Near packings of graphs

Andrzej Žak*
Faculty of Applied Mathematics
AGH University of Science and Technology
Kraków, Poland
zakandrz@agh.edu.pl

Submitted: Dec 18, 2012; Accepted: May 16, 2013; Published: May 24, 2013
Mathematics Subject Classifications: 05C70

Abstract
A packing of a graph $G$ is a set \{$G_1, G_2$\} such that $G_1 \cong G$, $G_2 \cong G$, and $G_1$ and $G_2$ are edge disjoint subgraphs of $K_n$. Let $\mathcal{F}$ be a family of graphs. A near packing admitting $\mathcal{F}$ of a graph $G$ is a generalization of a packing. In a near packing admitting $\mathcal{F}$, the two copies of $G$ may overlap so the subgraph defined by the edges common to both copies is a member of $\mathcal{F}$. In the paper we study three families of graphs (1) $\mathcal{E}_k$ – the family of all graphs with at most $k$ edges, (2) $\mathcal{D}_k$ – the family of all graphs with maximum degree at most $k$, and (3) $\mathcal{C}_k$ – the family of all graphs that do not contain a subgraph of connectivity greater than or equal to $k + 1$. By $m(n, \mathcal{F})$ we denote the maximum number $m$ such that each graph of order $n$ and size less than or equal to $m$ has a near-packing admitting $\mathcal{F}$. It is well known that $m(n, \mathcal{C}_0) = m(n, \mathcal{D}_0) = m(n, \mathcal{E}_0) = n - 2$ because a near packing admitting $\mathcal{C}_0$, $\mathcal{D}_0$ or $\mathcal{E}_0$ is just a packing. We prove some generalization of this result, namely we prove that $m(n, \mathcal{C}_k) \approx (k + 1)n$, $m(n, \mathcal{D}_1) \approx \frac{n}{2}$, $m(n, \mathcal{D}_2) \approx 2n$. We also present bounds on $m(n, \mathcal{E}_k)$. Finally, we prove that each graph of girth at least five has a near packing admitting $\mathcal{C}_1$ (i.e. a near packing admitting the family of the acyclic graphs).

1 Introduction

In this paper we use the term graph to refer to simple graphs without loops or multiple edges. The vertex and edge set of a graph $G$ is denoted by $V(G)$ and $E(G)$, respectively. The maximum degree of $G$ is denoted by $\Delta(G)$. A graph is called $k$-connected if any two of its vertices can be joined by $k$ internally vertex disjoint paths. A complete graph $K_1$ is

*The author was partially supported by the Polish Ministry of Science and Higher Education.
0-connected. By \( N_G(x) \) we denote the set of vertices adjacent with \( x \) in \( G \). For a vertex set \( X \), the set \( N_G(X) \) denotes the external neighbourhood of \( X \) in \( G \), i.e.

\[
N_G(X) = \{ y \in V(G) \setminus X : y \text{ is adjacent with some } x \in X \}.
\]

The degree of a vertex \( x \) is the number of vertices adjacent to \( x \) and is denoted by \( d_G(x) \).

**Definition 1.** Let \( G_1 \) and \( G_2 \) be two graphs such that \( |V(G_1)| = |V(G_2)| = n. A packing of \( G_1 \) and \( G_2 \) is a pair of edge-disjoint subgraphs \( \{H_1, H_2\} \) of \( K_n \) such that \( H_1 \cong G_1 \) and \( H_2 \cong G_2 \).

**Definition 2.** Let \( \mathcal{F} \) be any family of graphs and let \( G_1, G_2 \) be two graphs such that \( |V(G_1)| = |V(G_2)| = n. A near packing admitting \( \mathcal{F} \) of \( G_1 \) and \( G_2 \) is a pair of subgraphs \( \{H_1, H_2\} \) of \( K_n \) such that \( H_1 \cong G_1 \) and \( H_2 \cong G_2 \), and the subgraph having edges \( E(H_1) \cap E(H_2) \) is a member of \( \mathcal{F} \).

Given a graph \( G \) and a permutation \( \sigma \) of \( V(G) \), by \( \sigma(G) \) we denote the graph with \( V(\sigma(G)) = V(G) \) and such that \( \sigma(u)\sigma(v) \in E(\sigma(G)) \) if and only if \( uv \in E(G) \) for any \( u, v \in V(G) \). The spanning subgraph of \( G \) having edges \( E(G) \cap E(\sigma(G)) \) is denoted by \( G^\sigma \) (abbreviated to \( G^* \) if no confusion arises). Thus, in case when \( G_1 \cong G_2 \cong G \) the problem of finding a near packing admitting \( \mathcal{F} \) of \( G_1 \) and \( G_2 \) is equivalent to the problem of finding a near packing \( \sigma \) of \( V(G) \) such that \( G^\sigma \in \mathcal{F} \). Such a permutation \( \sigma \) of \( V(G) \) is called a near packing of \( G \) admitting \( \mathcal{F} \).

We consider three families of graphs: (1) \( \mathcal{E}_k \) being the family of all graphs with with at most \( k \) edges, (2) \( \mathcal{D}_k \) being the family of all graphs with maximum degree at most \( k \), and (3) \( \mathcal{C}_k \) being the family of all graphs that do not contain a subgraph of connectivity greater than or equal to \( k + 1 \). Notice that \( \mathcal{D}_0 = \mathcal{C}_0 = \mathcal{E}_0 \) is a family of edgeless graphs. Furthermore \( \mathcal{C}_1 \) is a family of acyclic graphs and \( \mathcal{C}_1 \cap \mathcal{D}_2 \) is a family of linear forests (i.e. disjoint unions of paths).

Let \( \mathcal{F} \) be any family of graphs. By \( m(n, \mathcal{F}) \) we denote the maximum number \( m \) such that each graph of order \( n \) and size less than or equal to \( m \) has a near-packing admitting \( \mathcal{F} \). A classic result in this area, obtained independently in [1, 2, 7], states that

**Theorem 3** ([1, 2, 7]). \( m(n, \mathcal{C}_0) = m(n, \mathcal{D}_0) = m(n, \mathcal{E}_0) = n - 2, \)

because a near packing admitting \( \mathcal{C}_0, \mathcal{D}_0 \) or \( \mathcal{E}_0 \) is just a packing. Our aim is to prove some generalizations of Theorem 3. For every \( k \geq 1 \), we determine \( m(n, \mathcal{C}_k) \) up to a constant depending only on \( k \). We find the problem concerning near packings admitting \( \mathcal{D}_k \) considerably harder. We determine only \( m(n, \mathcal{D}_1) \) up to a constant, while \( m(n, \mathcal{D}_2) \) is determined assymptotically. We also give bounds on \( m(n, \mathcal{E}_k) \).

The notion of a near packing was introduced by Eaton [3] in order to obtain some investigations concerning the following conjecture of Bollobás and Eldridge:

**Conjecture 4** ([1]). If \( |V(G_1)| = |V(G_2)| = n \) and \( (\Delta(G_1) + 1) \cdot (\Delta(G_2) + 1) \leq n + 1, \)

then there is a packing of \( G_1 \) and \( G_2 \).

The following theorem is a special case of a more general result proved by Eaton.
Lemma 7. Let $G$ be a graph and $k,l,q \geq 0$ integers. Suppose that $G$ contains an independent set $U \subset V(G)$ that satisfies the following conditions:

1. $d_G(u) \leq k$ for each $u \in U$,
2. $|N_G(u) \cap N_G(v)| \leq q$ for every $u,v \in U$.

If $|U| \geq \frac{2(k-q)}{l-q+1}$, then for every permutation $\sigma'$ of $V(G) \setminus U$ there exists a permutation $\sigma$ of $V(G)$ such that $\sigma|_{G-U} = \sigma'$ and $d_{G'}(u) \leq l$ for each $u \in U$.

Proof. Let $G' := G - U$ and $\sigma'$ be any permutation of $V(G')$. Below we show that we can extend $\sigma'$ to a permutation $\sigma$ as required of $G$.

For any $v \in V(G')$ let us define $\sigma(v) := \sigma'(v)$. Then let us consider a bipartite graph $B$ with partition sets $X := U \times \{0\}$ and $Y := U \times \{1\}$. For $u,v \in U$ the vertices $(u,0)$, $(v,1)$ are joined by an edge in $B$ if and only if $|\sigma'(N_G(u)) \cap N_G(v)| \leq l$. So, if $(u,0)$, $(v,1)$ are joined by an edge in $B$ we can put $\sigma(u) = v$. In other words, if $(u,0)$, $(v,1)$ are not neighbors in $B$, then $|\sigma'(N(u)) \cap N(v)| \geq l + 1$. Therefore, since $|N_G(u) \cap N_G(v)| \leq q$ and $d_G(u) \leq k$ for $u \in U$, we have $d_B((u,0)) \geq |U| - \frac{k-q}{l-q+1} \geq \frac{k-q}{l-q+1}$, by the assumption on $|U|$. Similarly, $d_B((v,1)) \geq \frac{k-q}{l-q+1}$.

Let $S \subset X$. If $|S| \leq |U| - \frac{k-q}{l-q+1}$ then obviously $|N_B(S)| \geq |S|$. Notice that if $|S| > |U| - \frac{k-q}{l-q+1}$ then $N_B(S) = Y$. Indeed, otherwise let $(v,1) \in Y$ be a vertex which has no neighbour in $S$. Thus,

$$d_B((v,1)) \leq |A| - |S| = |U| - |S| < |U| - \frac{k-q}{l-q+1} = \frac{k-q}{l-q+1},$$

a contradiction. Hence, in any case $|S| \leq |N(S)|$. Thus, by the Hall’s theorem there is a matching $M$ in $G$. Therefore we can define $\sigma(u) = v$ for $u,v \in U$ such that $(u,0)$, $(v,1)$ are incident with the same edge in $M$. \hfill \square

Theorem 5 ([3]). If $|V(G_1)| = |V(G_2)| = n$ and $(\Delta(G_1)+1) \cdot (\Delta(G_2)+1) \leq n + 1$, then there is a near packing admitting $\mathcal{D}_1$ of $G_1$ and $G_2$.

We also investigate another conjecture of graph packing by Faudree, Rousseau, Schelp and Schuster [4]:

Conjecture 6. For every non-star graph $G$ of girth at least 5, there is a packing of two copies of $G$.

In particular, Conjecture 6 is true for sufficiently large planar graphs [6]. On the other hand, the statement from the above conjecture is true if $G$ is a non-star graph of girth at least six [5]. In this paper we prove that the statement is true if the term ‘packing’ is replaced by the term ‘near packing admitting $G$’. This result is in some sense best possible, since for every permutation $\sigma$ of $V(K_{n,n})$ with $n \geq 3$, $K_{n,n}$ contains a cycle $C_4$.
where

Recall that $m$ near packings admitting $V$ denote a graph with vertex set $X$ pairwise disjoint and $E$ connecting any two vertices of $K$.

Proof. Let $K$ be a graph on the same vertex set as $G$, then $K$ has $m$ edges.

Let $K$ be a graph of order $n$ and size $m$ with $m \leq an - f(n)$, where $a$ is a real number and $f(n)$ is a non-decreasing function. If $U \subset V(G)$ and vertices from $U$ cover at least $a|U|$ edges, then

$$m' \leq an - f(n')$$

where $n'$ and $m'$ are respectively the order and the size of $G - U$.

Proof. Let $K$ denote the complement of a graph $G$.

We will show $G$ is $k$-connected. In what follows $\bar{G}$ denotes the complement of a graph $G$, i.e. a graph on the same vertex set as $G$ and with the property that $e \in E(G)$ if and only if $e \notin E(\bar{G})$.

Lemma 9. $m(n, C_k) \leq (k + 1)n - (k + 1)\frac{k+2}{2} - 1$.

Proof. Let $G = \overline{K_{k+1}} + K_{n-k-1}$. Clearly, $|E(G)| = (k + 1)n - (k + 1)\frac{k+2}{2}$. We will show that $G$ does not have a near packing admitting $C_k$. Consider an arbitrary permutation $\sigma$ of $V(G)$. Let $S \subset V(K_{k+1})$ be a maximal set of vertices with the property that $\sigma(S) \subset V(K_{k+1})$. Let $|S| = s$. Then, $G_\sigma$ contains a $K^+_{s,k+1-s,k+1-s}$ with $X_1 = S$, $Y = V(K_{k+1}) \setminus S$ and $X_2 \subset V(K_{n-k-1})$.

3 Near packings admitting $C_k$

Recall that $m(n, C_0) = n - 2$. We start with the following construction. Let $K^+_{s,k-s,k-s}$ denote a graph with vertex set $V(K^+_{s,k-s,k-s}) = X_1 \cup X_2 \cup Y$ such that $X_1, X_2, Y$ are pairwise disjoint and $|X_1| = s$, $|X_2| = |Y| = k - s$. Furthermore, $E(K^+_{s,k-s,k-s}) = E_1 \cup E_2$, where $E_1 = \{xy : x \in X_1 \cup X_2, y \in Y\}$ and $E_2 = \{xz : x \in X_1, z \in X_1 \cup X_2\}$. In other words, $K^+_{s,k-s,k-s}$ arises from a tripartite graph (with partition sets $X_1$, $X_2$ and $Y$) by adding all possible edges having two endpoints in $X_1$, see Figure 1. It is easily seen that any two vertices of $K^+_{s,k-s,k-s}$ are joined by at least $k$ internally vertex disjoint paths, so $K^+_{s,k-s,k-s}$ is $k$ connected. In what follows $G$ denotes the graph $G$.
Theorem 10. \( m(n, C_k) \geq (k + 1)n - 4k(k + 1)^2 - 2 \).

Proof. For \( k = 0 \) the result follows from Theorem 3. Fix \( k \geq 1 \) and let \( c_k = 4k(k + 1)^2 + 2 \). We will prove that each graph of order \( n \) and size at most \( (k + 1)n - c_k \) has a near packing admitting \( C_k \).

Suppose that \( G \) is a counterexample with minimum order \( n \). Let \( m \) denote the size of \( G \), so \( m \leq (k + 1)n - c_k \). Note that if \( n \leq 4(k + 1)^2 \), then

\[
m \leq (k + 1)n - c_k = kn - c_k + n
\leq k(4(k + 1)^2) - (4k(k + 1)^2 + 2) + n = n - 2.
\]

Hence \( G \) has a near packing admitting \( C_k \), by Theorem 3, which contradicts our assumption on \( G \). Thus, we may assume that \( n \geq 4(k + 1)^2 + 1 \). Furthermore, if \( \Delta(G) \leq 2(k + 1) - 1 \) then \((\Delta(G) + 1)^2 \leq 4(k + 1)^2 < n + 1 \). Hence, \( G \) has a near packing admitting \( C_k \) by Theorem 5 (because \( D_1 \subseteq C_k \)), a contradiction again. Therefore, we may assume that \( \Delta(G) \geq 2(k + 1) \). Let \( w \in V(G) \) with \( d_G(w) \geq 2(k + 1) \).

Suppose first that \( G \) contains a vertex \( u \) with \( d_G(u) \leq k \). By Proposition 8 and by the minimality assumption, \( G' := G - \{u, w\} \) has a near packing \( \sigma' \) admitting \( C_k \). We claim that \( \sigma := (u, w)\sigma' \) is a near packing of \( G \) admitting \( C_k \). Indeed, since \( d_G(u) \leq k \) then \( d_G((u) \leq k \) as well as \( d_G(w) \leq k \). Hence, neither \( u \) nor \( w \) can be in a subgraph of \( G^{*} \) of connectivity \( k + 1 \) or more. Moreover, since \( \sigma|_G \) is a near packing of \( G' \) admitting \( C_k \), then \( G^{*} - \{u, w\} \) does not contain a subgraph of connectivity \( k + 1 \) or more, neither. Therefore, \( \sigma \) is a near packing of \( G \) admitting \( C_k \).

Thus, we may assume that \( d_G(u) \geq k + 1 \) for every \( u \in V(G) \). Let \( S \) be a maximum set of vertices of \( G \) such that \( S \) is independent, \( k + 1 \leq d_G(u) \leq 2k + 1 \) for each \( u \in S \), and \( |N_G(u) \cap N_G(w)| \leq k \) for every \( u, w \in S \). Since \( S \) is independent, by Proposition 8 and by the minimality assumption, \( G - S \) has a near packing \( \sigma'' \) admitting \( C_k \). By Lemma 7 (with \( k, l, q \) replaced by \( 2k + 1, k, k \), respectively), if \( |S| \geq 2k + 2 \) then there is a permutation \( \sigma \) of \( G \), such that \( \sigma|_{G - S} = \sigma'' \) and \( d_G^{-1}(u) \leq k \) for every \( u \in S \). Simirarly as before, we can see that \( \sigma \) is a near packing of \( G \) admitting \( C_k \), a contradiction.

Therefore \( |S| \leq 2k + 1 \) and so \( |N_G(S)| \leq (2k + 1)^2 \). Let \( V_j = \{v \in V(G) \setminus N_G(S) : d_G(v) = j\} \). Note that by the definition of \( S \), we have \( |N_G(S) \cap N_G(u)| \geq k + 1 \) for every \( u \in V_{k+1} \cup \cdots \cup V_{2k+1} \). Hence, vertices from \( N_G(S) \) are incident (in common) to at least \((k + 1)(|V_{k+1}| + \cdots + |V_{2k+1}|)\) edges. Thus,

\[
(2k + 2)n - 8k(k + 1)^2 - 4 \geq 2m
= \sum_{u \in N_G(S)} d_G(u) + \sum_{v \in V(G) \setminus N_G(S)} d_G(v)
\geq (k + 1)(|V_{k+1}| + \cdots + |V_{2k+1}|) + (k + 1)|V_{k+1}| + \cdots + (k + 1)|V_{2k+1}|
+ (2k + 2)(n - |V_{k+1}| + \cdots |V_{2k+1}| - |N_G(S)|)
\geq (2k + 2)n - (2k + 2)(2k + 1)^2,
\]
a contradiction. Hence, we deduce no counterexample to Theorem 10 exists. \qed

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Theorem 11. Every graph of girth at least 5 has a near packing admitting $C_1$.

Proof. Let $G$ be a minimum counterexample to Theorem 11. Let $u \in V(G)$ with $d_G(u) = \Delta(G)$. Let $G' = G - u$ and $U = N_G(u)$. By the girth assumption, $U$ is an independent set in $G'$ (as well as in $G$), and $N_G(x) \cap N_G(y) = \emptyset$ for every $x, y \in U$. By the minimality assumption $G'' := G' - U$ has a near packing $\sigma''$ admitting $C_1$. Moreover, $|U| = \Delta(G)$ and $d_G(u) \leq \Delta(G) - 1$. Hence, by Lemma 7 (with $k = \Delta(G) - 1, l = 1, q = 0$), $G''$ has a near packing $\sigma'$ such that $\sigma'|_{G''} = \sigma''$ and $d_{G''}(u) \leq 1$ for each $u \in U$. Thus, since $G''$ is acyclic, $G''$ is also acyclic. Let $u$ be any vertex from $U$. It is easy to see that the permutation $\sigma$ such that $\sigma(u) = x, \sigma(x) = u$ and $\sigma(y) = \sigma'(y)$ for every $y \in V(G) \setminus \{u, x\}$ is a near packing of $G$ admitting $C_1$, a contradiction. $$\square$$

4 Near packings admitting $D_k$

Recall that $m(n, D_0) = n - 2$.

Lemma 12. $m(n, D_k) \leq \left\lfloor \frac{(k+2)(n-1)}{2} \right\rfloor - 1$.

Proof. Let $H$ be a $k$-regular graph of order $n - 1$ provided that $k$ is even or $n - 1$ is even. Otherwise, let $H$ be a graph having all but one vertices having degree $k$ and one vertex having degree $k + 1$. Let $G = K_1 + H$ and $V(K_1) = \{u\}$. It is easily seen that for any permutation $\sigma$ of $V(G)$, the vertex $u$ (as well as its image) has degree at least $k + 1$ in $G'_*$. Thus, $G$ does not have a near packing admitting $D_k$. Furthermore, $E(G) = \frac{(k+1)(n-1)+n-1}{2} = \frac{(k+2)(n-1)}{2}$ if $k$ is even or $n - 1$ is even, or $E(G) = \frac{(k+1)(n-2)+(k+2)+n-1}{2} = \frac{(k+2)(n-1)+1}{2}$ otherwise. $\square$

We are tempted to propose the following conjecture

Conjecture 13.

$$\frac{k+2}{2}n - c_1(k) \leq m(n, D_k) \leq \frac{k+2}{2}n - c_2(k),$$

where $c_i(k)$ are constants depending only on $k$.

The next theorem confirms Conjecture 13 for $k = 1$.

Theorem 14. $m(n, D_1) \geq \frac{3}{2}n - 10$.

Proof. Let $G$ be a counterexample of minimum order $n$. Without loss of generality we assume that $m := |E(G)| = \frac{3}{2}n - 10$. Note that if $n \leq 16$ then $\frac{3}{2}n - 10 \leq n - 2$. Thus, by Theorem 3, $G$ has a packing which contradicts our assumption on $G$. Hence, we may assume that $n \geq 17$. Furthermore, if $\Delta(G) \leq 3$, then $(\Delta(G) + 1)^2 \leq 16 < n + 1$, so $G$ has a near packing admitting $D_1$ by Theorem 5. Thus, we may assume that $\Delta(G) \geq 4$. Let $w \in V(G)$ with $d_G(w) \geq 4$.

Suppose first that $G$ has a vertex $u$ with $d_G(u) = 0$. Then, by Proposition 8 and by the minimality assumption, $G_1 := G - \{u, w\}$ has a near packing $\sigma_1$ admitting $D_1$. Clearly, $(u, w)\sigma_1$ is a near packing of $G$ admitting $D_1$. 

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So we may assume that $G$ has no isolated vertex. Suppose now that $G$ has a vertex $u$ with $d_G(u) = 1$ and let $v$ be the neighbor of $u$. If $d_G(v) \geq 3$ then, by Proposition 8 and the minimality assumption, $G_2 := G - \{u, v\}$ has a near packing $\sigma_2$ admitting $D_1$. Clearly, $(u, v)\sigma_2$ is a near packing admitting $D_1$ of $G$. Similarly, if $d_G(v) = 1$ then $(u)(w, v)\sigma_3$ is a near packing admitting $D_1$ of $G$ where $\sigma_3$ is a near packing admitting $D_1$ of $G - \{u, v, w, x\}$ ($\sigma_4$ exists by the minimality assumption). Thus we may assume that $d_G(v) = 2$. Let $x$ be the neighbor of $v$ different from $u$. If $x \neq w$ then $(u)(v, w, x)\sigma_4$ is a near packing admitting $D_1$ of $G$ where $\sigma_4$ is a near packing admitting $D_1$ of $G - \{u, v, w, x\}$ ($\sigma_5$ exists by the minimality assumption).

Therefore, we may assume that $d_G(u) \geq 2$ for each $u \in V(G)$. Let $S \subseteq V(G)$ be a maximal set such that $S$ is independent in $G$, $d_G(v) = 2$ for every $v \in S$, and $N_G(u) \cap N_G(v) = \emptyset$ for every $u, v \in S$. Note that $S \neq \emptyset$. By Proposition 8 and by the minimality assumption, $G - S$ has a near packing $\sigma'$ admitting $D_1$. Note that if $|S| \geq 4$, then by Lemma 7 (with $k = 2$, $q = 0$ and $l = 0$), there exists a near packing of $G$ admitting $D_1$, a contradiction with the assumption on $G$. Thus, $|S| \leq 3$ and so $|N_G(S)| \leq 6$. Let $V_j = \{v \in V(G) \setminus N_G(S) : d_G(v) = j\}$. Note that by the definition of $S$, we have $|N_G(S) \cap N_G(u)| \geq 1$ for every $u \in V_2$. Therefore,

$$3n - 20 = 2m = \sum_{u \in N_G(S)} d_G(u) + \sum_{v \in V(G) \setminus N_G(S)} d_G(v) \geq |V_2| + 2|V_2| + 3(n - |V_2| - |N_G(S)|) \geq 3n - 18,$$

a contradiction. Hence, we deduce no counterexample to Theorem 14 exists. \hfill \Box

The following result provides some evidence for Conjecture 13 in case when $k = 2$.

**Theorem 15** ([8]). $m(n, D_2) \geq 2n - 10n^{2/3} - 7$.

## 5 Near packings admitting $E_k$

The join $G = G_1 + G_2$ of graphs $G_1$ and $G_2$ with disjoint vertex sets $V_1$ and $V_2$ and edge sets $E_1$ and $E_2$ is the graph union $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ together with all the edges joining $V_1$ and $V_2$.

**Lemma 16.** If $n \geq 2k + 2$ then $m(n, E_{2k}) \leq \left\lfloor \frac{(k+2)(n-1)}{2} \right\rfloor - 1$.

**Proof.** Let $H$ be a $k$-regular graph of order $n - 1$ provided that $k$ is even or $n - 1$ is even. Otherwise, let $H$ be a graph with all but one vertices having degree $k$ and one vertex having degree $k + 1$. Let $G = K_1 + H$ and $V(K_1) = \{u\}$. It is easily seen that for any permutation $\sigma$ of $V(G)$, the vertex $u$ as well as $\sigma(u)$ has degree at least $k + 1$ in $G_\sigma$. Thus, if $u \neq \sigma(u)$ then $G_\sigma^*$ has at least $2k + 1$ edges. If $u = \sigma(u)$ then $u$ has degree $n - 1$ in $G_\sigma^*$. Since $n \geq 2k + 2$, $G_\sigma^*$ has at least $2k + 1$ edges. Therefore, $G$ does not have a near
packing admitting $\mathcal{E}_{2k}$. Furthermore, $E(G) = \frac{(k+1)(n-1)+n-1}{2} = \frac{(k+2)(n-1)}{2}$ if $k$ is even or $n-1$ is even, or $E(G) = \frac{(k+1)(n-2)+(k+2)+(n-1)}{2} = \frac{(k+2)(n-1)+1}{2}$ otherwise. \hfill \Box

**Theorem 17.** $m(n, \mathcal{E}_k) \geq \sqrt{\frac{k}{2}n(n-1)}$.

Proof. Let $G$ be a graph of order $n$ and size $m$. We will prove that if $m \leq \sqrt{\frac{k}{2}n(n-1)}$ then there is a near-packing of $G$ admitting $\mathcal{E}_k$. Consider the probability space whose $n!$ points are the permutations of $V(G)$. For any two edges $e, f \in E(G)$ let $X_{ef}$ denote the indicator random variable with value 1 if $f$ is an image of $e$. Then

$$E(X_{ef}) = \text{Prob}(X_{ef} = 1) = \frac{2(n-2)!}{n!} = \left(\frac{n}{2}\right)^{-1}.$$ 

Let $X = \sum_{e,f \in E(G)} X_{ef}$. Thus, by the linearity of expectation, we have

$$E(X) = \sum_{e,f \in E(G)} E(X_{ef}) \leq m^2 \left(\frac{n}{2}\right)^{-1} \leq k.$$ 

This implies that there exists a permutation $\sigma$ of $V(G)$ such that $G^*_{\sigma}$ has at most $k$ edges. Thus, $\sigma$ is a near packing of $G$ admitting $\mathcal{E}_k$. \hfill \Box

**References**


