u^{13}v, u^{12}v, u^4v, u^{11}, u^{13}, u^{12}, u^9, u^{15}, u^{10}, u^{14}.$$

When considering which groups are most likely to give a counterexample to Bailey's conjecture those with many involutions and/or large elementary abelian 2-groups as subgroups

Table 3: Partial 2-sequencing for S_{8t}

α	$u^{2t-1}, u^{2t+2}v, u^3v, u^{2t+4}v, u^5v, \ldots, u^{4t-2}v, u^{2t-1}v$
β	$uv, u^2v, u^3v, \ldots, u^{2t-2}v, u^{2t-1}v$
u^{2t}	v
v	$u^{2t}v$
γ	$u^{4t-1}, u^{2t+2}, u^{4t-3}, u^{2t+4}, \ldots, u^{2t+3}, u^{4t-2}, u$
u^tv	u^{2t}

Table 4: Partial 2-sequencing for M_{8t}

α	$u^{2t-1}, u^{2t+2}v, u^{4t-3}v, u^{2t+4}v, u^{4t-5}v, \ldots, u^{4t-2}v, u^{2t+1}v$
β	$uv, u^{4t-2}v, u^3v, u^{4t-4}, \ldots, u^{2t+2}v, u^{2t-1}v$
u^{2t}	v
v	$u^{2t}v$
γ	$u, u^{2t-2}, u^3, u^{2t-4}, \ldots, u^{2t-3}, u^2, u^{2t-1}$
u^tv	u^{2t}

are natural contenders. Theorem 3 and Corollary 1 imply that many such contenders are indeed terraced; groups of the form $\mathbb{Z}_2^{2s} \times D_{8t}$ for all s and for $t \geqslant 2$, for example.

A computer search for extendable terraces for small groups has been implemented in GAP [5]. Neither of the two non-abelian groups of order 8 has an extendable terrace. Extendable terraces were found for all twelve non-abelian groups of orders 12, 16 and 20 not covered by Theorem 3. The notation $G_{n/p}$ indicates that the group has order n and is in position p in GAP's small group library. Where the group has a common name that is indicated as well, and we use the more familiar permutation notation for the alternating group A_4. The value for j in the definition of an extendable terrace is also given.

Order 12:

$$\begin{aligned}
G_{12/1} &= \langle a, b : a^6 = e, b^2 = a^3, ab = ba^{-1} \rangle \cong Q_{12},\ j = 5 \\
& \quad e, a^2, a, b, a^3b, a^3, a^4b, a^5, a^5b, ab, a^2b, a^4 \\
G_{12/3} &\cong A_4,\ j = 7 \\
& \quad (), (123), (234), (124), (134), (14)(23), (12)(34), (13)(24), (142), (143), (243), (132) \\
G_{12/4} &= \langle a, b : a^6 = b^2 = e, ab = ba^{-1} \rangle \cong D_{12},\ j = 6 \\
& \quad e, a, a^2b, a^5, a^4b, ab, a^3, a^3b, a^5b, b, a^4, a^2
\end{aligned}$$

Order 16:

$$
\begin{aligned}
G_{16/3} &= \langle a,b,c : a^4=b^2=c^2=e, ab=bac, [a,c]=[b,c]=e\rangle,\ j=6\\
&\quad e,a,a^3,a^3c,a^2b,bc,a^2c,b,ab,ac,abc,c,a^3bc,a^2bc,a^3b,a^2\\
G_{16/4} &= \langle a,b : a^4=b^4=e, ab=ba^{-1}\rangle,\ j=13\\
&\quad e,a^2b,a^3b^3,a^3,ab^3,a^2,a,a^3b^2,b,b^3,a^2b^3,a^3b,a^2b^2,ab,ab^2,b^2\\
G_{16/9} &= \langle a,b : a^8=e, b^2=a^4, ab=ba^{-1}\rangle \cong Q_{16},\ j=5\\
&\quad e,a^2b,ab,a,a^7,a^6,b,a^3b,a^7b,a^5b,a^2,a^5,a^6b,a^3,a^4b,a^4\\
G_{16/11} &= \langle a,b,c : a^4=b^2=c^2=e, ab=ba^{-1}, [a,c]=[b,c]=e\rangle \cong D_8\times\mathbb{Z}_2,\ j=8\\
&\quad e,a^3c,a^3,a,b,a^3b,bc,a^2c,a^2b,c,ab,a^3bc,a^2bc,ac,abc,a^2\\
G_{16/12} &= \langle a,b,c : a^4=c^2=e, b^2=a^2, ab=ba^{-1}, [a,c]=[b,c]=e\rangle \cong Q_8\times\mathbb{Z}_2,\ j=10\\
&\quad e,b,ab,ac,bc,c,a,a^3c,a^3b,a^2c,a^3bc,abc,a^2b,a^2bc,a^3,a^2\\
G_{16/13} &= \langle a,b,c : a^2=b^2=c^4=e, ab=bac^2, [a,c]=[b,c]=e)\rangle,\ j=8\\
&\quad e,c,ac^2,abc^2,bc^3,ac,bc,b,c^3,ab,bc^2,ac^3,abc,abc^3,a,c^2
\end{aligned}
$$

Order 20:

$$
\begin{aligned}
G_{20/1} &= \langle a,b : a^{10}=e, b^2=a^5, ab=ba^{-1}\rangle \cong Q_{20},\ j=9\\
&\quad e,b,a^6b,ab,a^4b,a,a^3b,a^7,a^6,a^9,a^2b,a^2,a^4,a^8b,a^7b,a^8,a^5b,a^9b,a^3,a^5\\
G_{20/3} &= \langle a,b : a^5=b^4=e, ab=ba^2\rangle,\ j=17\\
&\quad e,a^2,ab,a^3,a^4b^3,ab^3,a^2b^2,a^2b,b^2,a,a^4b^2,b,a^3b,a^3b^3,a^3b^2,ab^2,a^2b^3,a^4b,b^3,a^4\\
G_{20/4} &= \langle a,b : a^{10}=b^2=e, ab=ba^{-1}\rangle \cong D_{20},\ j=5\\
&\quad e,a^6,a^4b,a^5,a^8b,a^3b,b,a^4,a^5b,a^8,a^2b,a^9b,a^7b,a^6b,a,ab,a^9,a^7,a^3,a^2
\end{aligned}
$$

The smallest order for which Bailey's conjecture is not settled is 64. The abelian case for this order was proven in [9, 11]. Of the 256 non-abelian groups of order 64, only three were known to have terraces prior to this work [8]. The extendable terraces for groups of order 16 above imply that at least 25 further non-abelian groups of order 64 are terraced (this is the number of groups that have commuting elements in *all* pairs of commuting cosets of some central Klein 4-group to use in Theorem 2). Combining this with the known ones and the new terraces here for S_{64} and M_{64} gives a total of 30. There are 208 non-abelian groups of order 64 that have a central Klein 4-group with at least one pair of commuting cosets that contain a pair of commuting elements; many of these may fall to Theorem 2 if an appropriate extendable terrace for the quotient group of order 16 can be found.

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