Note on upper density of quasi-random hypergraphs

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Abstract

In 1964, Erdős proved that for any $\alpha > 0$, an l-uniform hypergraph G with $n \geqslant n_0(\alpha, l)$ vertices and $\alpha\binom{n}{l}$ edges contains a large complete l-equipartite subgraph. This implies that any sufficiently large G with density $\alpha > 0$ contains a large subgraph with density at least $l!/l^l$.

In this note we study a similar problem for l-uniform hypergraphs Q with a weak quasi-random property (i.e. with edges uniformly distributed over the sufficiently large subsets of vertices). We prove that any sufficiently large quasi-random l-uniform hypergraph Q with density $\alpha > 0$ contains a large subgraph with density at least $\frac{(l-1)!}{l^{l-1}-1}$. In particular, for $l = 3$, any sufficiently large such Q contains a large subgraph with density at least $\frac{1}{4}$ which is the best possible lower bound.

We define jumps for quasi-random sequences of l-graphs and our result implies that every number between 0 and $\frac{(l-1)!}{l^{l-1}-1}$ is a jump for quasi-random l-graphs. For $l = 3$ this interval can be improved based on a recent result of Glebov, Král' and Volec. We prove that every number between [0, 0.3192) is a jump for quasi-random 3-graphs.

Keywords: hypergraphs; quasi-random; density; jumps

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1 Introduction

For fixed $l \geqslant 2$, an *l-graph* $G = (V, E)$ is an l-uniform hypergraph with vertex set V and edge set $E \subseteq \binom{V}{l}$, or a subset of the l-tuples of V. For $K \subseteq V$ and $|K| = k$, we denote the l-subgraph of G *induced* by K as $G[K] = (K, E \cap \binom{K}{l})$. The *density* of such an l-graph G is defined by $d(G) = |E|/\binom{|V|}{l}$.

Let $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be a sequence of l-graphs with $G_n = (V_n, E_n)$ such that $|V_n| \to \infty$ as $n \to \infty$. We define the density $d(\mathcal{G})$ of a sequence $\mathcal{G}$ as $d(\mathcal{G}) = \lim_{n\to\infty} d(G_n)$ (if the limit exists). We will consider only graph sequences for which the limit $d(G_n)$ exists as $n \to \infty$.

Setting

$$\sigma_k(\mathcal{G}) = \max_n \max_{K \in \binom{V_n}{k}} d(G_n[K]),$$

a simple averaging argument yields that $\{\sigma_k(\mathcal{G})\}_{k=2}^{\infty}$ is a non-increasing non-negative sequence and so the limit $\overline{d}(\mathcal{G}) = \lim\limits_{k\to\infty} \sigma_k(\mathcal{G})$ exists. We call this limit $\overline{d}(\mathcal{G})$ the *upper density* of $\mathcal{G}$.

The result we present in this note are motivated by a theorem of Erdős from [2]:

Theorem 1.1. *For every $\epsilon > 0, l \geqslant 2$ and t, there exists n such that any l-graph with n vertices and ϵn^l edges contains a complete l-partite l-graph $K^{(l)}_{t,t,\ldots,t}$. Consequently, for any sequence $\mathcal{G}$ of l-graphs with $d(\mathcal{G}) > 0$, $\overline{d}(\mathcal{G}) \geqslant l!/l^l$.*

In this note we are interested in a similar problem if we restrict to quasi-random l-graphs.

Definition 1.2. Given $\epsilon > 0$ and $\alpha > 0$, we define an (α, ϵ)-*quasi-random* hypergraph to be an l-graph $Q = (V, E)$ with the property that for all $W \subseteq V$, $d(Q[W]) = \alpha(1 \pm \epsilon)$ for $|W| \geqslant \epsilon n$ where $|V| = n$. A sequence $\mathcal{Q} = \{Q_n\}_{n=1}^{\infty}$ of (α, ϵ_n)-quasi-random l-graphs is *quasi-random* if ϵ_n is decreasing and $\epsilon_n \to 0$ as $n \to \infty$.

Note that for $l = 2$ quasi-random graphs must contain arbitrarily large cliques as $\epsilon_n \to 0$ and thus any quasi-random sequence of 2-graphs with $d(\mathcal{Q}) > 0$ necessarily satisfies $\overline{d}(\mathcal{Q}) = 1$. In this note we prove a related result for $l \geqslant 3$:

Theorem 1.3. *For a sequence $\mathcal{Q}$ of quasi-random l-graphs with $d(\mathcal{Q}) > 0$,*

(i) $\overline{d}(\mathcal{Q}) \geqslant \frac{(l-1)!}{l^{l-1}-1}$ and

(ii) when $l = 3$ there exists a quasi-random sequence of 3-graphs with $\overline{d}(\mathcal{Q}) = \frac{1}{4}$.

For $l > 3$, however, we do not know if $\overline{d}(\mathcal{Q}) \geqslant \frac{(l-1)!}{l^{l-1}-1}$ could not be replaced by a larger number. Our results for $l = 3$ are shown in the Section 2.1 and a similar construction may be applied to generalize the result for all l-graphs, proving Theorem 1.3*(*i).

A number α is a *jump* if there exists a constant $c = c(\alpha)$ such that given any sequence of l-graphs $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ if $\overline{d}(\mathcal{G}) > \alpha$, then $\overline{d}(\mathcal{G}) \geqslant \alpha + c$. It follows from the Erdős-Stone Theorem that all non-negative numbers less than 1 are jumps for graphs and it follows from Theorem 1.1 that all non-negative numbers less than $\frac{l!}{l^l}$ are jumps for l-graphs. Erdős conjectured that, analogous to graphs, all numbers less than 1 are jumps for l-graphs as well. This conjecture was disproved by Frankl and Rödl in [5] who showed that there are an infinite number of non-jumps for all $l \geqslant 3$. However, these non-jumps were found to occur at relatively large densities. While the smallest case of determining whether $\frac{l!}{l^l}$ is a jump is still open and likely a difficult problem, our result shows that under the further assumption of quasi-randomness that $\frac{l!}{l^l}$ is indeed a jump for all $l \geqslant 3$.

We extend the concept of jumps to sequences of quasi-random l-graphs:

Definition 1.4. A number α is a *jump for quasi-random l-graphs* if there exists a constant $c = c(\alpha)$ such that given any sequence of quasi-random l-graphs $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ if $\overline{d}(\mathcal{G}) > \alpha$, then $\overline{d}(\mathcal{G}) \geqslant \alpha + c$.

Theorem 1.3*(i)* implies that every number between 0 and $\frac{(l-1)!}{l^{l-1}-1}$ is a jump for quasi-random l-graphs. Further we will show that for $l = 3$ this interval can be improved from $[0, \frac{1}{4})$ to $[0, 0.3192)$ given the following question of Erdős [3] is answered positively:

Question 1.5. *Let $c > 0$ and $\mathcal{Q} = \{Q_n\}_{n=1}^{\infty}$ be a quasi-random sequence of 3-graphs. If $d(\mathcal{Q}) = \frac{1}{4} + c$, then does each Q_n contain $K_4^{(3)} - e$ as $n \to \infty$?*

More formally, we prove in Section 3:

Theorem 1.6. *A positive answer to Question 1.5 implies that any quasi-random sequence $\mathcal{Q}$ with $d(\mathcal{Q}) > \frac{1}{4}$ satisfies $\overline{d}(\mathcal{Q}) > 0.3192$.*

Very recently, Glebov, Kráľ' and Volec in [6] proved Question 1.5 in the positive using Razborov's flag-algebra method [10]. This result confirms our assertion in Theorem 1.6.

We include our remarks and questions for future study for quasi-random l-graphs with $l > 3$ and other possibilites for jumps for quasi-random 3-graphs in Section 4.

2 Proof of Theorem 1.3

2.1 The lower bound

Our proof is based on the following lemma proved in [1] and [9]:

Lemma 2.1. *For all $\alpha > 0$ and $\epsilon > 0$, there exists $\delta > 0$, $m > 0$ and $n_0 > 0$ such that if $Q = (V, E)$ is an (α, δ)-quasi-random l-graph with $|V| = n \geqslant n_0$ vertices then $Q[M]$ is (α, ϵ)-quasi-random for at least $\frac{1}{2}\binom{n}{m}$ m-sets $M \in \binom{n}{m}$.*

Going forward in this subsection, we restrict to $l = 3$ for simplicity. Essentially the same statements may be applied to general l-graphs.

Given a 3-graph F, $\alpha > 0$ and $\epsilon > 0$, we write $(\alpha, \epsilon) \to F$ to denote the fact that every (α, ϵ)-quasi-random 3-graph R contains F. Let F and H be 3-graphs. For F, H, and $v \in V(F)$, we define F_H^v to be the 3-graph as follows:

(i) $V(F_H^v) = V(F) \cup V(H) - v$ and

(ii) $E(F_H^v) = E(F - v) \cup E(H) \cup \bigcup_{u \in V(H)} \{\{a, b, u\} \colon \{a, b, v\} \in E(F)\}$

In other words, to obtain F_H^v from F, replace v with $V(H)$ and add all the edges in H as well as the edges of type $\{a, b, u\}$ where $u \in V(H)$ and $\{a, b, v\} \in E(F)$. In this construction we will assume that F and H are vertex disjoint and thus $|V(F_H^v)| = |V(F)| + |V(H)| - 1$ and $|E(F_H^v)| = |E(F)| + |E(H)| + |V(H) - 1||\{e \in E(F), v \in e\}|$.

Using the notation stated above, we observe the following:

Lemma 2.2. *For all $\alpha > 0$, $\epsilon > 0$, $\gamma > 0$ and 3-graphs F and H, there exists $\delta = \delta(\alpha, \epsilon, \gamma) > 0$ such that if $(\alpha, \epsilon) \to F$ and $(\alpha, \gamma) \to H$, then $(\alpha, \delta) \to F_H^v$.*

Proof. Let $|V(F)| = f$ and let $v \in V(F)$. Given $\alpha > 0$ and $\epsilon > 0$ such that $(\alpha, \epsilon) \to F$, let $\delta_{L(2.1)}$ and $m = m(\alpha, \epsilon)$ be the constants ensured by Lemma 2.1. Consider an (α, δ)-quasi-random hypergraph Q on n vertices. Set $\delta \leqslant \min\left(\delta_{L(2.1)}, \frac{\gamma}{2m^f}\right)$. We want to show that Q must contain F_H^v. By Lemma 2.1, $R = Q[M]$ is (α, ϵ)-quasi-random for at least $\frac{1}{2}\binom{n}{m}$ M's. By assumption ($(\alpha, \epsilon) \to F$) each such (α, ϵ)-quasi-random $Q[M]$ contains a copy of F. Consequently, the number of $Q[M]$'s with each containing a copy of F is at least $\frac{1}{2}\binom{n}{m}$. On the other hand, each copy of F is in at most $\binom{n-f}{m-f}$ different $Q[M]$'s. Thus, we have at least $\frac{1}{2}\binom{n}{m} / \binom{n-f}{m-f} = \frac{\binom{n}{f}}{2\binom{m}{f}} > \frac{1}{2}\left(\frac{n}{m}\right)^f = cn^f$ distinct copies of F in Q, where $c = c(m(\alpha, \epsilon), f) = \frac{1}{2m^f}$. Set $V(F) = \{u_1, u_2, \ldots, u_{f-1}, v\}$ and let $F^{copy} = F^c$ be a copy of F in Q with $V(F^c) = \{u_1^c, u_2^c, \ldots, u_{f-1}^c, v^c\}$ so that $u_i \to u_i^c$ for $i = 1, 2, \ldots, f-1$ and $v \to v^c$ is an isomorphism.

For each of the cn^f copies F^c of F, consider an ordered $(f-1)$-tuple $(u_1^c, u_2^c, \ldots, u_{f-1}^c)$. Since the total number of $(f-1)$-tuples of vertices of Q is bounded by $n(n-1)\ldots(n-(f-1)) \leqslant n^{f-1}$ we infer that there exists an $(f-1)$-tuple of vertices $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_{f-1}$ of Q contained in $cn^f/n^{f-1} \sim cn$ copies F^c of F. Consider a set S, $|S| = cn = \frac{n}{cm^f}$, of vertices $\overline{v}$ each of which together with $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_{f-1}$ induces a copy F^c of F. Due to the (α, δ)-quasi-randomness of Q and the fact that $\delta \leqslant \frac{\gamma}{2m^f} = c\gamma$, $Q[S]$ is (α, γ)-quasi-random and, therefore, due to the assumption of Lemma 2.2, contains a copy of H with vertex set $V(H) = \{v_1, \ldots, v_{|V(H)|}\}$. Since each v_i ($1 \leqslant i \leqslant |V(H)|$) together with $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_{f-1}$ span a copy F^c of F, we infer that $\{\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_{f-1}, v_1, \ldots, v_{|V(H)|}\}$ spans a copy of F_H^v. Thus, $(\alpha, \delta) \to F_H^v$. □

Before we prove Theorem 1.3*(i)* for $l = 3$, we construct an auxilliary sequence of 3-graphs $\mathcal{G} = \{G_i\}_{i=1}^{\infty}$ with density tending to $\frac{1}{4}$. We will then show that G_i is in Q_n for n large enough. Let G_1 be a 3-graph with three vertices and one edge. For $i > 1$, let G_i be the 3-graph obtained by taking 3 vertex disjoint copies of G_{i-1}, and adding all edges with exactly one vertex in each copy. For instance, G_2 has 9 vertices and $3 + 3^3 = 30$ edges.

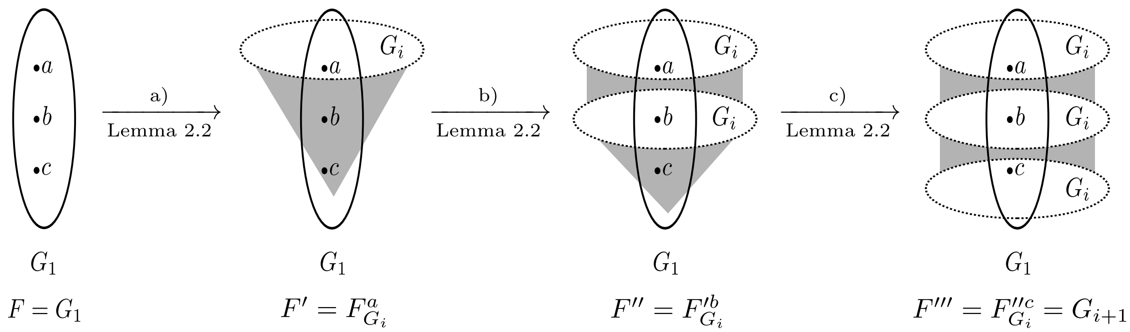

Figure 1: Three applications of Lemma 2.2 prove Claim 2.3

Since $|V(G_i)| = 3|V(G_{i-1})| = 3^i$ and $|E(G_i)| = |V(G_{i-1})|^3 + 3|E(G_{i-1})| = 3^{3(i-1)}(1 + \frac{1}{9} + \dots \frac{1}{9^{i-1}}) = 3^{i-1}\frac{(3^i-1)(3^i+1)}{8}$, the density of G_i as $i \to \infty$ is

$$\lim_{i\to\infty} d(G_i) = \lim_{i\to\infty} \frac{3^{i-1}\frac{(3^i-1)(3^i+1)}{8}}{\binom{3^i}{3}} = \lim_{i\to\infty} \frac{1}{4}\left(\frac{3^i+1}{3^i-2}\right) = \frac{1}{4}.$$

Consider an arbitrary sequence of (α, δ_n)-quasi-random 3-graphs $\mathcal{Q} = \{Q_n\}_{n=1}^{\infty}$ with $d(Q_n) = \alpha(1 \pm \delta_n) > 0$ where $\delta_n \in (0, 1)$, δ_n is decreasing and $\delta_n \to 0$ as $n \to \infty$. We will show that there exists $n_1 < n_2 < n_3 < \dots$ such that for $n \geqslant n_i$, Q_n contains G_i. Based on our density calculation of G_i above, $\overline{d}(\mathcal{Q}) \geqslant \frac{1}{4}$.

Since Q_n contains G_1 whenever $\delta_n < \alpha$, it remains to show the following claim by induction on i:

Claim 2.3. *Assuming* $(\alpha, \delta_{n_i}) \to G_i$*, there exists* n_{i+1} *such that* $(\alpha, \delta_{n_{i+1}}) \to G_{i+1}$

Proof. Our goal is to find n_{i+1} so that $(\alpha, \delta_n) \to G_{i+1}$ for all $n \geqslant n_{i+1}$. This will be achieved in three applications of Lemma 2.2 as shown in Figure 1. We will construct hypergraphs F', F'', F''' with $G_i \subseteq F' \subseteq F'' \subseteq F''' = G_{i+1}$ and n', n'', n''' with $n_i < n' < n'' < n''' = n_{i+1}$ such that

$$(\alpha, \delta_n) \to F^{(i)} \text{ for all } n \geqslant n^{(i)} \tag{*}$$

Set $V(G_1) = \{a, b, c\}$, $H = G_i$, and $\gamma = \delta_{n_i}$. Below we will describe appropriate choices of F, ϵ and v to obtain graphs $F^{(i)}$, $i = 1, 2, 3$ satisfying (*).

a) Set $F = G_1$, $\epsilon = \delta_1$ and $v = a$. Since $(\alpha, \delta_1) \to G_1$ and $(\alpha, \delta_{n_i}) \to G_i$, by Lemma 2.2 there exists $\delta' = \delta(\alpha, \delta_1, \delta_{n_i})$ such that $(\alpha, \delta') \to F^a_{G_i}$.

b) Set $F' = F^a_{G_i}$, $\epsilon = \delta'$ and $v = b$. Since $(\alpha, \delta') \to F'$ and $(\alpha, \delta_{n_i}) \to G_i$, by Lemma 2.2 there exists $\delta'' = \delta(\alpha, \delta', \delta_{n_i})$ such that $(\alpha, \delta'') \to F'^b_{G_i}$.

c) Set $F'' = F'^b_{G_i}$, $\epsilon = \delta''$ and $v = c$. Since $(\alpha, \delta'') \to F''$ and $(\alpha, \delta_{n_i}) \to G_i$, by Lemma 2.2 there exists $\delta''' = \delta(\alpha, \delta'', \delta_{n_i})$ such that $(\alpha, \delta''') \to F''^c_{G_i}$.

Observe that $F''' = F''^c_{G_i} = G_{i+1}$. Consequently $(\alpha, \delta_n) \to G_{i+1}$ for all n with $\delta_n \leqslant \delta'''$. $\square$

In a similar way to Claim 2.3 one can show a slightly more general fact stated below as Proposition 2.5. First we define the lexicographic product of two 3-graphs:

Definition 2.4. The *lexicographic product* of two 3-graphs F and H with vertex set U and W respectively is a 3-graph $F \cdot H$ with vertex set $U \times W$ and with $\{(u_1, w_1), (u_2, w_2), (u_3, w_3)\} \in E(F \cdot H)$ if $\{u_1, u_2, u_3\} \in E(F)$ or if $u_1 = u_2 = u_3$ and $\{w_1, w_2, w_3\} \in E(H)$.

Proposition 2.5. *For all $\alpha > 0$, $\epsilon > 0$, $\gamma > 0$ and 3-graphs F and H there exists $\delta = \delta(\alpha, \epsilon, \gamma) > 0$ such that $(\alpha, \epsilon) \to F$ and $(\alpha, \gamma) \to F$ implies $(\alpha, \delta) \to F \cdot H$.*

2.2 The upper bound for l=3

It remains to show there exists a sequence of quasi-random 3-graphs with upper density $\frac{1}{4}$.

Proof. Consider a random tournament T_n on n vertices in which pairs are assigned arc direction with probability $\frac{1}{2}$. Let R_n be a 3-graph with $V(R_n) = V(T_n)$ and $E(R_n)$ consisting of vertex sets of all directed 3-cycles (this 3-graph was first considered by Erdős and Hajnal in [4] in the context of Ramsey theory).

It is well known (see [3]) that R_n is $(\frac{1}{4}, \delta_n)$-quasi-random with $\delta_n \to 0$ as $n \to \infty$. On the other hand it follows from the well known result of Kendall and Babington Smith [7] that any tournament on n vertices has at most $\frac{1}{24}(n^3 - n)$ directed 3-cycles (cf. [8]) and so no subgraph of any R_n has density larger than $\frac{1}{4} + o(1)$. Thus the upper density of the sequence $\mathcal{R} = \{R_n\}_{n=1}^{\infty}$ is at most $\frac{1}{4} + o(1)$ establishing *(ii)* of Theorem 1.3. □

3 Proof of Theorem 1.6

For $l = 3$, Theorem 1.3*(i)* implies that every number in $[0, \frac{1}{4})$ is a jump for quasi-random 3-graphs. In this section, we prove that $\frac{1}{4}$ is a jump as well and, more precisely, any number in $[\frac{1}{4}, 0.3192)$ is a jump for quasi-random 3-graphs given Question 1.5 is answered positively. To this end, we use a recent result of Glebov, Král' and Volec who in [6] confirmed Question 1.5 using a computer aided proof based on Razborov's flag-algebra method [10].

Proof. Given a sequence of quasi-random 3-graphs $\mathcal{Q} = \{Q_n\}_{n=1}^{\infty}$ with $\overline{d}(\mathcal{Q}) > \frac{1}{4}$, any Q_n with $n \geqslant n_0$ contains $K_4^{(3)} - e$ by [6]. In a way similar to the proof of Theorem 1.3*(i)* we will first construct a sequence of 3-graphs $\mathcal{F} = \{F_n\}_{n=1}^{\infty}$ such that $F_n \subseteq Q_n$ and $\lim_{n\to\infty} d(F_n) = \frac{3}{10}$. Subsequently we will alter it to a sequence of 3-graphs $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ in which $\lim_{n\to\infty} d(G_n) \approx 0.3192$.

Let $F_1 = K_4^{(3)} - e$ with $V(F_1) = \{a_1, a_2, a_3, b\}$ and $E(F_1) = \{\{a_1, a_2, b\}, \{a_1, a_3, b\}, \{a_2, a_3, b\}\}$. Let A_i $(1 \leqslant i \leqslant 3)$ and B be copies of $K_4^{(3)} - e$. We obtain F_2 by taking four vertex disjoint copies of F_1, with vertex set A_i, $1 \leqslant i \leqslant 3$, and B and adding edges of type $\{a_i, a_j, b\}$ where $a_i \in A_i, a_j \in A_j, b \in B, 1 \leqslant i < j \leqslant 3$. Note that $|V(F_2)| = 4^2 = 16$ and $|E(F_2)| = 3(4) + 4^3(3)$. In other words $F_2 = F_1 \cdot F_1$ is the lexicographic product of two copies of F_1. We continue in this fashion to construct the sequence $\mathcal{F}$. For $i > 1$, let $F_i = F_1 \cdot F_{i-1}$ be the 3-graph obtained by taking four vertex disjoint copies of F_{i-1}, and adding edges in a similar way as described above. Since $|V(F_i)| = 4|V(F_{i-1}| = 4^i$ and $|E(F_i)| = 3|V(F_{i-1})| + 4^3|E(F_{i-1})| = 3 \cdot 4^{i-1}(1 + 4^2 + \ldots + 4^{2(i-1)}) = \frac{4^{i-1}}{5}(16^i - 1)$, the density of F_i as $i \to \infty$ is

$$\lim_{i\to\infty} d(F_i) = \lim_{i\to\infty} \frac{\frac{4^{i-1}}{5}(16^i - 1)}{\binom{4^i}{3}} = \frac{3}{10}.$$

In a similar way as in the proof of Theorem 1.3*(i)*, one can show that for all i there exists n such that F_i is contained in Q_n. Thus, every number between 0 and $\frac{3}{10}$ is a jump for quasi-random 3-graphs.

One can improve $\frac{3}{10}$ to 0.3192 by considering conveniently chosen "blow ups" of F_i. We will describe this in more detail now. Setting $V(F_i) = \{1, 2, \ldots, \nu_i\}$, we first observe (similarly as in Lemma 2.2) that for each i, there exists an n_i so that 3-graphs Q_n, $n \geqslant n_i$, contain $c_i|V(Q_n)|^{\nu_i}$ copies of F_i. Hence by Theorem 1.1, Q_n contains a t-blowup $F_i * t$ of F_i, more precisely, a graph with vertex set $\bigcup_{j=1}^{\nu_i} W_j$, $|W_1| = \cdots = |W_{\nu_i}| = t$ and $\{\widetilde{a}, \widetilde{b}, \widetilde{c}\} \in E(F_i * t)$ if $\{a, b, c\} \in E(F_i)$. In order to maximize the density, we consider graphs F_i with different vertices "blown up" to sets of different cardinalities.

More precisely, set $\alpha = \frac{2}{5}(4\sqrt{6} - 9) \approx 0.2154$ and to each vertex $\overline{x} = (x_1, \ldots, x_i) \in V(F_i)$ assign a weight $w(\overline{x}) = (1 - 3\alpha)^j \alpha^{i-j}$ where j represents the number of b's among entries of $\overline{x}$ and for t large consider a blow-up G_i of F_i with each vertex $\overline{x}$ "blown-up" by $w(\overline{x}) * t$ vertices. Using this iterated construction, one can calculate that every number between 0 and $\frac{1}{19}(9 - 2\sqrt{6}) \approx 0.3192$, where $\frac{1}{19}(9 - 2\sqrt{6}) = \lim_{i\to\infty} d(G_i)$, is a jump for quasi-random 3-graphs. $\square$

4 Other remarks and questions

In Section 2.2 we considered $\mathcal{R} = \{R_n\}_{n=1}^{\infty}$, a sequence of quasi-random 3-graphs formed by random tournaments T_n, and observed that $d(\mathcal{R}) = \overline{d}(\mathcal{R}) = \frac{1}{4}$. There are other quasi-random sequences of 3-graphs with density equal to upper density. Consider the quasi-random sequences $\mathcal{Q} = \{Q_n\}_{n=1}^{\infty}$ described in [11]: Let χ be a random $(k-1)$-coloring of pairs of $\{1, \ldots, n\}$ and define the edges of Q_n to be all triples $\{i, u, v\}$ such that $\chi(\{i, u\}) \neq \chi(\{i, v\})$. It can be shown that $d(\mathcal{Q}) = \overline{d}(\mathcal{Q}) = 1 - \frac{1}{k-1}$. In summary, if $\alpha \in \{\frac{1}{4}, \frac{1}{2}, \frac{2}{3}, \ldots\}$, then there is a sequence of quasi-random 3-graphs with $d(\mathcal{Q}) = \overline{d}(\mathcal{Q})$. Are there any others?

We proved that a sequence of quasi-random l-graphs $\mathcal{Q}$ with $d(\mathcal{Q}) > 0$ has $\overline{d}(\mathcal{Q}) \geqslant \frac{(l-1)!}{l^{l-1}-1}$. In particular, we showed that this bound is the best possible when $l = 3$. For $l = 4$, it is not clear to the authors if there exists a quasi-random sequence of 4-graphs with upper density equal to $\frac{3!}{4^3-1} = \frac{2}{21}$.

Theorem 1.3*(i)* implies that every quasi-random sequence of l-graphs with positive density has upper density at least $\frac{(l-1)!}{l^{l-1}-1}$. For $l = 3$ this is the best possible, but we were unable to show an analogous fact for $l > 3$. One can observe that $\frac{(l-1)!}{l^{l-1}-1}$ cannot be replaced by a number larger than $\frac{(l-1)!}{(l-1)^{l-1}}$. In order to see this, consider the quasi-random sequence $\mathcal{Q} = \{Q_n\}_{n=1}^{\infty}$ with vertex set $V(Q_n) = \{1, \ldots, n\} = [n]$. Let χ be a random $(l-1)$-coloring of pairs of $[n]$. Define the edge set $\{i, v_1, \ldots, v_{l-1}\} \in E(Q_n)$ if and only if all pairs $\{i, v_1\}$, $\ldots$, $\{i, v_{l-1}\}$ have different color. One can observe that $d(\mathcal{Q}) = \overline{d}(\mathcal{Q}) = \frac{(l-1)!}{(l-1)^{l-1}}$.

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