

On the Topology of the Cambrian Semilattices

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Abstract

For an arbitrary Coxeter group W , Reading and Speyer defined Cambrian semilattices \mathcal{C}_γ as sub-semilattices of the weak order on W induced by so-called γ -sortable elements. In this article, we define an edge-labeling of \mathcal{C}_γ , and show that this is an EL-labeling for every closed interval of \mathcal{C}_γ . In addition, we use our labeling to show that every finite open interval in a Cambrian semilattice is either contractible or spherical, and we characterize the spherical intervals, generalizing a result by Reading.

1 Introduction

In [6, Theorem 9.6], Björner and Wachs observed that the Tamari lattice \mathcal{T}_n , introduced in [27], can be regarded as the subposet of the weak-order lattice on the symmetric group \mathfrak{S}_n , consisting of 312-avoiding permutations. More precisely, there exists an order-preserving surjection $\sigma : \mathfrak{S}_n \rightarrow \mathcal{T}_n$ such that \mathcal{T}_n is isomorphic to the subposet of the weak-order lattice on \mathfrak{S}_n consisting of the bottom elements in the fibers of σ . In [19], the map σ was realized as a map from \mathfrak{S}_n to the triangulations of an $(n+2)$ -gon, where the partial order on the latter is given by diagonal flips. It was shown that the fibers of σ induce a congruence relation on the weak-order lattice on \mathfrak{S}_n , and that the Tamari lattice is isomorphic to the lattice quotient induced by this congruence. Moreover, it was observed that different embeddings of the $(n+2)$ -gon in the plane yield different lattice quotients of the weak-order lattice on \mathfrak{S}_n . The realization of \mathfrak{S}_n as the Coxeter group A_{n-1} was then used to connect the embedding of the $(n+2)$ -gon in the plane with a Coxeter element of A_{n-1} . This connection eventually led to the definition of Cambrian lattices, which can analogously be defined for an arbitrary finite Coxeter group W as lattice quotients

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of the weak-order lattice on W with respect to certain lattice congruences induced by orientations of the Coxeter diagram of W , see [21].

As suggested in [26, Appendix B], and later in [15, Theorem 1], the Hasse diagram of the Tamari lattice corresponds to the 1-skeleton of the classical associahedron. (Due to the connection to the symmetric group, which was elaborated in [15], the classical associahedron is also referred to as *type A-associahedron*.) In [7, 8, 10, 24], generalized associahedra were defined for all crystallographic Coxeter groups which generalize the type A -associahedron. The Cambrian lattices provide another viewpoint for the generalized associahedra, namely that the fan associated to a Cambrian lattice of crystallographic type is the normal fan of the generalized associahedron of the same type, see [22] for the details of this construction. Moreover, since the Cambrian lattices are defined for all finite Coxeter groups, this connection defines a generalized associahedron for the non-crystallographic types as well, see [22, Corollary 8.1].

In [23], Reading and Speyer generalized the construction of Cambrian lattices to infinite Coxeter groups. Since there exists no longest element in an infinite Coxeter group, the weak order constitutes only a meet-semilattice. Using the realization of the Cambrian lattices in terms of Coxeter-sortable elements, which was first described in [21] and later extended in [23], the analogous construction as in the finite case yields a sub-semilattice of the weak-order semilattice, the so-called *Cambrian semilattice*.

This article is dedicated to the investigation of the topological properties of the order complex of the proper part of closed intervals in a Cambrian semilattice. One (order-theoretic) tool to investigate these properties is EL-shellability, which was introduced in [1], and further developed in [4–6]. The fact that a poset is EL-shellable implies a number of properties of the associated order complex: this order complex is Cohen-Macaulay, it is homotopy equivalent to a wedge of spheres and the dimensions of its homology groups can be computed from the labeling. The first main result of the present article is the following.

Theorem 1.1. *Every closed interval in \mathcal{C}_γ is EL-shellable for every (possibly infinite) Coxeter group W and every Coxeter element $\gamma \in W$.*

We prove this result uniformly using the realization of \mathcal{C}_γ in terms of Coxeter-sortable elements, and thus our proof does not require W to be finite or even crystallographic. For finite crystallographic Coxeter groups, Theorem 1.1 is implied by [12, Theorem 4.17]. Ingalls and Thomas considered in [12] the category of finite dimensional representations of an orientation of the Coxeter diagram of a finite crystallographic Coxeter group W , and considered the corresponding Cambrian lattices as a poset of torsion classes of this category. However, their approach cannot be applied directly to non-crystallographic or to infinite Coxeter groups. Moreover, we remark that the labeling from Ingalls' and Thomas' approach is different from our labeling.

Finally, using the fact that every closed interval of \mathcal{C}_γ is EL-shellable, we are able to determine the homotopy type of the proper parts of these intervals by counting the number of falling chains with respect to our labeling. It turns out that every open interval is either contractible or spherical, *i.e.* homotopy equivalent to a sphere. We can further

characterize which intervals of \mathcal{C}_γ are contractible and which are spherical, as our second main result shows. Recall that a closed interval $[x, y]$ in a lattice is called *nuclear* if y is the join of atoms of $[x, y]$.

Theorem 1.2. *Let W be a (possibly infinite) Coxeter group and let $\gamma \in W$ be a Coxeter element. Every finite open interval in the Cambrian semilattice \mathcal{C}_γ is either contractible or spherical. Furthermore, a finite open interval $(x, y)_\gamma$ is spherical if and only if the corresponding closed interval $[x, y]_\gamma$ is nuclear.*

For finite Coxeter groups, Theorem 1.2 is implied by concatenating [18, Theorem 1.1] and [18, Propositions 5.6 and 5.7]. Reading's approach in the cited article was to investigate fan posets of central hyperplane arrangements. He showed that for a finite Coxeter group W the Cambrian lattices can be viewed as fan posets of a fan induced by regions of the Coxeter arrangement of W which are determined by orientations of the Coxeter diagram of W . The tools Reading developed in [18] apply to a much larger class of fan posets, but cannot be applied directly to infinite Coxeter groups.

The proofs of Theorems 1.1 and 1.2 are obtained completely within the framework of Coxeter-sortable elements and thus have the advantage that they are uniform and direct.

This article is organized as follows. In Section 2, we recall the necessary order-theoretic concepts, as well as the definition of EL-shellability. Furthermore, we recall the definition of Coxeter groups, and the construction of the Cambrian semilattices. In Section 3, we define a labeling of the Hasse diagram of a Cambrian semilattice and give a case-free proof that this labeling is indeed an EL-labeling for every closed interval of this semilattice, thus proving Theorem 1.1. In Section 4, we prove Theorem 1.2, by counting the falling maximal chains with respect to our labeling and by applying [5, Theorem 5.9] which relates the number of falling maximal chains in a poset to the homotopy type of the corresponding order complex. The characterization of the spherical intervals of \mathcal{C}_γ follows from Theorem 4.3.

2 Preliminaries

In this section, we recall the necessary definitions, which are used throughout the article. For further background on posets, we refer to [9] or to [25], where in addition some background on lattices and lattice congruences is provided. An introduction to poset topology can be found in either [2] or [28]. For more background on Coxeter groups, we refer to [3] and [11].

2.1 Posets and EL-Shellability

Let (P, \leq_P) be a finite partially ordered set (*poset* for short). We say that P is *bounded* if it has a unique minimal and a unique maximal element, which we usually denote by $\hat{0}$ and $\hat{1}$, respectively. For $x, y \in P$, we say that y *covers* x (and write $x \lessdot_P y$) if $x \leq_P y$ and there is no $z \in P$ such that $x <_P z <_P y$. We denote the set of all covering relations of P by $\mathcal{E}(P)$.

For $x, y \in P$ with $x \leq_P y$, we define the closed interval $[x, y]$ to be the set $\{z \in P \mid x \leq_P z \leq_P y\}$. Similarly, we define the open interval (x, y) to be the set $\{z \in P \mid x <_P z <_P y\}$. A chain $c : x = p_0 \leq_P p_1 \leq_P \cdots \leq_P p_s = y$ is called *maximal* if $(p_i, p_{i+1}) \in \mathcal{E}(P)$ for every $0 \leq i \leq s - 1$.

Let (P, \leq_P) be a bounded poset and let $c : \hat{0} = p_0 \leq_P p_1 \leq_P \cdots \leq_P p_s = \hat{1}$ be a maximal chain of P . Given another poset (Λ, \leq_Λ) , a map $\lambda : \mathcal{E}(P) \rightarrow \Lambda$ is called an *edge-labeling* of P . We denote the sequence $(\lambda(p_0, p_1), \lambda(p_1, p_2), \dots, \lambda(p_{s-1}, p_s))$ of edge-labels of c by $\lambda(c)$. The chain c is called *rising* (respectively *falling*) if $\lambda(c)$ is a strictly increasing (respectively weakly decreasing) sequence. For two words (p_1, p_2, \dots, p_s) and (q_1, q_2, \dots, q_t) in the alphabet Λ , we write $(p_1, p_2, \dots, p_s) \leq_{\Lambda^*} (q_1, q_2, \dots, q_t)$ if and only if either

$$\begin{aligned} p_i &= q_i, & \text{for } 1 \leq i \leq s \text{ and } s \leq t, & \text{ or} \\ p_i &<_\Lambda q_i, & \text{for the least } i \text{ such that } p_i \neq q_i. \end{aligned}$$

A maximal chain c of P is called *lexicographically first* among the maximal chains of P if for every other maximal chain c' of P we have $\lambda(c) \leq_{\Lambda^*} \lambda(c')$. An edge-labeling of P is called *EL-labeling* if for every closed interval $[x, y]$ in P there exists a unique rising maximal chain which is lexicographically first among all maximal chains in $[x, y]$. A bounded poset that admits an EL-labeling is called *EL-shellable*.

Let us further recall that the Möbius function μ of P is the map $\mu : P \times P \rightarrow \mathbb{Z}$ defined recursively by

$$\mu(x, y) = \begin{cases} 1, & x = y \\ -\sum_{x \leq_P z <_P y} \mu(x, z), & x <_P y \\ 0, & \text{otherwise.} \end{cases}$$

A remarkable property of EL-shellable posets is that we can compute the value of the Möbius function for every closed interval of P from the labeling, as is stated in the following proposition¹.

Proposition 2.1 ([5, Proposition 5.7]). *Let (P, \leq_P) be an EL-shellable poset, and let $x, y \in P$ with $x \leq_P y$. Then,*

$$\begin{aligned} \mu(x, y) &= \text{number of even length falling maximal chains in } [x, y] \\ &\quad - \text{number of odd length falling maximal chains in } [x, y]. \end{aligned}$$

2.2 Coxeter Groups and Weak Order

Let W be a (possibly infinite) group being generated by the finite set $S = \{s_1, s_2, \dots, s_n\}$. Let $m = (m_{i,j})_{1 \leq i, j \leq n}$ be a symmetric $(n \times n)$ -matrix, where the entries are either positive

¹Actually, Proposition 5.7 in [5] is stated for posets admitting a so-called *CR-labeling*. EL-shellable posets are a particular instance of this class of posets, and for the scope of this article it is sufficient to restrict our attention to these.

integers or the formal symbol ∞ , and which satisfies $m_{i,i} = 1$ for all $1 \leq i \leq n$, and $m_{i,j} \geq 2$ otherwise. (We use the convention that ∞ is formally larger than any natural number.) We call W a *Coxeter group* if it has the presentation

$$W = \langle S \mid (s_i s_j)^{m_{i,j}} = \varepsilon \text{ for } 1 \leq i, j \leq n \rangle,$$

where $\varepsilon \in W$ denotes the identity, and we call S the set of *simple generators of W* . We interpret the case $m_{i,j} = \infty$ as stating that there is no relation between the generators s_i and s_j , and call the matrix m the *Coxeter matrix of W* .

Since S is a generating set of W , we can write every element $w \in W$ as a product of the elements in S , and we call such a word a *reduced word for w* if it has minimal length. More precisely, define the *word length* on W (with respect to S) as

$$\ell_S : W \rightarrow \mathbb{N}, \quad w \mapsto \min\{k \mid w = s_{i_1} s_{i_2} \cdots s_{i_k} \text{ and } s_{i_j} \in S \text{ for all } 1 \leq j \leq k\}.$$

If $\ell_S(w) = k$, then every product of k generators which yields w is a reduced word for w . Define the (*right*) *weak order of W* by

$$u \leq_S v \quad \text{if and only if} \quad \ell_S(v) = \ell_S(u) + \ell_S(u^{-1}v).$$

The poset (W, \leq_S) is a ranked meet-semilattice, the so-called *weak-order semilattice of W* , and ℓ_S is its rank function. Moreover, (W, \leq_S) is *finitary* meaning that every principal order ideal of (W, \leq_S) is finite. In the case where W is finite, there exists a unique longest word w_o of W , and (W, \leq_S) is thus a lattice.

It is often convenient to have the following alternative characterization of the (right) weak order on W . Let $T = \{wsw^{-1} \mid w \in W, s \in S\}$, and define for $w \in W$, the (*left*) *inversion set of w* as

$$\text{inv}(w) = \{t \in T \mid \ell_S(tw) \leq \ell_S(w)\}.$$

It is the statement of [3, Proposition 3.1.3] that $u \leq_S v$ if and only if $\text{inv}(u) \subseteq \text{inv}(v)$, and [3, Corollary 1.4.5] states that $\ell_S(w) = |\text{inv}(w)|$ for all $w \in W$. Thus, every $w \in W$ is uniquely determined by its (left) inversion set.

2.3 Coxeter-Sortable Elements

Let $\gamma = s_1 s_2 \cdots s_n \in W$ be a Coxeter element, and define the half-infinite word

$$\gamma^\infty = s_1 s_2 \cdots s_n | s_1 s_2 \cdots s_n | \cdots .$$

The vertical bars in the representation of γ^∞ are “dividers”, which have no influence on the structure of the word, but shall serve for a better readability. Clearly, every reduced word for $w \in W$ can be considered as a subword of γ^∞ . Among all reduced words for w , there is a unique reduced word, which is lexicographically first considered as a subword of γ^∞ . This reduced word is called the *γ -sorting word of w* .

Example 2.2. Consider the Coxeter group $W = \mathfrak{S}_5$, generated by $S = \{s_1, s_2, s_3, s_4\}$, where s_i corresponds to the transposition $(i, i + 1)$ for all $i \in \{1, 2, 3, 4\}$ and let $\gamma = s_1 s_2 s_3 s_4$. Clearly, s_1 and s_4 commute. Hence, $w_1 = s_1 s_2 | s_1 s_4$ and $w_2 = s_1 s_2 s_4 | s_1$ are reduced words for the same element $w \in W$. Considering w_1 and w_2 as subwords of γ^∞ , we find that w_2 is a lexicographically smaller subword of γ^∞ than w_1 is. There are six other reduced words for w , namely

$$\begin{aligned} w_3 &= s_1 s_4 | s_2 | s_1, & w_4 &= s_4 | s_1 s_2 | s_1, & w_5 &= s_4 | s_2 | s_1 s_2, \\ w_6 &= s_2 s_4 | s_1 s_2, & w_7 &= s_2 | s_1 s_4 | s_2, & w_8 &= s_2 | s_1 s_2 s_4. \end{aligned}$$

It is easy to see that among these w_2 is the lexicographically first subword of γ^∞ , and hence w_2 is the γ -sorting word of w .

In the following, we consider only γ -sorting words, and we write

$$w = s_1^{\delta_{1,1}} s_2^{\delta_{1,2}} \cdots s_n^{\delta_{1,n}} | s_1^{\delta_{2,1}} s_2^{\delta_{2,2}} \cdots s_n^{\delta_{2,n}} | \cdots | s_1^{\delta_{l,1}} s_2^{\delta_{l,2}} \cdots s_n^{\delta_{l,n}}, \quad (1)$$

where $\delta_{i,j} \in \{0, 1\}$ for $1 \leq i \leq l$ and $1 \leq j \leq n$. For each $i \in \{1, 2, \dots, l\}$, we say that

$$b_i = \{s_j \mid \delta_{i,j} = 1\} \subseteq S$$

is the i -th block of w . We consider the blocks of w sometimes as sets and sometimes as subwords of γ , depending on how much structure we need. We say that w is γ -sortable if and only if $b_1 \supseteq b_2 \supseteq \cdots \supseteq b_l$, and we denote the set of γ -sortable elements of W by C_γ .

Example 2.3. Let us continue the previous example. We have seen that $w_2 = s_1 s_2 s_4 | s_1$ is a γ -sorting word in W , and $b_1 = \{s_1, s_2, s_4\}$, and $b_2 = \{s_1\}$. Since $b_2 \subseteq b_1$, we see that w_2 is indeed γ -sortable.

The γ -sortable elements of W are characterized by a recursive property which we will describe next. A generator $s \in S$ is called *initial in γ* if it is the first letter in some reduced word for γ . For some subset $J \subseteq S$, let W_J denote the parabolic subgroup of W generated by the set J , and let $W^J = \{w \in W \mid w <_S ws \text{ for all } s \in J\}$. It is the statement of [3, Proposition 2.4.4 (i)] that w can be written uniquely as $w = w^J \cdot w_J$ for $w^J \in W^J$ and $w_J \in W_J$. It is straightforward to verify that the map $w \mapsto w_J$ is defined by the property that $\text{inv}(w_J) = \text{inv}(w) \cap W_J$, see for instance [23, p. 10] for a proof. For $s \in S$ we abbreviate $\langle s \rangle = S \setminus \{s\}$.

Proposition 2.4 ([23, Proposition 2.29]). *Let W be a Coxeter group, let $\gamma \in W$ be a Coxeter element and let s be initial in γ . Then, an element $w \in W$ is γ -sortable if and only if*

- (i) $s \leq_S w$ and sw is $s\gamma s$ -sortable, or
- (ii) $s \not\leq_S w$ and w is an $s\gamma$ -sortable element of $W_{\langle s \rangle}$.

Remark 2.5. The property of being γ -sortable does not depend on the choice of a reduced word for γ , see [23, Section 2.7]. For $w \in W$, let w_1 and w_2 be the γ -sorting words of w with respect to two different reduced words γ_1 and γ_2 for γ . Since γ_1 and γ_2 differ only in commutations of letters, it is clear that w_1 and w_2 differ also only in commutations of letters, with no commutations across dividers. Hence, the i -th block of w_1 , considered as a subset of S , is equal to the i -th block of w_2 , considered as a subset of S . However, the i -th block of w_1 , considered as a subword of γ_1 , is different from the i -th block of w_2 , considered as a subword of γ_2 .

2.4 Cambrian Semilattices

In [23, Section 7] the *Cambrian semilattice* \mathcal{C}_γ is defined as the sub-semilattice of the weak order on W consisting of all γ -sortable elements. That \mathcal{C}_γ is well-defined follows from the following theorem.

Theorem 2.6 ([23, Theorem 7.1]). *Let A be a collection of γ -sortable elements of W . If A is nonempty, then $\bigwedge_S A$ is γ -sortable. If A has an upper bound, then $\bigvee_S A$ is γ -sortable.*

It turns out that \mathcal{C}_γ is not only a sub-semilattice of the weak order, but also a quotient semilattice. The key role in the proof of this property is played by the projection π_\downarrow^γ which maps every word $w \in W$ to the unique largest γ -sortable element below w . More precisely if s is initial in γ , then define

$$\pi_\downarrow^\gamma(w) = \begin{cases} s\pi_\downarrow^{s\gamma s}(sw), & \text{if } s \leq_S w \\ \pi_\downarrow^{s\gamma}(w_{\langle s \rangle}), & \text{if } s \not\leq_S w, \end{cases} \quad (2)$$

and set $\pi_\downarrow^\gamma(\varepsilon) = \varepsilon$, see [23, Section 6]. The most important properties of this map are stated in the following theorems.

Theorem 2.7 ([23, Theorem 6.1]). *The map π_\downarrow^γ is order-preserving.*

Theorem 2.8 ([23, Theorem 7.3]). *For some subset $A \subseteq W$, if A is nonempty, then $\bigwedge_\gamma \pi_\downarrow^\gamma(A) = \pi_\downarrow^\gamma(\bigwedge_S A)$ and if A has an upper bound, then $\bigvee_\gamma \pi_\downarrow^\gamma(A) = \pi_\downarrow^\gamma(\bigvee_S A)$.*

Hence, π_\downarrow^γ is a semilattice homomorphism from the weak order on W to \mathcal{C}_γ , and \mathcal{C}_γ can be considered as the quotient semilattice of the weak order modulo the semilattice congruence θ_γ induced by the fibers of π_\downarrow^γ . This semilattice congruence is called *Cambrian congruence*. Since the lack of a maximal element is the only obstruction for the weak order to be a lattice, it follows immediately that the restriction of π_\downarrow^γ (and hence θ_γ) to closed intervals of the weak order yields a lattice homomorphism (and hence a lattice congruence). Figure 1 shows the Hasse diagram of the weak order on the Coxeter group A_3 and the congruence classes of θ_γ for $\gamma = s_1s_2s_3$.

In the remainder of this article, we switch frequently between the weak-order semilattice on W and the Cambrian semilattice \mathcal{C}_γ . In order to point out properly which semilattice we consider, we denote the order relation of the weak-order semilattice by

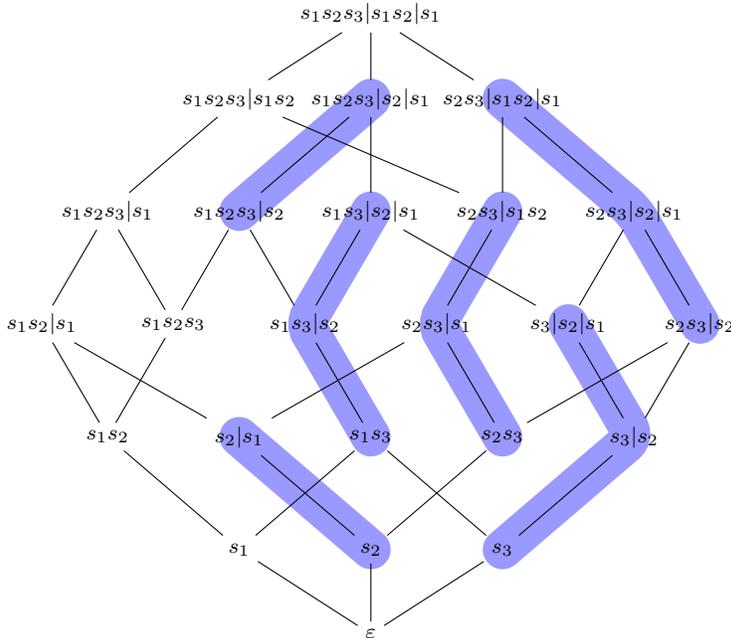


Figure 1: The Cambrian congruence on the weak-order lattice on A_3 induced by the Coxeter element $s_1s_2s_3$. The non-singleton congruence classes are highlighted.

\leq_S , and the order relation of \mathcal{C}_γ by \leq_γ . Analogously, we denote a closed (respectively open) interval in the weak-order semilattice by $[u, v]_S$ (respectively $(u, v)_S$), and a closed (respectively open) interval in \mathcal{C}_γ by $[u, v]_\gamma$ (respectively $(u, v)_\gamma$). Moreover, we denote the meet (respectively join) in the weak-order semilattice by \wedge_S (respectively \vee_S), and the meet (respectively join) in \mathcal{C}_γ by \wedge_γ (respectively \vee_γ).

3 EL-Shellability of the Closed Intervals in \mathcal{C}_γ

In this section, we define an edge-labeling of \mathcal{C}_γ , discuss some of its properties and eventually prove Theorem 1.1.

3.1 The Labeling

In the remainder of this article, we focus on a fixed reduced word for γ , namely $\gamma = s_1s_2 \cdots s_n$. Define for every $w \in W$ the set of positions of the γ -sorting word of w as

$$\alpha_\gamma(w) = \{(i-1) \cdot n + j \mid \delta_{i,j} = 1\} \subseteq \mathbb{N},$$

where the $\delta_{i,j}$'s are the exponents from (1).

Example 3.1. Let $W = \mathfrak{S}_4$, $\gamma = s_1s_2s_3$ and consider $u = s_1s_2s_3|s_2$, and $v = s_2s_3|s_2|s_1$. Then, $\alpha_\gamma(u) = \{1, 2, 3, 5\}$, and $\alpha_\gamma(v) = \{2, 3, 5, 7\}$, where $u \in \mathcal{C}_\gamma$, while $v \notin \mathcal{C}_\gamma$.

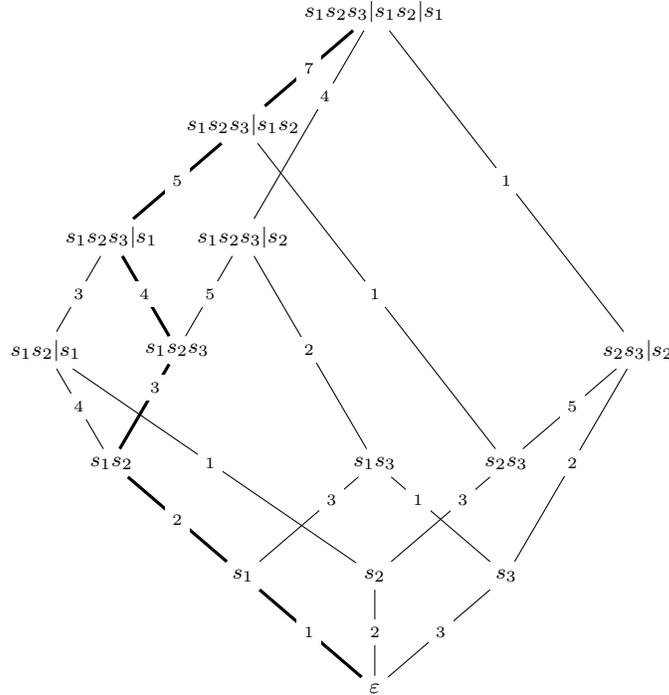


Figure 2: An A_3 -Cambrian lattice with the labeling as defined in (3).

It is not hard to see that an element $w \in W$ lies in C_γ if and only if the following holds: if $i \in \alpha_\gamma(w)$ and $i > n$, then $i - n \in \alpha_\gamma(w)$. In the previous example, we see that $\alpha_\gamma(u)$ contains both 5 and 2, while $\alpha_\gamma(v)$ does not contain $7 - 3 = 4$.

Lemma 3.2. *Let $u, v \in W$ with $u \leq_S v$. Then $\alpha_\gamma(u)$ is a subset of $\alpha_\gamma(v)$.*

Proof. The γ -sorting word of an element $w \in W$ is a reduced word for w . Thus, it follows immediately from the definition of the weak order that any letter appearing in the γ -sorting word of u has to appear also in the γ -sorting word of every element that is greater than w in the weak order. Thus, if $u, v \in C_\gamma$ with $u \leq_\gamma v$, then $\alpha_\gamma(u) \subseteq \alpha_\gamma(v)$. \square

Denote by $\mathcal{E}(C_\gamma)$ the set of covering relations of C_γ , and define an edge-labeling of C_γ by

$$\lambda_\gamma : \mathcal{E}(C_\gamma) \rightarrow \mathbb{N}, \quad (u, v) \mapsto \min\{i \mid i \in \alpha_\gamma(v) \setminus \alpha_\gamma(u)\}. \quad (3)$$

Figures 2 and 3 show the Hasse diagrams of a Cambrian lattice C_γ of the Coxeter groups A_3 and B_3 , respectively, together with the labels defined by the map λ_γ .

Remark 3.3. By definition, the Cambrian semilattice C_γ does not depend on a reduced word for γ , see Remark 2.5. In contrast to that, the labeling defined in (3) *does* depend on the reduced word for γ . That means, if γ_1 and γ_2 are two different reduced words for γ , the Cambrian semilattices C_{γ_1} and C_{γ_2} are isomorphic, but the labeling of C_{γ_1} is structurally different from the labeling of C_{γ_2} . However, we show that the property that λ is an EL-labeling does not depend on the reduced word for γ .

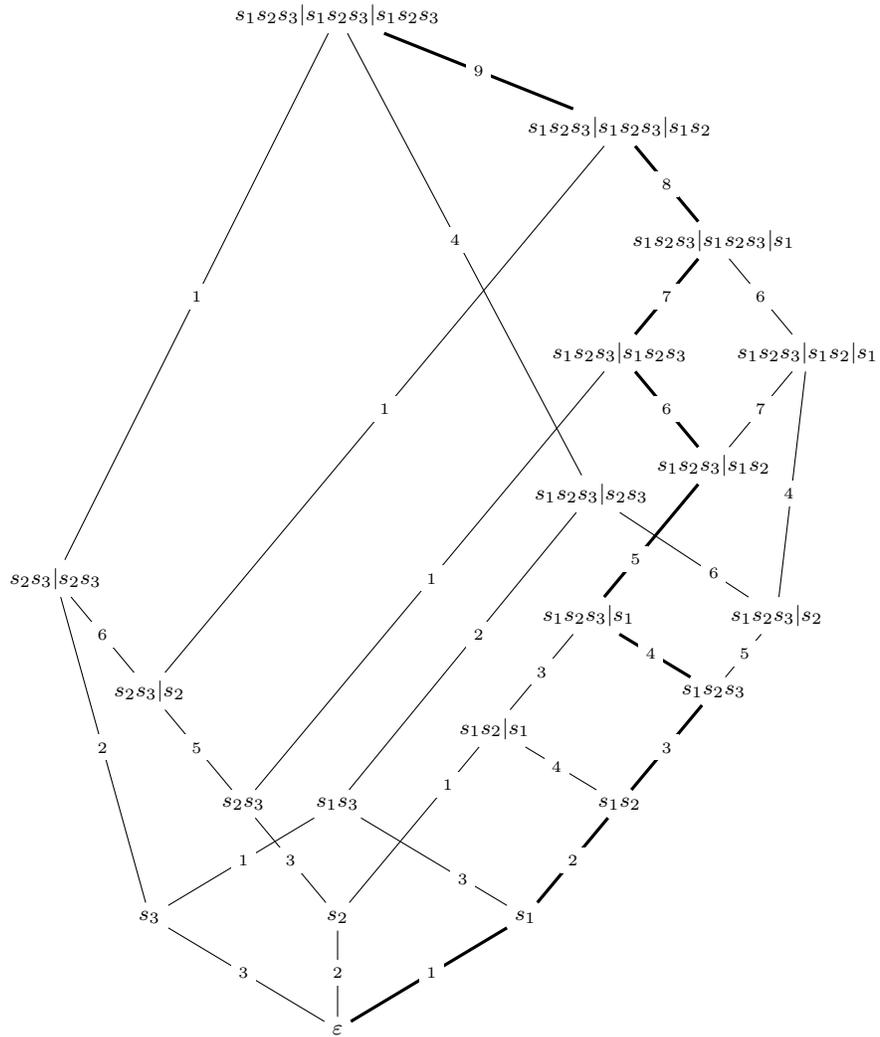


Figure 3: A B_3 -Cambrian lattice, with the labeling as defined in (3).

3.2 Properties of the Labeling

Lemma 3.4. *Let C_γ be a Cambrian semilattice, and let $u, v \in C_\gamma$ such that $u \leq_\gamma v$. Let $i_0 = \min\{i \mid i \in \alpha_\gamma(v) \setminus \alpha_\gamma(u)\}$. Then, the following hold.*

(i) *The label i_0 appears in every maximal chain of the interval $[u, v]_\gamma$.*

(ii) *The labels of a maximal chain in $[u, v]_\gamma$ are distinct.*

Proof. (i) Suppose that this is not the case. Then there exists a maximal chain $u = x_0 \leq_\gamma x_1 \leq_\gamma \dots \leq_\gamma x_{t-1} \leq_\gamma x_t = v$ with $\lambda_\gamma(x_i, x_{i+1}) \neq i_0$ for every $i \in \{0, 1, \dots, t-1\}$. Hence, $i_0 \in \alpha_\gamma(u)$ if and only if $i_0 \in \alpha_\gamma(v)$, which contradicts the definition of i_0 .

(ii) Let $u = x_0 \leq_\gamma x_1 \leq_\gamma \dots \leq_\gamma x_t = v$ be a maximal chain in $[u, v]_\gamma$. Assume that there are $i, j \in \{0, 1, \dots, t-1\}$ with $i < j$ such that $\lambda_\gamma(x_i, x_{i+1}) = k = \lambda_\gamma(x_j, x_{j+1})$. By definition, $k \in \alpha_\gamma(x_{i+1})$, and $k \notin \alpha_\gamma(x_j)$. Since $x_{i+1} \leq_S x_j$, we can conclude from Lemma 3.2 that $\alpha_\gamma(x_{i+1}) \subseteq \alpha_\gamma(x_j)$, which yields a contradiction. \square

The γ -sortable elements of W are defined recursively as described in Proposition 2.4. Before we investigate how our labeling behaves with respect to this recursion, we need to recall one more result. For $s \in S$, let $W_{\geq_S s} = \{w \in W \mid s \leq_S w\}$ and let $W_{\not\leq_S s} = \{w \in W \mid s \not\leq_S w\}$.

Proposition 3.5 ([23, Proposition 2.18]). *Let $w \in W$ and $s \in S$. Then $\ell_S(sw) < \ell_S(w)$ if and only if $s \leq_S w$ if and only if $s \in \text{inv}(w)$. Left multiplication by s is a poset isomorphism from $(W_{\not\leq_S s}, \leq_S)$ to $(W_{\geq_S s}, \leq_S)$. If $w \leq_S w', s \leq_S w'$ and $s \not\leq_S w$, then $w' = sw$.*

Lemma 3.6. *Let W be a Coxeter group and let $\gamma = s_1 s_2 \dots s_n$ be a Coxeter element of W . For $u, v \in C_\gamma$ with $u \leq_\gamma v$, we have*

$$\lambda_\gamma(u, v) = \begin{cases} 1, & \text{if } s_1 \not\leq_S u \text{ and } s_1 \leq_S v, \\ \lambda_{s_1 \gamma s_1}(s_1 u, s_1 v) + 1, & \text{if } s_1 \leq_S u, \\ \lambda_{s_1 \gamma}(u_{(s_1)}, v_{(s_1)}) + k, & \text{if } s_1 \not\leq_S v \text{ and the first position where } u \text{ and } v \\ & \text{differ is in their } k\text{-th block.} \end{cases}$$

Proof. Let first $s_1 \not\leq_S u$ and $s_1 \leq_S v$. By definition of the weak order, s_1 does not occur in the first position of any reduced word for u , in particular it does not occur in the first position of the γ -sorting word of u . Hence, $1 \notin \alpha_\gamma(u)$. On the other hand, s_1 does occur in the first position of the γ -sorting word of v , and hence $1 \in \alpha_\gamma(v)$. By definition this implies $\lambda_\gamma(u, v) = 1$.

Let now $s_1 \leq_S u$. Then, $s_1 \leq_S v$, and with Proposition 2.4, we find that $s_1 u$ and $s_1 v$ are $s_1 \gamma s_1$ -sortable. It follows from Propositions 2.4 and 3.5 and from the definition of the weak order that $s_1 u \leq_{s_1 \gamma s_1} s_1 v$. Say $\lambda_{s_1 \gamma s_1}(s_1 u, s_1 v) = k$. By construction, the $s_1 \gamma s_1$ -sorting word of $s_1 u$ is precisely the subword of u starting at the second position. Thus, the $s_1 \gamma s_1$ -sorting word of $s_1 u$ is the leftmost subword of γ^∞ where the first position is empty,

and likewise for s_1v . If the first position of $(s_1\gamma s_1)^\infty$ where s_1u and s_1v differ is k , then the first position of γ^∞ where u and v differ is $k+1$. Hence, $\lambda_\gamma(u, v) = \lambda_{s_1\gamma s_1}(s_1u, s_1v) + 1$.

Finally, let $s_1 \not\leq_S v$. Then, $s_1 \not\leq_S u$, and with Proposition 2.4, we find that $u = u_{\langle s_1 \rangle}$ and $v = v_{\langle s_1 \rangle}$ are $s_1\gamma$ -sortable elements of the parabolic subgroup $W_{\langle s_1 \rangle}$ of W , and the Cambrian lattice $\mathcal{C}_{s_1\gamma}$ is an order ideal in \mathcal{C}_γ . Say that the first position filled in $v_{\langle s_1 \rangle}$ but not in $u_{\langle s_1 \rangle}$ is in the k -th block of $v_{\langle s_1 \rangle}$. Considering $u_{\langle s_1 \rangle}$ and $v_{\langle s_1 \rangle}$ as subwords of γ^∞ adds the letter s_1 with exponent 0 to each block of $u_{\langle s_1 \rangle}$ and $v_{\langle s_1 \rangle}$. Since the first difference of $u_{\langle s_1 \rangle}$ and $v_{\langle s_1 \rangle}$ is in the k -th block, the first difference of u and v is still in the k -th block, but each block has an additional first letter. Hence $\lambda_\gamma(u, v) = \lambda_{s_1\gamma}(u_{\langle s_1 \rangle}, v_{\langle s_1 \rangle}) + k$. \square

Example 3.7. Let $W = B_3$ generated by $S = \{s_1, s_2, s_3\}$ satisfying $(s_1s_2)^3 = (s_2s_3)^4 = (s_1s_3)^2 = \varepsilon$ and $s_1^2 = s_2^2 = s_3^2 = \varepsilon$, and let $\gamma = s_1s_2s_3$ be a Coxeter element of B_3 . Thus, \mathcal{C}_γ is the lattice depicted in Figure 3.

Consider $u_1 = s_2s_3|s_2s_3$ and $v_1 = s_1s_2s_3|s_1s_2s_3|s_1s_2s_3$. With the definition of our labeling follows $\lambda_\gamma(u_1, v_1) = 1$ immediately.

Let now $u_2 = s_1s_2s_3|s_1s_2$ and $v_2 = s_1s_2s_3|s_1s_2s_3$. Then, $s_1u_2 = s_2s_3s_1|s_2$ and $s_1v_2 = s_2s_3s_1|s_2s_3$ considered as $s_1\gamma s_1$ -sorting words. We have

$$\lambda_{s_1\gamma s_1}(s_1u_2, s_1v_2) = 5, \quad \text{and} \quad \lambda_\gamma(u_2, v_2) = 6.$$

Finally, let $u_3 = s_2s_3|s_2$ and $v_3 = s_2s_3|s_2s_3$. The $(s_1\gamma)^\infty$ -sorting words of $(u_3)_{\langle s_1 \rangle}$ and $(v_3)_{\langle s_1 \rangle}$ written as in (1) are

$$(u_3)_{\langle s_1 \rangle} = s_2^1s_3^1|s_2^1s_3^0, \quad \text{and} \quad (v_3)_{\langle s_1 \rangle} = s_2^1s_3^1|s_2^1s_3^1.$$

The corresponding γ -sorting words of u_3 and v_3 are

$$u_3 = s_1^0s_2^1s_3^1|s_1^0s_2^1s_3^0, \quad \text{and} \quad v_3 = s_1^0s_2^1s_3^1|s_1^0s_2^1s_3^1.$$

Hence, $\lambda_{s_1\gamma}((u_3)_{\langle s_1 \rangle}, (v_3)_{\langle s_1 \rangle}) = 4$ and $\lambda_\gamma(u_3, v_3) = 6$.

3.3 Proof of Theorem 1.1

We will prove Theorem 1.1 by showing that the map λ_γ defined in (3) is an EL-labeling for every closed interval in \mathcal{C}_γ . In particular we show the following.

Theorem 3.8. *Let $u, v \in \mathcal{C}_\gamma$ with $u \leq_\gamma v$. Then the map λ_γ defined in (3) is an EL-labeling for $[u, v]_\gamma$.*

For the proof of Theorem 3.8, we need two more technical results. The first one is [23, Proposition 2.20] which was first proved in [13, Lemmas 4.2 (iii) and 4.5]. The second result uses many of the deep results on Cambrian semilattices developed in [23]. For $w \in W$, we say that $t \in \text{inv}(w)$ is called a *cover reflection* of w if there exists some $s \in S$ with $tw = ws$. We denote by $\text{cov}(w)$ the set of all cover reflections of w .

Proposition 3.9 ([23, Proposition 2.20]). *Let $J \subseteq S$ and $A \subseteq W$ and define $A_J = \{w_J \mid w \in A\}$. If A is nonempty, then $\bigwedge_S(A_J) = (\bigwedge_S A)_J$, and if A has an upper bound, then $\bigvee_S(A_J) = (\bigvee_S A)_J$.*

Lemma 3.10. *Let $\gamma = s_1 s_2 \cdots s_n$, and let $u, v \in C_\gamma$ with $u \leq_\gamma v$. If $s_1 \not\leq_\gamma u$ and $s_1 \leq_\gamma v$, then the join $s_1 \vee_\gamma u$ covers u in C_γ .*

Proof. First of all, since $s_1 \leq_\gamma v$ and $u \leq_\gamma v$, we conclude from Theorem 2.6 that $s_1 \vee_\gamma u$ exists, and we set $z = s_1 \vee_\gamma u$. Now, we observe that if $w, w' \in C_\gamma$, then it follows from Theorem 2.7 that

$$w \leq_S w' \quad \text{implies} \quad \pi_\downarrow^\gamma(w) \leq_\gamma \pi_\downarrow^\gamma(w') \quad \text{implies} \quad w \leq_\gamma w'. \quad (4)$$

By assumption, we have $s_1 \not\leq_\gamma u$, and by contraposition follows from (4) that $s_1 \not\leq_S u$. Hence, with (2) follows that $u = \pi_\downarrow^\gamma(u) = \pi_\downarrow^{s_1 \gamma}(u_{\langle s_1 \rangle}) \in W_{\langle s_1 \rangle}$, and Proposition 2.4 implies $u = u_{\langle s_1 \rangle} \in W_{\langle s_1 \rangle}$.

Since $u <_\gamma z$, there exists some $u' \in C_\gamma$ with $u \leq_\gamma u' <_\gamma z$. If $s_1 \leq_\gamma u'$, then u' is an upper bound for s_1 and u which contradicts $u' <_\gamma z$. Thus, we have $s_1 \not\leq_\gamma u'$, which with (4) implies $s_1 \not\leq_S u'$ again, and it follows from Proposition 2.4 that $u' = u'_{\langle s_1 \rangle} \in W_{\langle s_1 \rangle}$. Since C_γ is a sub-semilattice of the weak-order semilattice, the relation $u' <_\gamma z$ implies $u' <_S z$, and we obtain $u'_{\langle s_1 \rangle} \leq_S z_{\langle s_1 \rangle} = (s_1 \vee_\gamma u)_{\langle s_1 \rangle}$. In view of Proposition 3.9, this implies $u'_{\langle s_1 \rangle} \leq_S (s_1)_{\langle s_1 \rangle} \vee_\gamma u_{\langle s_1 \rangle}$. However, since $(s_1)_{\langle s_1 \rangle} = \varepsilon$, and since $u = u_{\langle s_1 \rangle}$ and $u' = u'_{\langle s_1 \rangle}$, we obtain $u' \leq_S u$. This implies $u = u'$ and thus the result. \square

Proof of Theorem 3.8. Let $[u, v]_\gamma$ be a closed interval of C_γ . Since the weak order on W is finitary, it follows that $[u, v]_\gamma$ is a finite lattice. We show that there exists a unique maximal rising chain which is the lexicographically first among all maximal chains in this interval.

We proceed by induction on length and rank, using the recursive structure of γ -sortable elements, see Proposition 2.4. We assume that $\ell_S(v) \geq 3$, and that W is a Coxeter group of rank ≥ 2 , since the result is trivial otherwise. Say that W is of rank n , and say that $\ell_S(v) = k$. Suppose that the induction hypothesis is true for all parabolic subgroups of W having rank $< n$ and suppose that for every closed interval $[u', v']_\gamma$ of C_γ with $\ell_S(v') < k$, there exists a unique rising maximal chain from u' to v' which is lexicographically first among all maximal chains in $[u', v']_\gamma$. We show that there is a unique rising maximal chain in the interval $[u, v]_\gamma$ which is lexicographically first among all maximal chains in $[u, v]_\gamma$. Let $\gamma = s_1 s_2 \cdots s_n$. We distinguish two cases: (1) $s_1 \not\leq_\gamma v$ and (2) $s_1 \leq_\gamma v$.

(1) Since $s_1 \not\leq_\gamma v$, it follows that no element of $[u, v]_\gamma$ contains the letter s_1 in its γ -sorting word. We consider the parabolic Coxeter group $W_{\langle s_1 \rangle}$ (generated by $S \setminus \{s_1\}$) and the Coxeter element $s_1 \gamma$. It follows from Proposition 2.4 that the interval $[u, v]_\gamma$ is isomorphic to the interval $[u_{\langle s_1 \rangle}, v_{\langle s_1 \rangle}]_{s_1 \gamma}$ in $W_{\langle s_1 \rangle}$. Since the rank of $W_{\langle s_1 \rangle}$ is $n - 1 < n$, by induction there exists a unique maximal rising chain $c' : u_{\langle s_1 \rangle} = (x_0)_{\langle s_1 \rangle} \leq_{s_1 \gamma} (x_1)_{\langle s_1 \rangle} \leq_{s_1 \gamma} \cdots \leq_{s_1 \gamma} (x_t)_{\langle s_1 \rangle} = v_{\langle s_1 \rangle}$ which is lexicographically first among all maximal chains in $[u_{\langle s_1 \rangle}, v_{\langle s_1 \rangle}]_{s_1 \gamma}$. Let $(x_{j_a})_{\langle s_1 \rangle} \leq_{s_1 \gamma} (x_{j_a+1})_{\langle s_1 \rangle}$ and $(x_{j_b})_{\langle s_1 \rangle} \leq_{s_1 \gamma} (x_{j_b+1})_{\langle s_1 \rangle}$ be two covering relations in c' with $j_a + 1 \leq j_b$. Say that the first block where $(x_{j_a})_{\langle s_1 \rangle}$ and $(x_{j_a+1})_{\langle s_1 \rangle}$ differ is the d_a -th block of their $s_1 \gamma$ -sorting word and say that the first block where $(x_{j_b})_{\langle s_1 \rangle}$ and $(x_{j_b+1})_{\langle s_1 \rangle}$ differ is the d_b -th block of their $s_1 \gamma$ -sorting word. Since c' is rising, we conclude that $d_a \leq d_b$, and Lemma 3.6 implies that the corresponding maximal

chain $c : u = x_0 \leq_\gamma x_1 \leq_\gamma \cdots \leq_\gamma x_t = v$ in $[u, v]_\gamma$ is rising. Similarly, it follows that c is the unique maximal rising chain and that it is lexicographically first among all maximal chains in $[u, v]_\gamma$.

(2a) Suppose first that $s_1 \leq_\gamma u$ as well. Then, s_1 is the first letter in the γ -sorting word of every element in $[u, v]_\gamma$. It follows from Propositions 2.4 and 3.5 that the interval $[u, v]_\gamma$ is isomorphic to the interval $[s_1u, s_1v]_{s_1\gamma s_1}$. Moreover, Lemma 3.6 implies that for a covering relation $x \leq_\gamma y$ in $[u, v]_\gamma$ we have $\lambda_\gamma(x, y) = \lambda_{s_1\gamma s_1}(s_1x, s_1y) + 1$. Say that $c' : s_1u = s_1x_0 \leq_{s_1\gamma s_1} s_1x_1 \leq_{s_1\gamma s_1} \cdots \leq_{s_1\gamma s_1} s_1x_t = s_1v$ is the unique rising maximal chain in $[s_1u, s_1v]_{s_1\gamma s_1}$. (This chain exists by induction, since $\ell_S(s_1v) < \ell_S(v)$.) Then, the chain $c : u = x_0 \leq_\gamma x_1 \leq_\gamma \cdots \leq_\gamma x_t = v$ is a maximal chain in $[u, v]_\gamma$ and clearly rising. Now consider some maximal chain $\bar{c} : u = \bar{x}_0 \leq_\gamma \bar{x}_1 \leq_\gamma \cdots \leq_\gamma \bar{x}_t = v$ in $[u, v]_\gamma$ with $\bar{c} \neq c$, and let $\bar{c}' : s_1u = s_1\bar{x}_0 \leq_{s_1\gamma s_1} s_1\bar{x}_1 \leq_{s_1\gamma s_1} \cdots \leq_{s_1\gamma s_1} s_1\bar{x}_t = s_1v$ be the corresponding chain in $[s_1u, s_1v]_{s_1\gamma s_1}$. Then, it follows from Propositions 2.4 and 3.5 that $\bar{c}' \neq c'$, and the induction hypothesis implies that \bar{c}' is not rising and that \bar{c}' is lexicographically larger than c' . Lemma 3.6 implies now the same for \bar{c} . Thus, c is the unique rising chain in $[u, v]_\gamma$ and every other maximal chain in $[u, v]_\gamma$ is lexicographically larger than c .

(2b) Suppose now that $s_1 \not\leq_\gamma u$. Since $s_1 \leq_\gamma v$ and $u \leq_\gamma v$ the join $u_1 = s_1 \vee_\gamma u$ exists and lies in $[u, v]_\gamma$. Lemma 3.10 implies that $u \leq_\gamma u_1$. Consider the interval $[u_1, v]_\gamma$. Then $s_1 \leq_\gamma u_1$ and analogously to (2a) we can find a unique maximal rising chain $c' : u_1 = x_1 \leq_\gamma x_2 \leq_\gamma \cdots \leq_\gamma x_t = v$ in $[u_1, v]_\gamma$ which is lexicographically first. Moreover, $\min\{i \mid i \in \alpha_\gamma(v) \setminus \alpha_\gamma(u_1)\} > 1$, since $s_1 \leq_\gamma u_1 \leq_\gamma v$. By definition of our labeling, the label 1 cannot appear as a label in any chain in the interval $[u_1, v]_\gamma$. On the other hand, it follows from Lemma 3.6 that $\lambda_\gamma(u, u_1) = 1$. Thus, the chain $c : u = x_0 \leq_\gamma x_1 \leq_\gamma x_2 \leq_\gamma \cdots \leq_\gamma x_t = v$ is maximal and rising in $[u, v]_\gamma$. Suppose that there is another element u' that covers u in $[u, v]_\gamma$ such that $\lambda_\gamma(u, u') = 1$. Then, by definition of λ_γ , it follows that s_1 appears in the γ -sorting word of u' . In particular, since s_1 is initial in γ , we deduce that $s_1 \leq_\gamma u'$. Therefore u' is above both s_1 and u in \mathcal{C}_γ . By the uniqueness of joins and the definition of u_1 it follows that $u_1 = u'$. Thus, c is the lexicographically smallest maximal chain in $[u, v]_\gamma$. Finally, Lemma 3.4 implies that c is the unique maximal rising chain. \square

Proof of Theorem 1.1. This follows by definition from Theorem 3.8. \square

Remark 3.11. In the case where W is finite and crystallographic, Ingalls and Thomas have shown in [12, Theorem 4.17] that \mathcal{C}_γ is trim. Trimness is a lattice property that generalizes distributivity to ungraded lattices. Then, by definition of trimness, it follows that \mathcal{C}_γ is left-modular, meaning that there exists a maximal chain $c : x_1 \leq_\gamma x_2 \leq_\gamma \cdots \leq_\gamma x_n$ satisfying $(y \vee_\gamma x_i) \wedge_\gamma z = y \vee_\gamma (x_i \wedge_\gamma z)$, for all $y <_\gamma z$ and $i \in \{1, 2, \dots, n\}$. According to [14], this property yields another EL-labeling of \mathcal{C}_γ , defined by

$$\xi(y, z) = \min\{i \mid y \vee_\gamma x_i \wedge_\gamma z = z\},$$

for all $y, z \in L$ with $y <_\gamma z$. It is not hard to show that this labeling is structurally different from our labeling. (The difference between the two labelings can already be observed in the Cambrian lattice depicted in Figure 2.)

Remark 3.12. In the case where W is finite and $\gamma = s_1 s_2 \cdots s_n$, [20, Remark 2.1] states that the γ -sortable elements constitute a spanning tree of the Hasse diagram of \mathcal{C}_γ , which is rooted at the identity. The edges of this spanning tree correspond to covering relations $u \lessdot_\gamma v$ in \mathcal{C}_γ such that the γ -sorting word of u is a prefix of the γ -sorting word of v . This spanning tree is related to the labeling λ_γ in the following way: let $w \in W$, with $\ell_S(w) = k$, and let $(i_0, i_1, \dots, i_{k-1})$ be the sequence of edge-labels of the unique rising chain in $[\varepsilon, w]_\gamma$. In view of Theorem 3.8, and [20, Remark 2.1], we notice that the unique path from ε to w in the spanning tree of \mathcal{C}_γ corresponds to the unique rising chain in $[\varepsilon, w]_\gamma$. Hence, the γ -sorting word of w is $s_{i_0} s_{i_1} \cdots s_{i_{k-1}}$, where s_{i_j} is the i_j -th letter of γ^∞ , and the length of the rising chain in $[\varepsilon, w]_\gamma$ is precisely $\ell_S(w)$. Moreover, it follows from the proof of Theorem 3.8 that the length of the unique rising chain in an interval $[u, v]_\gamma$ equals $\ell_S(v) - \ell_S(u)$.

In view of Theorem 3.8, we can carry out the same construction even in the case of infinite Coxeter groups.

4 Applications

In [18], Reading investigated, among other things, the topological properties of open intervals in so-called *fan posets*. A fan poset is a certain partial order defined on the maximal cones of a complete fan of regions of a real hyperplane arrangement. For a finite Coxeter group W and a Cambrian congruence θ , the *Cambrian fan* \mathcal{F}_θ is the complete fan induced by certain cones in the Coxeter arrangement \mathcal{A}_W of W . More precisely, each such cone is a union of regions of \mathcal{A}_W which correspond to elements of W lying in the same congruence class of θ . It is the assertion of [18, Theorem 1.1] that a Cambrian lattice of W is the fan poset associated to the corresponding Cambrian fan. The following theorem is a concatenation of [18, Theorem 1.1] and [18, Propositions 5.6 and 5.7]. In fact, Propositions 5.6 and 5.7 in [18] imply this result for a much larger class of fan posets.

Theorem 4.1. *Let W be a finite Coxeter group and let $\gamma \in W$ be a Coxeter element. Every open interval in the Cambrian lattice \mathcal{C}_γ is either contractible or spherical.*

It is well-known that the reduced Euler characteristic of the order complex of an open interval (x, y) in a poset equals $\mu(x, y)$, see for instance [25, Proposition 3.8.6]. Hence, it follows immediately from Theorem 4.1 that for γ -sortable elements x and y in a finite Coxeter group W satisfying $x \leq_\gamma y$, we have $|\mu(x, y)| \leq 1$, as was already remarked in [19, pp. 4-5]. In light of Proposition 2.1 and Theorem 3.8, we can extend this statement to compute the Möbius function of closed intervals in the Cambrian semilattice \mathcal{C}_γ , by counting the falling maximal chains with respect to the labeling defined in (3), as our next theorem shows.

Theorem 4.2. *Let W be a (possibly infinite) Coxeter group and $\gamma \in W$ a Coxeter element. For $u, v \in \mathcal{C}_\gamma$ with $u \leq_\gamma v$, we have $|\mu(u, v)| \leq 1$.*

Proof. In view of Proposition 2.1 it is enough to show that the interval $[u, v]_\gamma$ has at most one maximal falling chain. We use similar arguments as in the proof of Theorem 3.8 and

proceed by induction on length and rank. Again, we may assume that $\ell_S(v) = k \geq 3$ and that W is a Coxeter group of rank $n \geq 2$, since the result is trivial otherwise. Suppose that the induction hypothesis is true for all parabolic subgroups of W with rank $< n$ and suppose that for every closed interval $[u', v']_\gamma$ of \mathcal{C}_γ with $\ell_S(v') < k$, there exists at most one falling maximal chain. We will show that there is at most one maximal falling chain in the interval $[u, v]_\gamma$ as well. Let $\gamma = s_1 s_2 \cdots s_n$. We distinguish two cases: (1) $s_1 \not\leq_\gamma v$ and (2) $s_1 \leq_\gamma v$.

(1) The result follows directly by induction on the rank of W by following the steps of case (1) in the proof of Theorem 3.8.

(2a) Suppose in addition that $s_1 \leq_\gamma u$. The result follows directly by induction on the length of v by following the steps of case (2a) in the proof of Theorem 3.8.

(2b) Suppose now that $s_1 \not\leq_\gamma u$. Thus, it follows from Lemma 3.4 that the label 1 occurs in every maximal chain from u to v , and that a maximal chain $u = x_0 \leq_\gamma x_1 \leq_\gamma \cdots \leq_\gamma x_{t-1} \leq x_t = v$ of $[u, v]_\gamma$ can be falling only if $\lambda_\gamma(x_{t-1}, v) = 1$. Hence, if there is no element $v_1 \in (u, v)_\gamma$, with $v_1 \leq v$ satisfying $\lambda_\gamma(v_1, v) = 1$, then the interval $[u, v]_\gamma$ has no maximal falling chain, which means that $\mu(u, v) = 0$. Otherwise, consider the interval $[u, v_1]_\gamma$. By the choice of v_1 , it follows that every maximal falling chain in $[u, v_1]_\gamma$ can be extended to a maximal falling chain in the interval $[u, v]_\gamma$. Conversely, every maximal falling chain in $[u, v]_\gamma$ can be restricted to a maximal falling chain in $[u, v_1]_\gamma$. Therefore, since $\ell_S(v_1) < \ell_S(v)$, we deduce from the induction hypothesis that the interval $[u, v_1]_\gamma$ has at most one maximal falling chain. Thus $|\mu(u, v)| \leq 1$. \square

In addition Propositions 5.6 and 5.7 in [18] characterize the open intervals in a (finite) Cambrian lattice which are contractible, and those which are spherical in the following way: an interval $[u, v]_\gamma$ in \mathcal{C}_γ is called *nuclear* if the join of the upper covers of u (in $[u, v]_\gamma$) is precisely v . Reading showed that the nuclear intervals are precisely the spherical intervals. With the help of our labeling, we can generalize this characterization to infinite Coxeter groups.

Theorem 4.3. *Let $u, v \in C_\gamma$ with $u \leq_\gamma v$ and let k denote the number of atoms of the interval $[u, v]_\gamma$. Then, $\mu(u, v) = (-1)^k$ if and only if $[u, v]_\gamma$ is nuclear.*

For the proof of Theorem 4.3, we need the following lemma.

Lemma 4.4. *Let $\gamma = s_1 s_2 \cdots s_n$, and let $u, v \in C_\gamma$ with $u \leq_\gamma v$. Suppose further that $s_1 \not\leq_\gamma u$, while $s_1 \leq_\gamma v$. Then the following are equivalent:*

1. *The interval $[u, v]_\gamma$ is nuclear.*
2. *There exists an element $v' \in [u, v]_\gamma$ satisfying $s_1 \not\leq_\gamma v' \leq_\gamma v$, and the interval $[u, v']_\gamma$ is nuclear.*

Proof. Let $A = \{w \in C_\gamma \mid u \leq_\gamma w \leq_\gamma v\}$ be the set of atoms of the interval $[u, v]_\gamma$. Since $s_1 \leq_\gamma v$ and $u \leq_\gamma v$, we conclude from Theorem 2.6 that the join $s_1 \vee_\gamma u$ exists, and we set $z = s_1 \vee_\gamma u$. It follows from Lemma 3.10 that $u \leq_\gamma z$, and hence $z \in A$. We set $A_z = A \setminus \{z\}$ and remark that if $w \in A_z$, then $s_1 \not\leq_\gamma w$. (Indeed, suppose that there exists

some $z' \in A_z$ with $s_1 \leq_\gamma z'$. Since $u <_\gamma z'$, this implies $s_1 \vee_\gamma u \leq_\gamma z'$, and hence $z \leq_\gamma z'$. Since z and z' both cover u , this implies $z = z'$, which contradicts $z \notin A_z$.) In particular we have $A_z \subseteq W_{\langle s_1 \rangle}$. (Indeed, suppose that there is some $z' \in A_z$ with $\ell_S(z') = k$ and $z' \notin w_{\langle s_1 \rangle}$. This means that the γ -sorting word of z' contains the letter s_1 at least once. Since $z' \in C_\gamma$ this means that s_1 must occur in the first position of the γ -sorting word of z' . Then $\ell_S(z') = k = (k-1) + 1 = \ell_S(s_1 z') + \ell_S(s_1)$ which implies $s_1 \leq_S z'$, and with (4) follows $s_1 \leq_\gamma z'$ which is a contradiction.)

(1) \Rightarrow (2) Suppose that $[u, v]_\gamma$ is nuclear and let $v' = \bigvee_\gamma A_z$. Again, Theorem 2.6 ensures that v' exists and that it satisfies $u \leq_\gamma v' \leq_\gamma v$. Since $A_z \subseteq W_{\langle s_1 \rangle}$, it follows from Proposition 3.9 that $v' = \bigvee_\gamma A_z \in W_{\langle s_1 \rangle}$ which means that $s_1 \not\leq_\gamma v'$, and A_z is thus the set of atoms of the interval $[u, v']_\gamma$. Hence, $[u, v']_\gamma$ is nuclear. It remains to show that $v' <_\gamma v$. It follows from $u \leq_\gamma v'$ and the associativity of \vee_γ that

$$v = \bigvee_\gamma A = z \vee_\gamma \left(\bigvee_\gamma A_z \right) = z \vee_\gamma v' = (s_1 \vee_\gamma u) \vee_\gamma v' = s_1 \vee_\gamma (u \vee_\gamma v') = s_1 \vee_\gamma v'.$$

From above, we know that $s_1 \not\leq_\gamma v'$ and we can apply Lemma 3.10 which implies immediately that $v' <_\gamma s_1 \vee_\gamma v' = v$.

(2) \Rightarrow (1) Suppose now that there exists an element $v' \in [u, v]_\gamma$ satisfying $s_1 \not\leq_\gamma v' <_\gamma v$, and suppose that the interval $[u, v']_\gamma$ is nuclear. Let A' denote the set of atoms of $[u, v']_\gamma$. Since $s_1 \not\leq_\gamma v'$ and $s_1 \leq_\gamma z$, it follows that $z \notin A'$, thus $A' \subseteq A_z$. Furthermore, from $s_1 \leq_\gamma v$, from $v' <_\gamma v$, and from Lemma 3.10, we deduce that $s_1 \vee_\gamma v' = v$. Now we have

$$z \vee_\gamma v' = (s_1 \vee_\gamma u) \vee_\gamma v' = s_1 \vee_\gamma (u \vee_\gamma v') = s_1 \vee_\gamma v' = v,$$

since $u \leq_\gamma v'$. Thus, we can write $v = \bigvee_\gamma (A' \cup \{z\})$. Finally, we show that $v = \bigvee_\gamma A$. Let $z' \in A \setminus A'$. Since $z' \leq_\gamma v$, it follows that

$$\bigvee_\gamma (A' \cup \{z, z'\}) = \bigvee_\gamma (A' \cup \{z\}) \vee_\gamma z' = v \vee_\gamma z' = v,$$

and hence $v = \bigvee_\gamma A$. This implies that $[u, v]_\gamma$ is nuclear. □

We remark that under the hypothesis of Lemma 4.4, the element $v' = \bigvee A_z$ constructed in the part (1) \Rightarrow (2) of the proof is the unique element in $[u, v]_\gamma$ satisfying condition (2). The uniqueness of v' is a consequence of the uniqueness of the join $\bigvee_\gamma A_z$.

Proof of Theorem 4.3. In view of Proposition 2.1, we need to show that $[u, v]_\gamma$ has a falling chain if and only if $[u, v]_\gamma$ is nuclear. We use similar arguments as in the proof of Theorem 3.8 and proceed by induction on length and rank. Again we may assume that $\ell_S(v) = k \geq 3$ and that W is a Coxeter group of rank $n \geq 2$, since the result is trivial otherwise. Suppose that the induction hypothesis is true for all parabolic subgroups of W with rank $< n$ and suppose that for every closed interval $[u', v']_\gamma$ of \mathcal{C}_γ with $\ell_S(v') < k$ there exists a falling maximal chain if and only if $[u', v']_\gamma$ is nuclear. Let $\gamma = s_1 s_2 \cdots s_n$. We distinguish two cases: (1) $s_1 \not\leq_\gamma v$ and (2) $s_1 \leq_\gamma v$.

(1) The result follows directly by induction on the rank of W by following the steps of case (1) in the proof of Theorem 4.2.

(2a) Suppose in addition that $s_1 \leq_\gamma u$. The result follows directly by induction on the length of v by following the steps of case (2a) in the proof of Theorem 4.2.

(2b) Suppose now that $s_1 \not\leq_\gamma u$. If $[u, v]_\gamma$ is nuclear, then Lemma 4.4 implies that there exists a unique element $v' \in \mathcal{C}_\gamma$ with $u \leq_\gamma v' \leq_\gamma v$ such that $[u, v']_\gamma$ is nuclear, and $s_1 \not\leq_\gamma v'$. Thus, we can apply induction on the rank of W and obtain a maximal falling chain $c' : u = x_0 \leq_\gamma x_1 \leq_\gamma \cdots \leq_\gamma x_{t-1} = v'$. Lemma 3.4 implies that $1 \notin \lambda_\gamma(c')$, and Lemma 3.6 implies that $\lambda_\gamma(v', v) = 1$. Thus, the chain $c : u = x_0 \leq_\gamma x_1 \leq_\gamma \cdots \leq_\gamma x_{t-1} \leq_\gamma x_t = v$ is a falling maximal chain in $[u, v]_\gamma$, and Theorem 4.2 implies its uniqueness.

Conversely, suppose that there exists a maximal falling chain $c : u = x_0 \leq_\gamma x_1 \leq_\gamma \cdots \leq_\gamma x_t = v$ in $[u, v]_\gamma$. In view of Lemma 3.4, we notice that $\lambda_\gamma(x_{t-1}, v) = 1$, which implies $s_1 \not\leq_\gamma x_{t-1}$. Clearly $\ell_S(x_{t-1}) < k$ and the chain $c' : u = x_0 \leq_\gamma x_1 \leq_\gamma \cdots \leq_\gamma x_{t-1}$ is falling, thus by induction we can conclude that the interval $[u, x_{t-1}]_\gamma$ is nuclear. Since $s_1 \not\leq_\gamma x_{t-1} \leq_\gamma v$, it follows from Lemma 4.4 that $[u, v]_\gamma$ is nuclear. This completes the proof of the theorem. \square

Proof of Theorem 1.2. Theorem 1.1 implies that every closed interval $[u, v]_\gamma$ of \mathcal{C}_γ is EL-shellable. Theorem 5.9 in [5] states that the dimension of the i -th homology group of the order complex of $(u, v)_\gamma$ corresponds to the number of falling chains in $[u, v]_\gamma$ having length $i + 2$. Theorem 4.2 implies that there is at most one falling chain in $[u, v]_\gamma$. Hence, either all homology groups of the order complex of $(u, v)_\gamma$ have dimension 0 (then, $(u, v)_\gamma$ is contractible) or there exists exactly one homology group of dimension 1 (then, $(u, v)_\gamma$ is spherical). Finally, the characterization of the spherical intervals is an immediate consequence of Theorem 4.3. \square

Remark 4.5. In the case of finite Coxeter groups, the statements of Theorems 1.1 and 1.2 can be generalized straightforwardly to the increasing flip order of subword complexes for so-called *realizing words*, as was pointed out to us by Stump (private conversation). In [16, Section 5.3], Pilaud and Stump define an acyclic, directed, edge-labeled graph on the facets of the subword complex, the so-called *increasing flip graph*. The transitive closure of this graph is then a partial order, the *increasing flip order*. In the case of realizing words, the Hasse diagram of the increasing flip order coincides with the increasing flip graph which then yields two natural edge-labelings of this poset. One can show that these labelings are indeed EL-labelings and that every interval has at most one falling chain with respect to either of these labelings. This has recently been done in [17].

It is the statement of [16, Corollary 6.31] that the Cambrian lattices of finite Coxeter groups correspond to the increasing flip order of special subword complexes. In addition, the construction of [16] as briefly described in the previous paragraph provides a nice geometric description of the statements of Theorems 1.1 and 1.2. We remark that both EL-labelings of Pilaud and Stump are structurally different from our labeling. This is pointed out in [17, Example 5.12] for the so-called positive labeling, and it can be seen for the so-called negative labeling in the Cambrian lattice of the symmetric group \mathfrak{S}_4 with respect to the Coxeter element $\gamma = (2\ 3)(1\ 2)(3\ 4)$.

We conclude this section with a short example of an infinite Coxeter group.

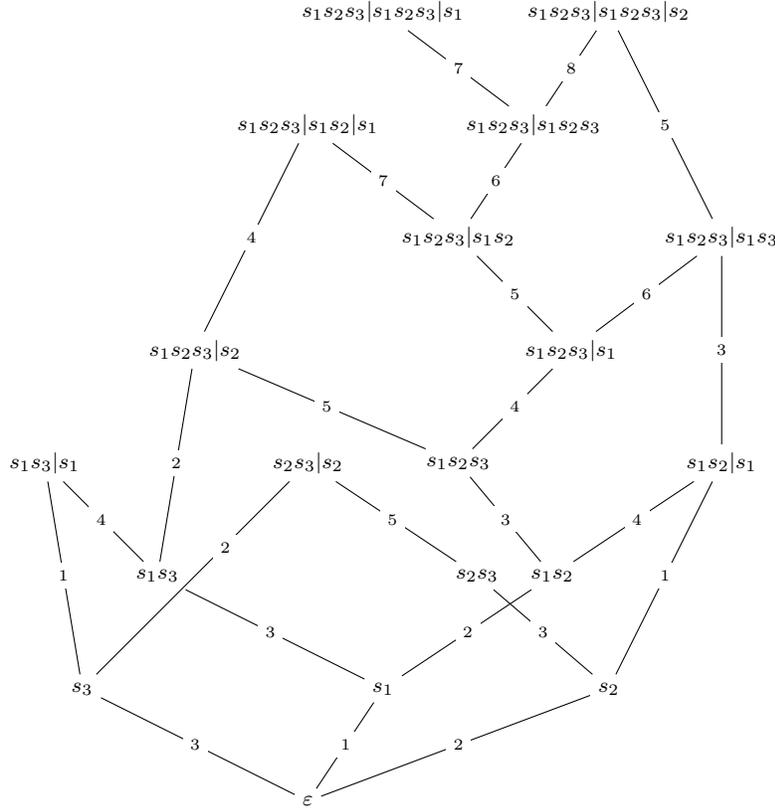


Figure 4: The first seven ranks of an \tilde{A}_2 -Cambrian semilattice, with the labeling as defined in (3).

Example 4.6. Consider the affine Coxeter group \tilde{A}_2 , which is generated by the set $\{s_1, s_2, s_3\}$ satisfying $(s_1s_2)^3 = (s_1s_3)^3 = (s_2s_3)^3 = \varepsilon$, as well as $s_1^2 = s_2^2 = s_3^2 = \varepsilon$. Consider the Coxeter element $\gamma = s_1s_2s_3$. Figure 4 shows the sub-semilattice of the Cambrian semilattice \mathcal{C}_γ consisting of all γ -sortable elements of \tilde{A}_2 of length ≤ 7 . We encourage the reader to verify Theorem 3.8 and Theorem 4.2.

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