Note on upper density of quasi-random hypergraphs

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Abstract

In 1964, Erdős proved that for any \( \alpha > 0 \), an \( l \)-uniform hypergraph \( G \) with \( n \geq n_0(\alpha,l) \) vertices and \( \alpha(\binom{n}{l}) \) edges contains a large complete \( l \)-equipartite subgraph. This implies that any sufficiently large \( G \) with density \( \alpha > 0 \) contains a large subgraph with density at least \( l!/l^l \).

In this note we study a similar problem for \( l \)-uniform hypergraphs \( Q \) with a weak quasi-random property (i.e. with edges uniformly distributed over the sufficiently large subsets of vertices). We prove that any sufficiently large quasi-random \( l \)-uniform hypergraph \( Q \) with density \( \alpha > 0 \) contains a large subgraph with density at least \( \frac{(l-1)!}{l^{l-1}} \).

In particular, for \( l = 3 \), any sufficiently large such \( Q \) contains a large subgraph with density at least \( \frac{1}{4} \) which is the best possible lower bound.

We define jumps for quasi-random sequences of \( l \)-graphs and our result implies that every number between 0 and \( \frac{(l-1)!}{l^{l-1}} \) is a jump for quasi-random \( l \)-graphs. For \( l = 3 \) this interval can be improved based on a recent result of Glebov, Král’ and Volec. We prove that every number between \([0, 0.3192)\) is a jump for quasi-random 3-graphs.

Keywords: hypergraphs; quasi-random; density; jumps

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1 Introduction

For fixed $l \geq 2$, an $l$-graph $G = (V, E)$ is an $l$-uniform hypergraph with vertex set $V$ and edge set $E \subseteq \binom{V}{l}$, or a subset of the $l$-tuples of $V$. For $K \subseteq V$ and $|K| = k$, we denote the $l$-subgraph of $G$ induced by $K$ as $G[K] = (K, E \cap \binom{K}{l})$. The density of such an $l$-graph $G$ is defined by $d(G) = |E|/\binom{|V|}{l}$.

Let $G = \{G_n\}_{n=1}^{\infty}$ be a sequence of $l$-graphs with $G_n = (V_n, E_n)$ such that $|V_n| \to \infty$ as $n \to \infty$. We define the density $d(G)$ of a sequence $G$ as $d(G) = \lim_{n \to \infty} d(G_n)$ (if the limit exists). We will consider only graph sequences for which the limit $d(G_n)$ exists as $n \to \infty$.

Setting

$$\sigma_k(G) = \max_n \max_{K \in \binom{V_n}{k}} d(G_n[K]),$$

a simple averaging argument yields that $\{\sigma_k(G)\}_{k=2}^{\infty}$ is a non-increasing non-negative sequence and so the limit $\bar{d}(G) = \lim_{k \to \infty} \sigma_k(G)$ exists. We call this limit $\bar{d}(G)$ the upper density of $G$.

The result we present in this note are motivated by a theorem of Erdős from [2]:

**Theorem 1.1.** For every $\epsilon > 0, l \geq 2$ and $t$, there exists $n$ such that any $l$-graph with $n$ vertices and $en^l$ edges contains a complete $l$-partite $l$-graph $K^{(l)}_{t, t, \ldots, t}$. Consequently, for any sequence $G$ of $l$-graphs with $d(G) > 0$, $\bar{d}(G) \geq t!/t!$.

In this note we are interested in a similar problem if we restrict to quasi-random $l$-graphs.

**Definition 1.2.** Given $\epsilon > 0$ and $\alpha > 0$, we define an $(\alpha, \epsilon)$-quasi-random hypergraph to be an $l$-graph $Q = (V, E)$ with the property that for all $W \subseteq V$, $d(Q[W]) = \alpha(1 \pm \epsilon)$ for $|W| \geq \epsilon n$ where $|V| = n$. A sequence $Q = \{Q_n\}_{n=1}^{\infty}$ of $(\alpha, \epsilon_n)$-quasi-random $l$-graphs is quasi-random if $\epsilon_n$ is decreasing and $\epsilon_n \to 0$ as $n \to \infty$.

Note that for $l = 2$ quasi-random graphs must contain arbitrarily large cliques as $\epsilon_n \to 0$ and thus any quasi-random sequence of 2-graphs with $d(Q) > 0$ necessarily satisfies $d(Q) = 1$. In this note we prove a related result for $l \geq 3$:

**Theorem 1.3.** For a sequence $Q$ of quasi-random $l$-graphs with $d(Q) > 0$,

(i) $\bar{d}(Q) \geq \frac{(l-1)!}{l-1}$ and

(ii) when $l = 3$ there exists a quasi-random sequence of 3-graphs with $\bar{d}(Q) = \frac{1}{4}$.

For $l > 3$, however, we do not know if $\bar{d}(Q) \geq \frac{(l-1)!}{l-1}$ could not be replaced by a larger number. Our results for $l = 3$ are shown in the Section 2.1 and a similar construction may be applied to generalize the result for all $l$-graphs, proving Theorem 1.3(i).

A number $\alpha$ is a jump if there exists a constant $c = c(\alpha)$ such that given any sequence of $l$-graphs $G = \{G_n\}_{n=1}^{\infty}$ if $\bar{d}(G) > \alpha$, then $\bar{d}(G) \geq \alpha + c$. It follows from the Erdős-Stone Theorem that all non-negative numbers less than 1 are jumps for graphs and it follows...
from Theorem 1.1 that all non-negative numbers less than \( \frac{\alpha}{l} \) are jumps for \( l \)-graphs. Erdős conjectured that, analogous to graphs, all numbers less than 1 are jumps for \( l \)-graphs as well. This conjecture was disproved by Frankl and Rödl in [5] who showed that there are an infinite number of non-jumps for all \( l \geq 3 \). However, these non-jumps were found to occur at relatively large densities. While the smallest case of determining whether \( \frac{\alpha}{l} \) is a jump is still open and likely a difficult problem, our result shows that under the further assumption of quasi-randomness that \( \frac{\alpha}{l} \) is indeed a jump for all \( l \geq 3 \).

We extend the concept of jumps to sequences of quasi-random \( l \)-graphs:

**Definition 1.4.** A number \( \alpha \) is a jump for quasi-random \( l \)-graphs if there exists a constant \( c = c(\alpha) \) such that given any sequence of quasi-random \( l \)-graphs \( \mathcal{G} = \{G_n\}_{n=1}^{\infty} \) if \( \overline{d}(\mathcal{G}) > \alpha \), then \( \overline{d}(\mathcal{G}) \geq \alpha + c \).

Theorem 1.3(i) implies that every number between 0 and \( \frac{(l-1)!}{l^l} \) is a jump for quasi-random \( l \)-graphs. Further we will show that for \( l = 3 \) this interval can be improved from \([0, \frac{1}{4}] \) to \([0, 0.3192) \) given the following question of Erdős [3] is answered positively:

**Question 1.5.** Let \( c > 0 \) and \( \mathcal{Q} = \{Q_n\}_{n=1}^{\infty} \) be a quasi-random sequence of 3-graphs. If \( d(\mathcal{Q}) = \frac{1}{4} + c \), then does each \( Q_n \) contain \( K_{4(3)}^4 - e \) as \( n \to \infty \)?

More formally, we prove in Section 3:

**Theorem 1.6.** A positive answer to Question 1.5 implies that any quasi-random sequence \( \mathcal{Q} \) with \( d(\mathcal{Q}) > \frac{1}{4} \) satisfies \( \overline{d}(\mathcal{Q}) > 0.3192 \).

Very recently, Glebov, Král’ and Volec in [6] proved Question 1.5 in the positive using Razborov’s flag-algebra method [10]. This result confirms our assertion in Theorem 1.6.

We include our remarks and questions for future study for quasi-random \( l \)-graphs with \( l > 3 \) and other possibilities for jumps for quasi-random 3-graphs in Section 4.

## 2 Proof of Theorem 1.3

### 2.1 The lower bound

Our proof is based on the following lemma proved in [1] and [9]:

**Lemma 2.1.** For all \( \alpha > 0 \) and \( \epsilon > 0 \), there exists \( \delta > 0 \), \( m > 0 \) and \( n_0 > 0 \) such that if \( Q = (V, E) \) is an \((\alpha, \delta)\)-quasi-random \( l \)-graph with \( |V| = n \geq n_0 \) vertices then \( Q[M] \) is \((\alpha, \epsilon)\)-quasi-random for at least \( \frac{1}{2} \binom{n}{m} \) \( m \)-sets \( M \in \binom{V}{m} \).

Going forward in this subsection, we restrict to \( l = 3 \) for simplicity. Essentially the same statements may be applied to general \( l \)-graphs.

Given a 3-graph \( F \), \( \alpha > 0 \) and \( \epsilon > 0 \), we write \( (\alpha, \epsilon) \to F \) to denote the fact that every \((\alpha, \epsilon)\)-quasi-random 3-graph \( R \) contains \( F \). Let \( F \) and \( H \) be 3-graphs. For \( F \), \( H \), and \( v \in V(F) \), we define \( F_H^v \) to be the 3-graph as follows:
(i) \( V(F_H^w) = V(F) \cup V(H) - v \) and

(ii) \( E(F_H^w) = E(F - v) \cup E(H) \cup \bigcup_{u \in V(H)} \{\{a, b, u\} : \{a, b, v\} \in E(F)\} \)

In other words, to obtain \( F_H^w \) from \( F \), replace \( v \) with \( V(H) \) and add all the edges in \( H \) as well as the edges of type \( \{a, b, u\} \) where \( u \in V(H) \) and \( \{a, b, v\} \in E(F) \). In this construction we will assume that \( F \) and \( H \) are vertex disjoint and thus \( |V(F_H^w)| = |V(F)| \) and \( |E(F_H^w)| = |E(F)| \) and \( |V(H)| - 1\{v \in E(F), v \in e\} \).

Using the notation stated above, we observe the following:

**Lemma 2.2.** For all \( \alpha > 0 \), \( \epsilon > 0 \), \( \gamma > 0 \) and 3-graphs \( F \) and \( H \), there exists \( \delta = \delta(\alpha, \epsilon, \gamma) > 0 \) such that if \( (\alpha, \epsilon) \to F \) and \( (\alpha, \gamma) \to H \), then \( (\alpha, \delta) \to F_H^w \).

**Proof.** Let \( |V(F)| = f \) and let \( v \in V(F) \). Given \( \alpha > 0 \) and \( \epsilon > 0 \) such that \( (\alpha, \epsilon) \to F \), let \( \delta_{L(2,1)} \) and \( m = m(\alpha, \epsilon) \) be the constants ensured by Lemma 2.1. Consider an \((\alpha, \delta)\)-quasi-random hypergraph \( Q \) on \( n \) vertices. Set \( \delta \leq \min(\delta_{L(2,1)}, \frac{\gamma}{2m}) \). We want to show that \( Q \) must contain \( F_H^w \). By Lemma 2.1, \( R = Q[M] \) is \((\alpha, \epsilon)\)-quasi-random for at least \( \frac{1}{2}(\frac{n}{m})^{M} \)'s. By assumption \(((\alpha, \epsilon) \to F)\) each such \((\alpha, \epsilon)\)-quasi-random \( Q[M] \) contains a copy of \( F \). Consequently, the number of \( Q[M] \)'s with each containing a copy of \( F \) is at least \( \frac{1}{2}(\frac{n}{m})^{f} \). On the other hand, each copy of \( F \) is in at most \( (\frac{n-f}{m}) \) different \( Q[M] \)'s. Thus, we have at least \( \frac{1}{2}(\frac{n}{m})^{f} \geq \frac{1}{2}(\frac{n}{m})^{f} > \frac{1}{2}(\frac{n}{m})^{f} = cn^{f} \) distinct copies of \( F \) in \( Q \), where \( c = c(m(\alpha, \epsilon), f) = \frac{1}{2m} \). Set \( V(F) = \{u_1, u_2, \ldots, u_{f-1}, v\} \) and let \( F^\text{copy} = F^c \) be a copy of \( F \) in \( Q \) with \( V(F^c) = \{u^c_1, u^c_2, \ldots, u^c_{f-1}, v^c\} \) so that \( u_i \to u^c_i \) for \( i = 1, 2, \ldots, f-1 \) and \( v \to v^c \) is an isomorphism.

For each of the \( cn^{f} \) copies \( F^c \) of \( F \), consider an ordered \((f-1)\)-tuple \((u^c_1, u^c_2, \ldots, u^c_{f-1})\). Since the total number of \((f-1)\)-tuples of vertices of \( Q \) is bounded by \( n(n-1) \ldots (n-(f-1)) \leq n^{f-1} \) we infer that there exists an \((f-1)\)-tuple of vertices \( \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{f-1} \) of \( Q \) contained in \( cn^{f}/n^{f-1} \sim cn \) copies \( F^c \) of \( F \). Consider a set \( S, |S| = cn = \frac{n}{cm} \) of vertices \( \bar{v} \) each of which together with \( \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{f-1} \) induces a copy \( F^c \) of \( F \). Due to the \((\alpha, \delta)\)-quasi-randomness of \( Q \) and the fact that \( \delta \leq \frac{\gamma}{2m} = c\gamma \), \( Q[S] \) is \((\alpha, \gamma)\)-quasi-random and, therefore, due to the assumption of Lemma 2.2, contains a copy of \( H \) with vertex set \( V(H) = \{v_1, \ldots, v_{|V(H)|}\} \). Since each \( v_i \) \((1 \leq i \leq |V(H)|)\) together with \( \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{f-1} \) span a copy \( F^c \) of \( F \), we infer that \( \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{f-1}, v_1, \ldots, v_{|V(H)|}\} \) spans a copy of \( F_H^w \). Thus, \( (\alpha, \delta) \to F_H^w \). \( \square \)

Before we prove Theorem 1.3(1) for \( l = 3 \), we construct an auxiliary sequence of 3-graphs \( \mathcal{G} = \{G_l\}_{l=1}^{\infty} \) with density tending to \( \frac{1}{4} \). We will then show that \( G_l \) is in \( Q_n \) for \( n \) large enough. Let \( G_1 \) be a 3-graph with three vertices and one edge. For \( i > 1 \), let \( G_i \) be the 3-graph obtained by taking 3 vertex disjoint copies of \( G_{i-1} \), and adding all edges with exactly one vertex in each copy. For instance, \( G_2 \) has 9 vertices and 3 + 3² = 30 edges.
b) Set $F = F_1$ in three applications of Lemma 2.2 as shown in Figure 1. We will construct hypergraphs $F$, $F'$, $F''$, and $F'''$ to obtain graphs $G$, $G'$, $G''$, and $G'''$, respectively. This will be achieved based on density calculations of $G$, $G'$, $G''$, and $G'''$. Consider an arbitrary sequence of $(\alpha, \delta_n)$-quasi-random 3-graphs $Q = \{Q_n\}_{n=1}^{\infty}$ with $d(Q_n) = \alpha(1 \pm \delta_n) > 0$ where $\delta_n \in (0, 1)$, $\delta_n$ is decreasing and $\delta_n \to 0$ as $n \to \infty$. We will show that for each $n < n_3$ such that for $n \geq n_i$, $Q_n$ contains $G_i$. Based on our density calculation of $G_i$ above, $d(Q) \geq \frac{1}{4}$. Since $Q_n$ contains $G_i$ whenever $\delta_n < \alpha$, it remains to show the following claim by induction on $i$:

**Claim 2.3.** Assuming $(\alpha, \delta_n_i) \to G_i$, there exists $n_{i+1}$ such that $(\alpha, \delta_{n_{i+1}}) \to G_{i+1}$

*Proof.* Our goal is to find $n_{i+1}$ so that $(\alpha, \delta_n) \to G_{i+1}$ for all $n \geq n_{i+1}$. This will be achieved in three applications of Lemma 2.2 as shown in Figure 1. We will construct hypergraphs $F'$, $F''$, $F'''$ with $G_i \subseteq F' \subseteq F'' \subseteq F''' = G_{i+1}$ and $n', n'', n'''$ with $n_i < n' < n'' < n''' = n_{i+1}$ such that

$$(\alpha, \delta_n) \to F^{(i)}(n) \quad \text{for all } n \geq n^{(i)}$$

Set $V(G_1) = \{a, b, c\}$, $H = G_i$, and $\gamma = \delta_n$. Below we will describe appropriate choices of $F$, $\epsilon$ and $v$ to obtain graphs $F^{(i)}$, $i = 1, 2, 3$ satisfying (*).

a) Set $F = G_1$, $\epsilon = \delta_1$ and $v = a$. Since $(\alpha, \delta_1) \to G_1$ and $(\alpha, \delta_{n_i}) \to G_i$, by Lemma 2.2 there exists $\delta' = \delta(\alpha, \delta_1, \delta_{n_i})$ such that $(\alpha, \delta') \to F_{G_1}^a$.

b) Set $F' = F_{G_i}^a$, $\epsilon = \delta'$ and $v = b$. Since $(\alpha, \delta') \to F'$ and $(\alpha, \delta_{n_i}) \to G_i$, by Lemma 2.2 there exists $\delta'' = \delta(\alpha, \delta', \delta_{n_i})$ such that $(\alpha, \delta'') \to F_{G_i}^b$.

c) Set $F'' = F_{G_i}^b$, $\epsilon = \delta''$ and $v = c$. Since $(\alpha, \delta'') \to F''$ and $(\alpha, \delta_{n_i}) \to G_i$, by Lemma 2.2 there exists $\delta''' = \delta(\alpha, \delta'', \delta_{n_i})$ such that $(\alpha, \delta''') \to F_{G_i}^{mc}$.

Observe that $F''' = F_{G_i}^{mc} = G_{i+1}$. Consequently $(\alpha, \delta_n) \to G_{i+1}$ for all $n$ with $\delta_n \leq \delta'''$. □
In a similar way to Claim 2.3 one can show a slightly more general fact stated below as Proposition 2.5. First we define the lexicographic product of two 3-graphs:

**Definition 2.4.** The lexicographic product of two 3-graphs $F$ and $H$ with vertex set $U$ and $W$ respectively is a 3-graph $F \cdot H$ with vertex set $U \times W$ and with $\{(u_1, w_1), (u_2, w_2), (u_3, w_3)\} \in E(F \cdot H)$ if $\{u_1, u_2, u_3\} \in E(F)$ or if $u_1 = u_2 = u_3$ and $\{w_1, w_2, w_3\} \in E(H)$.

**Proposition 2.5.** For all $\alpha > 0$, $\epsilon > 0$, $\gamma > 0$ and 3-graphs $F$ and $H$ there exists $\delta = \delta(\alpha, \epsilon, \gamma) > 0$ such that $(\alpha, \epsilon) \to F$ and $(\alpha, \gamma) \to F$ implies $(\alpha, \delta) \to F \cdot H$.

### 2.2 The upper bound for $l=3$

It remains to show there exists a sequence of quasi-random 3-graphs with upper density $\frac{1}{4}$.

**Proof.** Consider a random tournament $T_n$ on $n$ vertices in which pairs are assigned arc direction with probability $\frac{1}{2}$. Let $R_n$ be a 3-graph with $V(R_n) = V(T_n)$ and $E(R_n)$ consisting of vertex sets of all directed 3-cycles (this 3-graph was first considered by Erdős and Hajnal in [4] in the context of Ramsey theory).

It is well known (see [3]) that $R_n$ is $(\frac{1}{4}, \delta_n)$-quasi-random with $\delta_n \to 0$ as $n \to \infty$. On the other hand it follows from the well known result of Kendall and Babington Smith [7] that any tournament on $n$ vertices has at most $\frac{1}{24}(n^3 - n)$ directed 3-cycles (cf. [8]) and so no subgraph of any $R_n$ has density larger than $\frac{1}{4} + o(1)$. Thus the upper density of the sequence $\mathcal{R} = \{R_n\}_{n=1}^{\infty}$ is at most $\frac{1}{4} + o(1)$ establishing (ii) of Theorem 1.3. \qed

### 3 Proof of Theorem 1.6

For $l = 3$, Theorem 1.3(i) implies that every number in $[0, \frac{1}{4})$ is a jump for quasi-random 3-graphs. In this section, we prove that $\frac{1}{4}$ is a jump as well and, more precisely, any number in $[\frac{1}{4}, 0.3192)$ is a jump for quasi-random 3-graphs given Question 1.5 is answered positively. To this end, we use a recent result of Glebov, Král’ and Volec who in [6] confirmed Question 1.5 using a computer aided proof based on Razborov’s flag-algebra method [10].

**Proof.** Given a sequence of quasi-random 3-graphs $Q = \{Q_n\}_{n=1}^{\infty}$ with $\mathcal{Q}(Q) > \frac{1}{4}$, any $Q_n$ with $n \geq n_0$ contains $K_4^{(3)} - e$ by [6]. In a way similar to the proof of Theorem 1.3(i) we will first construct a sequence of 3-graphs $\mathcal{F} = \{F_n\}_{n=1}^{\infty}$ such that $F_n \subseteq Q_n$ and $\lim_{n \to \infty} d(F_n) = \frac{3}{10}$. Subsequently we will alter it to a sequence of 3-graphs $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ in which $\lim_{n \to \infty} d(G_n) \approx 0.3192$.

Let $F_1 = K_4^{(3)} - e$ with $V(F_1) = \{a_1, a_2, a_3, b\}$ and $E(F_1) = \{\{a_1, a_2, b\}, \{a_1, a_3, b\}, \{a_2, a_3, b\}\}$. Let $A_i$ ($1 \leq i \leq 3$) and $B$ be copies of $K_4^{(3)} - e$. We obtain $F_2$ by taking four vertex disjoint copies of $F_1$, with vertex set $A_i$, $1 \leq i \leq 3$, and $B$ and adding edges of type $\{a_i, a_j, b\}$ where $a_i \in A_i, a_j \in A_j, b \in B, 1 \leq i < j \leq 3$. Note that $|V(F_2)| = 4^2 = 16$ and $|E(F_2)| = 3(4) + 4^3(3)$. In other words $F_2 = F_1 \cdot F_1$ is the lexicographic product of two copies of $F_1$. We continue in this fashion to construct the sequence $\mathcal{F}$. For $i > 1$, we continue with...
let $F_i = F_1 \cdot F_{i-1}$ be the 3-graph obtained by taking four vertex disjoint copies of $F_{i-1}$, and adding edges in a similar way as described above. Since $|V(F_i)| = 4|V(F_{i-1})| = 4^i$ and $|E(F_i)| = 3|V(F_{i-1})| + 4^i|E(F_{i-1})| = 3 \cdot 4^{i-1}(1 + 4^2 + \ldots + 4^{2(i-1)}) = \frac{4^{i-1}}{5}(16^i - 1)$, the density of $F_i$ as $i \to \infty$ is

$$\lim_{i \to \infty} d(F_i) = \lim_{i \to \infty} \frac{4^{i-1}/5(16^i - 1)}{(4^i)} = \frac{3}{10}. $$

In a similar way as in the proof of Theorem 1.3(i), one can show that for all $i$ there exists $n$ such that $F_i$ is contained in $Q_n$. Thus, every number between 0 and $\frac{3}{10}$ is a jump for quasi-random 3-graphs.

One can improve $\frac{3}{10}$ to 0.3192 by considering conveniently chosen “blow ups” of $F_i$. We will describe this in more detail now. Setting $V(F_i) = \{1, 2, \ldots, \nu_i\}$, we first observe (similarly as in Lemma 2.2) that for each $i$, there exists an $n_i$ so that 3-graphs $Q_n$, $n \geq n_i$, contain $c_i|V(Q_n)|^{\nu_i}$ copies of $F_i$. Hence by Theorem 1.1, $Q_n$ contains a $t$-blowup $F_i \cdot t$ of $F_i$, more precisely, a graph with vertex set $\bigcup_{j=1}^{\nu_i} W_j$, $|W_1| = \cdots = |W_{\nu_i}| = t$ and $\{\tilde{a}, \tilde{b}, \tilde{c}\} \in E(F_i \cdot t)$ if $\{a, b, c\} \in E(F_i)$. In order to maximize the density, we consider graphs $F_i$ with different vertices “blown up” to sets of different cardinalities.

More precisely, set $\alpha = \frac{3}{10}(4\sqrt{6} - 9) \approx 0.2154$ and to each vertex $\pi = (x_1, \ldots, x_i) \in V(F_i)$ assign a weight $w(\pi) = (1 - 3\alpha)^j\alpha^{i-j}$ where $j$ represents the number of $b$’s among entries of $\pi$ and for $t$ large consider a blow-up $G_i$ of $F_i$ with each vertex $\pi$ “blown-up” by $w(\pi) \cdot t$ vertices. Using this iterated construction, one can calculate that every number between 0 and $\frac{1}{19}(9 - 2\sqrt{6}) \approx 0.3192$, where $\frac{1}{19}(9 - 2\sqrt{6}) = \lim_{i \to \infty} d(G_i)$, is a jump for quasi-random 3-graphs.

4 Other remarks and questions

In Section 2.2 we considered $\mathcal{R} = \{R_n\}_{n=1}^{\infty}$, a sequence of quasi-random 3-graphs formed by random tournaments $T_n$, and observed that $d(\mathcal{R}) = \overline{d}(\mathcal{R}) = \frac{1}{4}$. There are other quasi-random sequences of 3-graphs with density equal to upper density. Consider the quasi-random sequences $\mathcal{Q} = \{Q_n\}_{n=1}^{\infty}$ described in [11]: Let $\chi$ be a random $(k - 1)$-coloring of pairs of $\{1, \ldots, n\}$ and define the edges of $Q_n$ to be all triples $\{i, u, v\}$ such that $\chi(\{i, u\}) \neq \chi(\{i, v\})$. It can be shown that $d(\mathcal{Q}) = \overline{d}(\mathcal{Q}) = 1 - \frac{1}{k-1}$. In summary, if $\alpha \in \{\frac{1}{4}, \frac{1}{2}, \frac{2}{3}, \ldots\}$, then there is a sequence of quasi-random 3-graphs with $d(\mathcal{Q}) = \overline{d}(\mathcal{Q})$. Are there any others?

We proved that a sequence of quasi-random $l$-graphs $\mathcal{Q}$ with $d(\mathcal{Q}) > 0$ has $\overline{d}(\mathcal{Q}) \geq \frac{l(l-1)!}{l^l - 1}$. In particular, we showed that this bound is the best possible when $l = 3$. For $l = 4$, it is not clear to the authors if there exists a quasi-random sequence of 4-graphs with upper density equal to $\frac{3!}{4^3 - 1} = \frac{2}{21}$.

Theorem 1.3(i) implies that every quasi-random sequence of $l$-graphs with positive density has upper density at least $\frac{l(l-1)!}{l^l - 1}$. For $l = 3$ this is the best possible, but we were
unable to show an analogous fact for \( l > 3 \). One can observe that \( \frac{(l-1)!}{l-1} \) cannot be replaced by a number larger than \( \frac{(l-1)!}{(l-1)^2} \). In order to see this, consider the quasi-random sequence \( \mathcal{Q} = \{Q_n\}_{n=1}^{\infty} \) with vertex set \( V(Q_n) = \{1, \ldots, n\} = [n] \). Let \( \chi \) be a random \((l-1)\)-coloring of pairs of \([n]\). Define the edge set \( \{i, v_1, \ldots, v_{l-1}\} \in E(Q_n) \) if and only if all pairs \( \{i, v_1\}, \ldots, \{i, v_{l-1}\} \) have different color. One can observe that \( d(\mathcal{Q}) = \overline{d}(\mathcal{Q}) = \frac{(l-1)!}{(l-1)^2} \).

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References


