Gromov hyperbolicity in strong product graphs

Walter Carballosa

Departamento de Matemáticas
Universidad Carlos III de Madrid
Av. de la Universidad 30, 28911 Leganés, Madrid, Spain

wcarball@math.uc3m.es

Rocío M. Casablanca

Departamento de Matemática Aplicada I
Universidad de Sevilla
Av. Reina Mercedes 2, 41012 Sevilla, Spain

rociomc@us.es

Amauris de la Cruz   José M. Rodríguez

Departamento de Matemáticas
Universidad Carlos III de Madrid
Av. de la Universidad 30, 28911 Leganés, Madrid, Spain

alcruz@econ-est.uc3m.es, jomaro@math.uc3m.es

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Abstract

If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X. The space X is δ-*hyperbolic* (in the Gromov sense) if any side of T is contained in a δ-neighborhood of the union of the two other sides, for every geodesic triangle T in X. If X is hyperbolic, we denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e. $\delta(X) = \inf\{\delta \geqslant 0 : X \text{ is } \delta\text{-hyperbolic}\}$. In this paper we characterize the strong product of two graphs $G_1 \boxtimes G_2$ which are hyperbolic, in terms of G_1 and G_2: the strong product graph $G_1 \boxtimes G_2$ is hyperbolic if and only if one of the factors is hyperbolic and the other one is bounded. We also prove some sharp relations between $\delta(G_1 \boxtimes G_2)$, $\delta(G_1)$, $\delta(G_2)$ and the diameters of G_1 and G_2 (and we find families of graphs for which the inequalities are attained). Furthermore, we obtain the exact values of the hyperbolicity constant for many strong product graphs.

Keywords: Strong Product Graphs; Geodesics; Gromov Hyperbolicity; Infinite Graphs

1 Introduction

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory; see, for instance, [5, 6, 9, 11, 12, 13, 14, 15, 16, 18, 19, 27, 28, 29, 30, 31, 33, 34, 35, 38, 39, 42, 43, 45, 48, 49]. It is well known that most networks can be modeled by a graph $G = (V, E)$, where V is the set of mainly elements and E is the set of communication links between them in the network. Different methods have been proposed for configuration processing and data generation. Some of them are structural models which can be seen as the product graph of two given graphs, known as factors or generators. Many properties of structural models can be obtained by considering the properties of their generators. The different kinds of products of graphs are an important research topic in Graph Theory. In particular, the strong product graph operation has been extensively investigated in relation to a wide range of subjects [2, 10, 32, 47]. A fundamental principle for network design is extendability. That is to say, the possibility of building larger versions of a network preserving certain desirable properties. For designing large-scale interconnection networks, the strong p roduct is a useful method to obtain large graphs from smaller ones whose invariants can be easily calculated [10, 32, 47].

The theory of Gromov hyperbolic spaces was used initially for the study of finitely generated groups, where it was demonstrated to have an enormous practical importance. This theory was applied principally to the study of automatic groups (see [36]), which plays an important role in sciences of the computation. Another important application of these spaces is the secure transmission of information by internet. In particular, the hyperbolicity plays an important role in the spread of viruses through the network (see [28, 29]). The hyperbolicity is also useful in the study of DNA data (see [9]).

Last years several researchers have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Gehring-Osgood j-metric is Gromov hyperbolic; and the Vuorinen j-metric is not Gromov hyperbolic except in the punctured space (see [21]). The study of Gromov hyperbolicity of the quasihyperbolic and the Poincaré metrics is the subject of [1, 3, 7, 22, 23, 24, 25, 39, 40, 43, 44, 45, 49]. In particular, the equivalence of the hyperbolicity of Riemannian manifolds and the hyperbolicity of a simple graph was proved in [39, 43, 45, 49], hence, it is useful to know hyperbolicity criteria for graphs.

Notations and terminology not explicitly given here can be found in [20]. We present now some basic facts about Gromov's spaces. Let (X, d) be a metric space and let $\gamma : [a, b] \longrightarrow X$ be a continuous function. We define the *length* of γ as

$$L(\gamma) := \sup \Big\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : \, a = t_0 < t_1 < \cdots < t_n = b \Big\} .$$

We say that γ is a *geodesic* if it is an isometry, i.e. $d(\gamma(t), \gamma(s)) = s - t$ for every $t < s$. We say that X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y; we denote by $[xy]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected. If X is a graph, we use the notation $[u, v]$ for the edge joining the vertices u and v; in what follows, by $u \sim v$ we mean that $[u, v] \in E(X)$.

In order to consider a graph G as a geodesic metric space, we must identify (by an isometry) any edge $[u, v] \in E(G)$ with a real interval with length $l := L([u, v])$; therefore, an inner point of the edge $[u, v]$ is a point of G. A connected graph G is naturally equipped with a distance defined on its points, induced by taking shortest paths in G. Then, G can be seen as a metric graph.

Throughout this paper we just consider non-oriented (finite or infinite) connected graphs with edges of length 1. These conditions guarantee that the graphs are geodesic metric spaces. We also consider simple graphs, that is without loops or multiple edges, which is not a restriction [6, Theorems 6 and 8].

If X is a geodesic metric space and $J = \{J_1, J_2, \dots, J_n\}$ is a polygon with sides $J_j \subseteq X$, we say that J is *δ-thin* if for every $x \in J_i$ we have that $d(x, \cup_{j \neq i} J_j) \leqslant \delta$. We denote by $\delta(J)$ the sharp thin constant of J, i.e., $\delta(J) := \inf\{\delta \geqslant 0 : \ J \ \text{is } \delta\text{-thin}\}$. If $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ (sometimes we write $T = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$). The space X is *δ-hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) := \sup\{\delta(T) : T \ \text{is a geodesic triangle in } X\}$. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geqslant 0$. If X is hyperbolic, then $\delta(X) = \inf\{\delta \geqslant 0 : \ X \ \text{is } \delta\text{-hyperbolic}\}$. A geodesic *bigon* is a geodesic triangle $\{x_1, x_2, x_3\}$ with $x_2 = x_3$. Therefore, every bigon in a δ-hyperbolic geodesic metric space is δ-thin.

There are several definitions of Gromov hyperbolicity. These different definitions are equivalent in the sense that if X is δ-hyperbolic with respect to the definition A, then it is δ'-hyperbolic with respect to the definition B for some δ' (see, e.g., [8, 20]). We have chosen this definition since it has a deep geometric meaning (see, e.g., [20]).

The following remarks are interesting examples of hyperbolic spaces. The real line $\mathbb{R}$ is 0-hyperbolic due to any point of a geodesic triangle in the real line belongs to two sides of the triangle simultaneously. The Euclidean plane $\mathbb{R}^2$ is not hyperbolic since the equilateral triangles can be drawn with arbitrarily large diameter. This argument can be generalized in a similar way to higher dimensions: a normed vector space E is hyperbolic if and only if $\dim E = 1$. Every arbitrary length metric tree is 0-hyperbolic due to all points of a geodesic triangle in a tree belong simultaneously to two sides of the triangle. Every bounded metric space X is $((\operatorname{diam} X)/2)$-hyperbolic. Every simply connected complete Riemannian manifold with sectional curvature verifying $K \leqslant -k^2$, for some positive constant k, is hyperbolic. We refer to [8, 20] for more background and further results.

Notice that the main examples of hyperbolic graphs are the trees. In fact, the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how "tree-like" the space is, since those spaces X with $\delta(X) = 0$ are precisely the metric trees. This is an interesting subject since, in many applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see, e.g., [17]).

If D is a closed connected subset of X, we always consider in D the *inner metric* obtained by the restriction of the metric in X, that is

$$d_D(z,w) := \inf \big\{ L_X(\gamma) : \gamma \subset D \text{ is a continuous curve joining } z \text{ and } w \big\} \geqslant d_X(z,w)\,.$$

Consequently, $L_D(\gamma) = L_X(\gamma)$ for every curve $\gamma \subset D$.

Given a Cayley graph (of a presentation with solvable word problem) there is an algorithm which allows to decide if it is hyperbolic. However, the problem of deciding whether a general geodesic metric space is hyperbolic or not is usually very difficult. Note that, first of all, we have to consider an arbitrary geodesic triangle T, and calculate the minimum distance from an arbitrary point P of T to the union of the other two sides of the triangle to which P does not belong to. Finally we have to take supremum over all the possible choices for P and over all the possible choices for T. Without disregarding the difficulty to solve this minimax problem, notice that in general the main obstacle is that we do not know the location of geodesics in the space. Therefore, it is interesting to obtain inequalities involving the hyperbolicity constant and to study the hyperbolicity of a particular class of graphs.

The papers [5, 9, 16, 35, 37, 41, 48] study the hyperbolicity of, respectively, complement of graphs, chordal graphs, line graphs, Cartesian product graphs, cubic graphs, short graphs and median graphs, respectively. Our aim in this work is to obtain interesting results about the hyperbolicity constant of strong product graphs.

The structure of this paper is as follows. First, in Section 2, we study several inequalities involving the distance in the strong product of graphs and we obtain the exact value of its diameter. Furthermore, we also study the relations between the geodesics of $G_1 \boxtimes G_2$ and geodesics in G_1 and G_2; it is not a trivial issue as Example 7 will show.

In Section 3, we prove several lower and upper bounds for the hyperbolicity constant of $G_1 \boxtimes G_2$, involving $\delta(G_1)$, $\delta(G_2)$ and the diameters of G_1 and G_2. One of the main results of this work is Theorem 23, which characterizes the hyperbolic strong product graphs $G_1 \boxtimes G_2$ in terms of G_1 and G_2. The graph $G_1 \boxtimes G_2$ is hyperbolic if and only if one of its factors is hyperbolic and the other one is bounded. We also find families of graphs for which many of the inequalities of this section are attained. Another main result in this paper is Theorem 19 which provides the precise value of $\delta(G_1 \boxtimes G_2)$ for a large class of graphs G_1, G_2; this kind of result is not usual at all in the theory of hyperbolic graphs.

We conclude this paper with Section 4 where the exact values of the hyperbolicity constant for many strong product graphs are calculated.

2 The distance in strong product graphs

In order to estimate the hyperbolicity constant of the strong product of two graphs G_1 and G_2, we must obtain lower and upper bound on the distances between any two arbitrary points in $G_1 \boxtimes G_2$. The lemmas of this section provide these estimations. We will use the strong product definition given by Sabidussi in [46].

Definition 1. Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ two graphs. The strong product $G_1 \boxtimes G_2$ of G_1 and G_2 has $V(G_1) \times V(G_2)$ as vertex set, so that two distinct vertices $(u_1; v_1)$ and $(u_2; v_2)$ of $G_1 \boxtimes G_2$ are adjacent if either $u_1 = u_2$ and $[v_1, v_2] \in E(G_2)$, or $[u_1, u_2] \in E(G_1)$ and $v_1 = v_2$, or $[u_1, u_2] \in E(G_1)$ and $[v_1, v_2] \in E(G_2)$.

Note that the strong product of two graphs is commutative. We use the notation $(u; v)$ instead of (u, v) to the points of the graph $G_1 \boxtimes G_2$. We consider that every edge of $G_1 \boxtimes G_2$ has length 1.

Next, we will bound the distances between any two different pair of points in the strong product graph. For this aim we must distinguish some cases depending on the situation of the considered points. Let $p \in G_1$ and $q \in G_2$ be two points of G_1 and G_2 respectively. The pair $(p; q)$ is an *inner point* in $G_1 \boxtimes G_2$, if $p \in G_1 \setminus V(G_1)$ and $q \in V(G_2)$ or $p \in V(G_1)$ and $q \in G_2 \setminus V(G_2)$ or $p \in G_1 \setminus V(G_1)$ and $q \in G_2 \setminus V(G_2)$ (i.e., $(p; q) \in G_1 \boxtimes G_2 \setminus V(G_1 \boxtimes G_2)$). Notice that the first and second cases of the inner points in $G_1 \boxtimes G_2$ are contained in the Cartesian product graph $G_1 \square G_2 \subset G_1 \boxtimes G_2$; so the first and second cases are the inner points of the Cartesian edges properly. In order to represent the inner points of the non Cartesian edges in $G_1 \boxtimes G_2$ we will consider the following assumptions. Let $[A_1, A_2] \in E(G_1)$ and $[B_1, B_2] \in E(G_2)$ be edges in G_1 and G_2, respectively. Let $p \in [A_1, A_2]$ and $q \in [B_1, B_2]$ be inner points of theses fixed edges; we have $(p; q) \in G_1 \boxtimes G_2 \setminus G_1 \square G_2$ if $L([pA_1]) = L([qB_1])$ or $L([pA_1]) = L([qB_2])$.

Notice that there are different points on $G_1 \boxtimes G_2$ with the same representation: the midpoints of $[(A_1; B_1), (A_2; B_2)]$ and $[(A_1; B_2), (A_2; B_1)]$. Then, this notation is ambiguous, but it is convenient.

The following lemmas provide bounds on the distance between any two pair of points in the strong product graph $(p_1; q_1), (p_2; q_2) \in G_1 \boxtimes G_2$.

The first one is a well known property about distances between vertices in the strong product of graphs proved in [26].

Lemma 2 (Lemma 5.1 in [26]). *Let G_1, G_2 be any graphs. If $p_1, p_2 \in V(G_1)$ and $q_1, q_2 \in V(G_2)$, then*

$$d_{G_1 \boxtimes G_2}((p_1; q_1), (p_2; q_2)) = \max\{d_{G_1}(p_1, p_2), d_{G_2}(q_1, q_2)\}.$$

Next, a lower bound on the distance between any two points in the strong product graph.

Proposition 3. *Let G_1, G_2 be any graphs. For every $(p_1; q_1), (p_2; q_2) \in G_1 \boxtimes G_2$ we have*

$$d_{G_1 \boxtimes G_2}((p_1; q_1), (p_2; q_2)) \geqslant \max\{d_{G_1}(p_1, p_2), d_{G_2}(q_1, q_2)\}. \quad (1)$$

Proof. By symmetry, it suffices to prove $d_{G_1 \boxtimes G_2}((p_1; q_1), (p_2; q_2)) \geqslant d_{G_1}(p_1, p_2)$. Seeking for a contradiction, assume that $d_{G_1 \boxtimes G_2}((p_1; q_1), (p_2; q_2)) < d_{G_1}(p_1, p_2)$.

Hence, there exist a geodesic Γ joining $(p_1; q_1)$ and $(p_2; q_2)$ in $G_1 \boxtimes G_2$ with $L(\Gamma) < d_{G_1}(p_1, p_2)$. Denote by $(A_1; B_1), \dots, (A_k; B_k)$ the vertices of $G_1 \boxtimes G_2$ in Γ; without loss of generality we can assume that Γ meets $(A_1;B_1),\dots,(A_k;B_k)$ in this order. Then, we have

$$\Gamma := [(p_1;q_1)(A_1;B_1)]\bigcup\left\{\bigcup_{j=1}^{k-1}[(A_j;B_j),(A_{j+1};B_{j+1})]\right\}\bigcup[(A_k;B_k)(p_2;q_2)].$$

By Definition 1, we obtain that

$$\gamma := [p_1A_1]\bigcup\left\{\bigcup_{j=1}^{k-1}[A_jA_{j+1}]\right\}\bigcup[A_kp_2]$$

is a path joining p_1 and p_2 such that $L(\gamma)\leqslant L(\Gamma)<d_{G_1}(p_1,p_2)$. This is the contradiction we were looking for. □

The following result provides an upper bound for the distance between a vertex and an inner point, as well as between two inner points in $G_1\boxtimes G_2$.

Proposition 4. *Let G_1, G_2 be any graphs.*

(i) If $(u;v)\in V(G_1\boxtimes G_2)$ and $(p;q)\in G_1\boxtimes G_2\setminus V(G_1\boxtimes G_2)$, then

$$d_{G_1\boxtimes G_2}((u;v),(p;q))\leqslant \max\{d_{G_1}(u,p),d_{G_2}(v,q)\}+1. \tag{2}$$

(ii) If $(p_1;q_1),(p_2;q_2)\in G_1\boxtimes G_2\setminus V(G_1\boxtimes G_2)$, then

$$d_{G_1\boxtimes G_2}((p_1;q_1),(p_2;q_2))\leqslant \max\{d_{G_1}(p_1,p_2),d_{G_2}(q_1,q_2)\}+2. \tag{3}$$

Proof. In order to prove (i), let us consider $[(u_1;v_1),(u_2;v_2)]\in E(G_1\boxtimes G_2)$ such that $(p;q)\in[(u_1;v_1),(u_2;v_2)]$. Let γ be a geodesic in $G_1\boxtimes G_2$ joining $(u;v)$ and $(p;q)$. Without loss of generality we can assume that $(u_1;v_1)\in\gamma$. Define $\varepsilon := d_{G_1\boxtimes G_2}((u_1;v_1),(p;q))$. By Lemma 2, we have

$$\begin{aligned} d_{G_1\boxtimes G_2}((u;v),(p;q)) &= \max\{d_{G_1}(u,u_1),d_{G_2}(v,v_1)\}+\varepsilon \\ &\leqslant \max\{d_{G_1}(u,p)+d_{G_1}(p,u_1),d_{G_2}(v,q)+d_{G_2}(q,v_1)\}+\varepsilon \\ &\leqslant \max\{d_{G_1}(u,p),d_{G_2}(v,q)\}+2\varepsilon. \end{aligned}$$

If $\varepsilon\leqslant 1/2$, then we have (2). If $\varepsilon>1/2$, then we have $\max\{d_{G_1}(u,u_2),d_{G_2}(v,v_2)\} = \max\{d_{G_1}(u,u_1),d_{G_2}(v,v_1)\}+1$; thus, $d_{G_1\boxtimes G_2}\big((u;v),(p;q)\big)=\max\{d_{G_1}(u,p),d_{G_2}(v,q)\}$.

In order to proof (ii), notice that if $(p_1;q_1),(p_2;q_2)$ belong to the same edge of $G_1\boxtimes G_2$, then we have the result since $d_{G_1\boxtimes G_2}((p_1;q_1),(p_2;q_2))<1$. Assume now that $(p_1;q_1),(p_2;q_2)$ belong to different edges of $G_1\boxtimes G_2$. Let us consider $(u_1;v_1),(u_2;v_2),(u_3;v_3),(u_4;v_4)\in V(G_1\boxtimes G_2)$ such that $(p_1;q_1)\in[(u_1;v_1),(u_2;v_2)]$ and $(p_2;q_2)\in[(u_3;v_3),(u_4;v_4)]$. Let γ^* be a geodesic in $G_1\boxtimes G_2$ joining $(p_1;q_1)$ and $(p_2;q_2)$. Without loss of generality we can assume that $(u_2;v_2),(u_3;v_3) \in \gamma^*$. Define $\varepsilon_1 := d_{G_1\boxtimes G_2}((u_2;v_2),(p_1;q_1))$ and $\varepsilon_2 := d_{G_1\boxtimes G_2}((u_3;v_3),(p_2;q_2))$. Then, we have

$$\begin{aligned} d_{G_1\boxtimes G_2}((p_1;q_1),(p_2;q_2)) &= \varepsilon_1 + \max\{d_{G_1}(u_2,u_3), d_{G_2}(v_2,v_3)\} + \varepsilon_2 \\ &\leqslant 2\varepsilon_1 + \max\{d_{G_1}(p_1,p_2), d_{G_2}(q_1,q_2)\} + 2\varepsilon_2. \end{aligned}$$

Notice that if $\varepsilon_1, \varepsilon_2 \leqslant 1/2$, then (3) holds directly. If $\varepsilon_1 > 1/2$ (the case $\varepsilon_2 > 1/2$ is analogous), then $\max\{d_{G_1}(u_1,u_3), d_{G_2}(v_1,v_3)\} = \max\{d_{G_1}(u_2,u_3), d_{G_2}(v_2,v_3)\} + 1$; thus, $d_{G_1\boxtimes G_2}\big((p_1;q_1),(u_3;v_3)\big) = \max\{d_{G_1}(p_1,u_3), d_{G_2}(q_1,v_3)\}$. Hence, we have

$$\begin{aligned} d_{G_1\boxtimes G_2}\big((p_1;q_1),(p_2;q_2)\big) &= \max\{d_{G_1}(p_1,u_3), d_{G_2}(q_1,v_3)\} + \varepsilon_2 \\ &\leqslant \max\{d_{G_1}(p_1,p_2) + d_{G_1}(p_2,u_3), d_{G_2}(q_1,q_2) + d_{G_2}(q_2,v_3)\} + \varepsilon_2 \\ &\leqslant \max\{d_{G_1}(p_1,p_2), d_{G_2}(q_1,q_2)\} + 2\varepsilon_2. \end{aligned}$$

This finishes the proof. □

The previous lemmas let us announce the following general result on the distances in the strong product of two graphs.

Theorem 5. *For all graphs G_1, G_2 we have:*

a) $d_{G_1\boxtimes G_2}((p_1;q_1),(p_2;q_2)) = \max\{d_{G_1}(p_1,p_2), d_{G_2}(q_1,q_2)\}$, *for every* $(p_1;q_1),(p_2;q_2) \in V(G_1 \boxtimes G_2)$,

b) $\max\{d_{G_1}(p_1,p_2), d_{G_2}(q_1,q_2)\} \leqslant d_{G_1\boxtimes G_2}((p_1;q_1),(p_2;q_2)) \leqslant \max\{d_{G_1}(p_1,p_2), d_{G_2}(q_1,q_2)\} +1$, *for every* $(p_1;q_1) \in V(G_1 \boxtimes G_2)$ *and* $(p_2;q_2) \in G_1 \boxtimes G_2$,

c) $\max\{d_{G_1}(p_1,p_2), d_{G_2}(q_1,q_2)\} \leqslant d_{G_1\boxtimes G_2}((p_1;q_1),(p_2;q_2)) \leqslant \max\{d_{G_1}(p_1,p_2), d_{G_2}(q_1,q_2)\} +2$, *for every* $(p_1;q_1),(p_2;q_2) \in G_1 \boxtimes G_2$.

Let us consider the projection $P_k : G_1 \boxtimes G_2 \longrightarrow G_k$ for $k \in \{1,2\}$.

Corollary 6. *Let $\{i,j\}$ be a permutation of $\{1,2\}$. Then, for every x,y in $G_1 \boxtimes G_2$,*

$$d_{G_i}(P_i(x),P_i(y)) \leqslant d_{G_1\boxtimes G_2}(x,y) \leqslant d_{G_i}(P_i(x),P_i(y)) + \operatorname{diam} G_j + 2. \quad (4)$$

These results provide information about the geodesics in $G_1 \boxtimes G_2$. Notice that, if γ is a geodesic joining x and y in $G_1 \boxtimes G_2$, then it is possible that $P_j(\gamma)$ does not contain a geodesic joining $P_j(x)$ and $P_j(y)$ in G_j, as the following example shows.

Example 7. Consider a cycle graph G_1 with vertices $\{v_1,\dots,v_n\}$ such that $v_i \sim v_{i+1}$ for every $i \in \{1,\dots,n-1\}$ and a path graph G_2 with vertices $\{w_1,\dots,w_n\}$ such that $w_i \sim w_{i+1}$ for every $i \in \{1,\dots,n-1\}$. By Lemma 2, we have that $\gamma := \cup_{i=1}^{n-1}[(v_i;w_i),(v_{i+1};w_{i+1})]$ is a geodesic joining $(v_1;w_1)$ and $(v_n;w_n)$ in $G_1 \boxtimes G_2$, but $P_1(\gamma) = \cup_{i=1}^{n-1}[v_i,v_{i+1}]$ does not contain the geodesic joining v_1 and v_n in G_1 (the edge $[v_1,v_n]$).

The following result allows to compute the diameter of the strong product of two graphs. We denote by E_1 the graph with just a single vertex.

Theorem 8. *Let G_1, G_2 be any graphs. Then we have*

$$\operatorname{diam} G_1 \boxtimes G_2 = \begin{cases} \max\{\operatorname{diam} G_1, \operatorname{diam} G_2\}, & \text{if } G_1 \text{ or } G_2 \text{ is an isomorphic graph to } E_1, \\ \max\{\operatorname{diam} V(G_1), \operatorname{diam} V(G_2)\} + 1, & \text{otherwise.} \end{cases}$$

Proof. Since for any graph G, $E_1 \boxtimes G$ is isomorphic to G we have the first equality. By Lemma 2, we have $\operatorname{diam} V(G_1 \boxtimes G_2) = \max\{\operatorname{diam} V(G_1), \operatorname{diam} V(G_2)\}$; hence,

$$\max\{\operatorname{diam} V(G_1), \operatorname{diam} V(G_2)\} \leqslant \operatorname{diam} G_1 \boxtimes G_2 \leqslant \max\{\operatorname{diam} V(G_1), \operatorname{diam} V(G_2)\} + 1.$$

Without loss of generality we can assume that $\operatorname{diam} V(G_1) \leqslant \operatorname{diam} V(G_2)$. If $\operatorname{diam} V(G_2) = \infty$, then the inequality holds. Hence, we can assume that G_1 and G_2 are bounded. Let B_1, B_2 be vertices of G_2 such that $d_{G_2}(B_1, B_2) = \operatorname{diam} V(G_2)$, and let A_1, A_2 be two adjacent vertices of G_1. Let M_1 (respectively, M_2) be the midpoint of $[(A_1; B_1), (A_2; B_1)]$ (respectively, $[(A_1; B_2), (A_2; B_2)]$). One can check that $d_{G_1 \boxtimes G_2}(M_1, M_2) = \operatorname{diam} V(G_2) + 1$.

This finish the proof. □

Note that, in particular, $\operatorname{diam} G_1 \boxtimes G_2 = \operatorname{diam} V(G_1 \boxtimes G_2) + 1$ if G_1 and G_2 are not isomorphic to E_1.

We say that a subgraph Γ of G is *isometric* if $d_\Gamma(x, y) = d_G(x, y)$ for every $x, y \in \Gamma$.

We can deduce several results from Theorem 8. The first one says that $\max\{\operatorname{diam} G_1, \operatorname{diam} G_2\}$ is a good approximation of the diameter of $G_1 \boxtimes G_2$.

Corollary 9. *For all graphs G_1, G_2 we have*

$$\max\{\operatorname{diam} G_1, \operatorname{diam} G_2\} \leqslant \operatorname{diam} G_1 \boxtimes G_2 \leqslant \max\{\operatorname{diam} G_1, \operatorname{diam} G_2\} + 1.$$

Proof. If V is a vertex of G_1 (respectively, G_2), then, by Proposition 3, we have that $\{V\} \boxtimes G_2$ (respectively, $G_1 \boxtimes \{V\}$) is an isometric subgraph of $G_1 \boxtimes G_2$. Hence, we obtain the first inequality. The second one is a consequence of Theorem 8 and the inequality $\operatorname{diam} V(G) \leqslant \operatorname{diam} G$. □

Furthermore, we characterize the graphs with $\operatorname{diam} G_1 \boxtimes G_2 = \max\{\operatorname{diam} G_1, \operatorname{diam} G_2\}$.

Corollary 10. *The equality* $\operatorname{diam} G_1 \boxtimes G_2 = \max\{\operatorname{diam} G_1, \operatorname{diam} G_2\}$ *holds if and only if G_1 or G_2 is isomorphic to E_1, or* $\operatorname{diam} G = \operatorname{diam} V(G) + 1$ *for $G \in \{G_1, G_2\}$ with* $\operatorname{diam} G = \max\{\operatorname{diam} G_1, \operatorname{diam} G_2\}$.

3 Bounds for the hyperbolicity constant.

Some bounds for the hyperbolicity constant of the strong product of two graphs are studied in this section. These bounds allow to prove Theorem 23, which characterizes the hyperbolic strong product graphs. The next well-known result will be useful.

Theorem 11 (Theorem 8 in [42])**.** *In any graph G the inequality $\delta(G) \leqslant \frac{1}{2} \operatorname{diam} G$ holds and it is sharp.*

Thanks to the Theorems 8 and 11 we obtain the following consequence.

Corollary 12. *For all graphs G_1, G_2, we have*

$$\delta(G_1 \boxtimes G_2) \leqslant \frac{\max\{\operatorname{diam} V(G_1), \operatorname{diam} V(G_2)\} + 1}{2},$$

and the inequality is sharp.

Theorems 32, 34 and 35 are families of examples for which the equality in the previous corollary is attained.

Taking into account that $E_1 \boxtimes G$ is an isomorphic graph to G, we have the following result.

Corollary 13. *For every graph G we have*

$$\delta(G \boxtimes E_1) = \delta(E_1 \boxtimes G) = \delta(G).$$

The next result will be useful.

Lemma 14 (Lemma 5 in [42])**.** *If Γ is an isometric subgraph of G, then $\delta(\Gamma) \leqslant \delta(G)$.*

All the previous results allow us to present the following theorem which provides some lower bounds for $\delta(G_1 \boxtimes G_2)$.

Theorem 15. *For all graphs G_1, G_2 we have:*

(a) $\delta(G_1 \boxtimes G_2) \geqslant \max\{\delta(G_1), \delta(G_2)\}$,

(b) $\delta(G_1 \boxtimes G_2) \geqslant \frac{1}{2} \min\{\operatorname{diam} V(G_1), \operatorname{diam} V(G_2)\}$,

(c) $\delta(G_1 \boxtimes G_2) \geqslant \frac{1}{2} \big(\operatorname{diam} V(G_1) + 1\big)$, *if* $0 < \operatorname{diam} V(G_1) < \operatorname{diam} V(G_2)$,

(d) $\delta(G_1 \boxtimes G_2) \geqslant \frac{1}{4} \min\{\operatorname{diam} V(G_1) + 2\delta(G_2), \operatorname{diam} V(G_2) + 2\delta(G_1)\}$.

Proof. Part (a) is immediate due to $G_1 \boxtimes \{v\}$ and $\{u\} \boxtimes G_2$ are isometric subgraphs of $G_1 \boxtimes G_2$ for every $(u;v) \in V(G_1 \boxtimes G_2)$. Then Lemma 14 gives that $\delta(G_1 \boxtimes G_2) \geqslant \delta(G_1 \boxtimes \{v\}) = \delta(G_1)$ and $\delta(G_1 \boxtimes G_2) \geqslant \delta(\{u\} \boxtimes G_2) = \delta(G_2)$. Hence, we obtain $\delta(G_1 \boxtimes G_2) \geqslant \max\{\delta(G_1), \delta(G_2)\}$.

Let $D := \min\{\operatorname{diam} V(G_1), \operatorname{diam} V(G_2)\}$.

Let us prove (b). If $D = 0$, then (b) holds; so, we just consider $D > 0$. If $D < \infty$, let us consider a geodesic square $K := \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ in $G_1 \Box G_2 \subset G_1 \boxtimes G_2$ with sides of length D; then $T := \{\gamma_1, \gamma_2, \gamma\}$ is a geodesic triangle in $G_1 \boxtimes G_2$, where γ is a "diagonal" geodesic joining the endpoints of $\gamma_1 \cup \gamma_2$. It is clear that the midpoint p of γ satisfies $d_{G_1 \boxtimes G_2}(p, \gamma_1 \cup \gamma_2) = D/2$; therefore $\delta(T) \geqslant D/2$ and, consequently, $\delta(G_1 \boxtimes G_2) \geqslant D/2$. If $D = \infty$, we can repeat the same argument for any integer N instead of D, and we obtain $\delta(G_1 \boxtimes G_2) \geqslant N/2$, for every N: hence, $\delta(G_1 \boxtimes G_2) = \infty = D/2$.

In order to prove (c), note that $D < \infty$. Let us consider a geodesic rectangle $R := \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ in $G_1 \Box G_2 \subset G_1 \boxtimes G_2$ with $L(\sigma_1) = L(\sigma_3) = \operatorname{diam} V(G_1)$ and $L(\sigma_2) = L(\sigma_4) = \operatorname{diam} V(G_1) + 1$. Denote by γ a geodesic in $G_1 \boxtimes G_2$ joining the endpoints of $\sigma_1 \cup \sigma_2$ which contains the edge in σ_4 incident to $\sigma_1 \cap \sigma_4$; we may choose γ such that it contains a diagonal of a geodesic square in $G_1 \boxtimes G_2$. Then $B := \{\sigma_1, \sigma_2, \gamma\}$ is a geodesic triangle in $G_1 \boxtimes G_2$. If p is the midpoint of γ, then

$$d_{G_1 \boxtimes G_2}(p, \sigma_1 \cup \sigma_2) = \frac{\operatorname{diam} V(G_1) + 1}{2}.$$

Consequently, $\delta(G_1 \boxtimes G_2) \geqslant \delta(B) \geqslant (\operatorname{diam} V(G_1) + 1)/2$.

Finally, (d). Let $E := \max\{\delta(G_1), \delta(G_2)\}$. Then from parts (a) and (b), we have

$$\begin{aligned}
\delta(G_1 \boxtimes G_2) \geqslant \max\left\{\frac{D}{2}, E\right\} &\geqslant \frac{1}{2}\left(\frac{D}{2} + E\right) \\
&= \frac{1}{4}\min\{\operatorname{diam} V(G_1) + 2E, \operatorname{diam} V(G_2) + 2E\} \\
&\geqslant \frac{1}{4}\min\{\operatorname{diam} V(G_1) + 2\delta(G_2), \operatorname{diam} V(G_2) + 2\delta(G_1)\}.
\end{aligned}$$

□

Theorems 34 and 35 provide a family of examples for which the equality in Theorem 15 (a) is attained.

Corollary 12 and Theorem 15 provide lower and upper bounds for $\delta(G_1 \boxtimes G_2)$ just in terms of distances in G_1 and G_2.

Corollary 16. *For all graphs G_1, G_2, we have*

$$\frac{1}{2}\min\{\operatorname{diam} V(G_1), \operatorname{diam} V(G_2)\} \leqslant \delta(G_1 \boxtimes G_2) \leqslant \frac{1}{2}\big(\max\{\operatorname{diam} V(G_1), \operatorname{diam} V(G_2)\} + 1\big).$$

From Theorem 15 we have obtained several interesting consequences. The following one is a qualitative result about the hyperbolicity of $G_1 \boxtimes G_2$.

Theorem 17. *If G_1 and G_2 are infinite graphs, then $G_1 \boxtimes G_2$ is not hyperbolic.*

Theorem 18. *Let G_1, G_2 be graphs with at least two vertices. Let m and M be the minimum and the maximum between* $\operatorname{diam} V(G_1)$ *and* $\operatorname{diam} V(G_2)$*, respectively. Then we have*

$$\delta(G_1 \boxtimes G_2) \geqslant \min\left\{m + \frac{1}{2}, \frac{M}{2}\right\}. \tag{5}$$

Proof. First of all, we prove

$$\delta(G_1 \boxtimes G_2) \geqslant \min\left\{m, \frac{M}{2}\right\}. \tag{6}$$

In order to prove this inequality, assume first that $2m \leqslant M$. If $m < \infty$, then let us consider a geodesic rectangle $R := \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ in $G_1 \Box G_2 \subset G_1 \boxtimes G_2$ with $L(\gamma_1) = L(\gamma_3) = 2m$ and $L(\gamma_2) = L(\gamma_4) = m$, and consider a geodesic γ joining the endpoints of γ_1 and containing the midpoint of γ_3; then $B := \{\gamma_1, \gamma\}$ is a geodesic bigon in $G_1 \boxtimes G_2$. If p is the midpoint of γ_3, then $d_{G_1 \boxtimes G_2}(p, \gamma_1) = m$; therefore $\delta(B) \geqslant m$, and consequently $\delta(G_1 \boxtimes G_2) \geqslant m$. If $m = \infty$, then we can repeat the same argument for any integer N instead of m, and we obtain $\delta(G_1 \boxtimes G_2) \geqslant N$, for every N; hence, $\delta(G_1 \boxtimes G_2) = \infty = m$.

If $2m > M$, then $M < \infty$ and we can repeat the previous argument with $\lfloor M/2 \rfloor$ instead of m, and we obtain the result when M is even. If M is odd, let us consider a geodesic rectangle $R := \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ in $G_1 \Box G_2 \subset G_1 \boxtimes G_2$ with $L(\gamma_1) = L(\gamma_3) = 2\lfloor M/2 \rfloor + 1 = M$ and $L(\gamma_2) = L(\gamma_4) = \lfloor M/2 \rfloor$; let p_1, p_2 be points on γ_3 such that $d_{G_1 \boxtimes G_2}(p_1, \gamma_4) = \lfloor M/2 \rfloor$ and $d_{G_1 \boxtimes G_2}(p_2, \gamma_2) = \lfloor M/2 \rfloor$; consider a geodesic γ joining the endpoints of γ_1 and containing p_1 and p_2; then $B := \{\gamma_1, \gamma\}$ is a geodesic bigon in $G_1 \boxtimes G_2$. Denote by p the midpoint of $[p_1 p_2] \subset \gamma_3$; so, $d_{G_1 \boxtimes G_2}(p, \gamma_1) = M/2$; therefore, $\delta(G_1 \boxtimes G_2) \geqslant \delta(B) \geqslant M/2$.

Since we have proved (6), in order to obtain (5), we can assume that $0 < 2m < M$; then we have $m < \infty$. If we replace $\lfloor M/2 \rfloor$ by m in the previous argument, we obtain $\delta(G_1 \boxtimes G_2) \geqslant m + 1/2$. □

Corollary 33 and Theorems 34 and 35 show that the inequality in Theorem 18 is sharp.

Theorem 19. *Let G_1, G_2 be any graphs. Let m and M be the minimum and the maximum between* $\operatorname{diam} V(G_1)$ *and* $\operatorname{diam} V(G_2)$*, respectively. If $2m \geqslant M$, then*

$$\frac{M}{2} \leqslant \delta(G_1 \boxtimes G_2) \leqslant \frac{M+1}{2}. \tag{7}$$

Furthermore, if $2m > M > 0$, then

$$\delta(G_1 \boxtimes G_2) = \frac{M+1}{2}. \tag{8}$$

Proof. If $M = 0$, then $\delta(G_1 \boxtimes G_2) = 0$ and (7) holds. If $M > 0$, then, by Corollary 12 and Theorem 18, the inequalities in (7) hold directly.

In order to prove (8), without loss of generality we can assume that $\operatorname{diam} V(G_1) = m$ and $\operatorname{diam} V(G_2) = M$. Assume first that M is an even number. Since $m > M/2$, let us consider $A_0, A_1, \dots, A_{M/2+1} \in V(G_1)$ and $B_0, B_1, \dots, B_M \in V(G_2)$ with $\gamma_1 := A_0 A_1 \dots A_{M/2+1}$ is a geodesic in G_1 and $\gamma_2 := B_0 B_1 \dots B_M$ is a geodesic in G_2. Denote by X (respectively, Y) the midpoint of $[(A_0; B_0), (A_1; B_0)]$ (respectively, $[(A_0; B_M), (A_1; B_M)]$). Let us consider

$$\Gamma^* := [X(A_0; B_0)] \bigcup \left\{ \bigcup_{i=1}^{M} [(A_0; B_{i-1}), (A_0; B_i)] \right\} \bigcup [(A_0; B_M)Y]$$

and

$$\begin{aligned} \Gamma' := & [X(A_1; B_0)] \bigcup \left\{ \bigcup_{i=1}^{M/2} [(A_i; B_{i-1}), (A_{i+1}; B_i)] \right\} \bigcup \\ & \bigcup \left\{ \bigcup_{j=M/2+1}^{M} [(A_{M+2-j}; B_{j-1}), (A_{M+1-j}; B_j)] \right\} \bigcup [(A_1; B_M)Y]. \end{aligned}$$

Then $B := \{\Gamma^*, \Gamma'\}$ is a geodesic bigon in $G_1 \boxtimes G_2$. If p is the midpoint of Γ', then $d_{G_1 \boxtimes G_2}(p, \Gamma^*) = (M+1)/2$; therefore, $\delta(G_1 \boxtimes G_2) \geqslant \delta(B) \geqslant (M+1)/2$. Then, Corollary 12 gives the equality.

Assume now that M is an odd number. Since $m \geqslant (M+1)/2$, let us consider $A_0, A_1, \dots, A_{(M+1)/2} \in V(G_1)$ and $B_0, B_1, \dots, B_M \in V(G_2)$ with $\gamma_1 := A_0 A_1 \dots A_{(M+1)/2}$ is a geodesic in G_1 and $\gamma_2 := B_0 B_1 \dots B_M$ is a geodesic in G_2. Denote by X (respectively, Y) the midpoint of $[(A_0; B_0), (A_1; B_0)]$ (respectively, $[(A_0; B_M), (A_1; B_M)]$). Let us consider

$$\Gamma^* := [X(A_0; B_0)] \bigcup \left\{ \bigcup_{i=1}^{M} [(A_0; B_{i-1}), (A_0; B_i)] \right\} \bigcup [(A_0; B_M)Y]$$

and

$$\begin{aligned} \Gamma' := & [X(A_1; B_0)] \bigcup \left\{ \bigcup_{i=1}^{(M-1)/2} [(A_i; B_{i-1}), (A_{i+1}; B_i)] \right\} \bigcup \\ & \bigcup [(A_{(M+1)/2}; B_{(M-1)/2}), (A_{(M+1)/2}; B_{(M+1)/2})] \bigcup \\ & \bigcup \left\{ \bigcup_{j=(M+1)/2}^{M} [(A_{M+1-j}; B_{j-1}), (A_{M-j}; B_j)] \right\} \bigcup [(A_1; B_M)Y]. \end{aligned}$$

Then $B := \{\Gamma^*, \Gamma'\}$ is a geodesic bigon in $G_1 \boxtimes G_2$. If p is the midpoint of Γ', then $d_{G_1 \boxtimes G_2}(p, \Gamma^*) = (M+1)/2$; therefore, $\delta(G_1 \boxtimes G_2) \geqslant \delta(B) \geqslant (M+1)/2$. Finally, Corollary 12 gives the equality. $\square$

Theorems 34 and 35 show that the first inequality in Theorem 19 is attained.

Let X be a metric space, Y a non-empty subset of X and ε a positive number. We call *ε-neighborhood* of Y in X, denoted by $V_\varepsilon(Y)$ to the set $\{x \in X : d_X(x, Y) \leqslant \varepsilon\}$.

The next result will be useful in order to prove the upper bound for $\delta(G_1 \boxtimes G_2)$ in Theorem 21 below.

Theorem 20 (Theorem 2.9 in [41])**.** *Let X be a δ-hyperbolic geodesic metric space, $u, v \in X$, b a non-negative constant, h a curve joining u and v with $L(h) \leqslant d(u, v) + b$, and $g = [uv]$. Then,*

$$h \subseteq V_{8\delta+b/2}(g), \qquad g \subseteq V_{16\delta+b}(h).$$

Theorem 21. *Let G_1, G_2 be any graphs. Then, we have*

$$\delta(G_1 \boxtimes G_2) \leqslant \frac{5}{2} \operatorname{diam} G_1 + 25\delta(G_2) + 5. \tag{9}$$

Proof. It suffices to prove (9) if G_1 is bounded and G_2 is hyperbolic, since otherwise the inequality $\delta(G_1 \boxtimes G_2) \leqslant \infty$ holds. Let us consider any fixed geodesic triangle $T = \{x, y, z\}$ in $G_1 \boxtimes G_2$ and $\alpha \in T$. In order to bound $\delta(T)$, without loss of generality we can assume that $\alpha \in [xy]$. Consider the projection $P_2 : G_1 \boxtimes G_2 \longrightarrow G_2$ and any geodesic $\gamma := [uv]$ in $G_1 \boxtimes G_2$. By Corollary 6, we obtain

$$L\big(P_2(\gamma)\big) \leqslant L(\gamma) = d_{G_1 \boxtimes G_2}(u, v) \leqslant d_{G_2}\big(P_2(v), P_2(v)\big) + b, \qquad \text{with } b = \operatorname{diam} G_1 + 2.$$

Then, by Theorem 20, there is $\alpha' \in [P_2(x)P_2(y)]$ such that

$$d_{G_2}(P_2(\alpha), \alpha') \leqslant 8\delta(G_2) + \frac{b}{2}. \tag{10}$$

Since G_2 is hyperbolic, there is $\beta' \in [P_2(y)P_2(z)] \cup [P_2(z)P_2(x)]$ such that

$$d_{G_2}(\alpha', \beta') \leqslant \delta(G_2). \tag{11}$$

By Theorem 20, there is $\beta'' \in P_2([yz] \cup [zx])$ such that

$$d_{G_2}(\beta', \beta'') \leqslant 16\delta(G_2) + b. \tag{12}$$

Consequently, by (10), (11) and (12) we obtain

$$d_{G_2}(P_2(\alpha), P_2([yz] \cup [zx])) \leqslant d_{G_2}(P_2(\alpha), \beta'') \leqslant 25\delta(G_2) + \frac{3b}{2}. \tag{13}$$

Finally, by Corollary 6 and (13) we obtain

$$d_{G_1 \boxtimes G_2}(\alpha, [yz] \cup [zx]) \leqslant d_{G_2}(P_2(\alpha), P_2([yz] \cup [zx])) + b \leqslant 25\delta(G_2) + \frac{5b}{2}.$$

This finishes the proof. $\square$

Theorems 15 and 21 provide lower and upper bounds of $\delta(G_1 \boxtimes G_2)$ in terms of linear combinations of hyperbolicity constants and diameters of its generator graphs, as the following result shows.

Corollary 22. *For all graphs G_1, G_2, we have*

$$\frac{1}{4}\min\{2\delta(G_1) + \operatorname{diam} V(G_2), 2\delta(G_2) + \operatorname{diam} V(G_1)\} \leqslant \delta(G_1 \boxtimes G_2)$$

$$\leqslant \frac{5}{2}\min\{\operatorname{diam} G_1 + 10\delta(G_2), \operatorname{diam} G_2 + 10\delta(G_1)\} + 5.$$

Corollary 22 allows to obtain the main result of this work: the characterization of the hyperbolic graphs $G_1 \boxtimes G_2$.

Theorem 23. *For all graphs G_1, G_2 we have that $G_1 \boxtimes G_2$ is hyperbolic if and only if G_1 is hyperbolic and G_2 is bounded or G_2 is hyperbolic and G_1 is bounded.*

Many parameters γ of graphs satisfy the inequality $\gamma(G_1 \boxtimes G_2) \geqslant \gamma(G_1) + \gamma(G_2)$. Therefore, one could think that the inequality $\delta(G_1 \boxtimes G_2) \geqslant \delta(G_1) + \delta(G_2)$ holds for all graphs G_1, G_2. However, this is false, as the following example shows:

Example 24. $\delta(P \boxtimes C_4) < \delta(P) + \delta(C_4)$, where P is the Petersen graph.

We have that $\operatorname{diam} V(P) = 2$, $\operatorname{diam} V(C_4) = 2$. Besides, Theorem 11 in [42] gives that $\delta(P) = 3/2$ and $\delta(C_4) = 1$. By Theorem 19, we obtain $\delta(P \boxtimes C_4) = 3/2 < 3/2 + 1 = \delta(P) + \delta(C_4)$.

The inequality $\delta(G_1 \boxtimes G_2) \leqslant \delta(G_1) + \delta(G_2)$ is also false, since $\delta(P_2 \boxtimes P_2) = \delta(K_4) = 1 > 2\delta(P_2) = 0$.

4 Computation of the hyperbolicity constant for some product graphs

This last section present the value of the hyperbolicity constant for many product of graphs.

The following results in [4] will be useful. Denote by $J(G)$ the set of vertices and midpoints of edges in G. As usual, by *cycle* we mean a simple closed curve, i.e., a path with different vertices, unless the last one, which is equal to the first vertex.

First, remark some previous results of [4] which will be useful.

Theorem 25 (Theorem 2.6 in [4])**.** *For every hyperbolic graph G, $\delta(G)$ is a multiple of 1/4.*

Theorem 26 (Theorem 2.7 in [4])**.** *For any hyperbolic graph G, there exists a geodesic triangle $T = \{x, y, z\}$ that is a cycle with $x, y, z \in J(G)$ and $\delta(T) = \delta(G)$.*

Remark 27. By Theorems 25 and 26, in order to compute the hyperbolicity constant of a graph G it suffices to consider $d_G(p,[xz]\cup[yz])$ where $T=\{x,y,z\}$ is a geodesic triangle that is a cycle with $x,y,z\in J(G)$ and $p\in[xy]$ satisfies $d_G(p,V(G))\in\{0,1/4,1/2\}$.

The following results characterize the hyperbolicity constant of the strong product of trees and certain graphs. These results are interesting by themselves and, furthermore, they will be useful in order to prove the last theorems of this paper.

Theorem 28. *Let T be any tree and G any graph with $0<\operatorname{diam}V(G)<\operatorname{diam}T/2$. Then, we have*

$$\delta(G\boxtimes T)=\operatorname{diam}V(G)+\frac{1}{2}.$$

Proof. On the one hand, Theorem 18 gives $\delta(G\boxtimes T)\geqslant\operatorname{diam}V(G)+1/2$. On the other hand, by Theorem 26 it suffices to consider geodesic triangles $\triangle=\{x,y,z\}$ in $G\boxtimes T$ which are cycles with $x,y,z\in J(G\boxtimes T)$. Let $(v;w)$ be a vertex in $[xy]$. If $d_{G\boxtimes T}((v;w),\{x,y\})\leqslant\operatorname{diam}V(G)$, then $d_{G\boxtimes T}((v;w),[yz]\cup[zx])\leqslant\operatorname{diam}V(G)$. Assume that $d_{G\boxtimes T}((v;w),\{x,y\})>\operatorname{diam}V(G)$. Let V_x (respectively, V_y) be the closest vertex to x (respectively, y) in $[xy]$. Note that $d_{G\boxtimes T}(V_x,V_y)=d_{G\boxtimes T}(V_x,(v;w))+d_{G\boxtimes T}((v;w),V_y)\geqslant 2\operatorname{diam}V(G)$. Consider the projection P_T on T. By Lemma 2 we have $d_{G\boxtimes T}(V_x,V_y)=d_T(P_T(V_x),P_T(V_y))$. Due to $d_T(P_T(V_x),P_T(V_y))\leqslant d_T(P_T(V_x),w)+d_T(w,P_T(V_y))$, we have $d_{G\boxtimes T}(V_x,(v;w))=d_T(P_T(V_x),w)$ and $d_{G\boxtimes T}((v;w),V_y)=d_T(w,P_T(V_y))$. Then, $w\in[P_T(x)P_T(y)]=P_T([xy])$. Since T is a tree, $w\in P_T\big([yz]\cup[zx]\big)$. Then, $([yz]\cup[zx])\cap(G\boxtimes\{w\})\neq\emptyset$ and $d_{G\boxtimes T}((v;w),[yz]\cup[zx])\leqslant\operatorname{diam}V(G)$. So, we have $d_{G\boxtimes T}((v;w),[yz]\cup[zx])\leqslant\operatorname{diam}V(G)$ for every vertex $(v;w)$ in $[xy]$. Since $x,y\in J(G\boxtimes T)$, $d_{G\boxtimes T}(p,[yz]\cup[zx])\leqslant\operatorname{diam}V(G)+1/2$ for every $p\in[xy]$. Hence, $\delta(\triangle)\leqslant\operatorname{diam}V(G)+1/2$, and we obtain $\delta(G\boxtimes T)\leqslant\operatorname{diam}V(G)+1/2$. □

Theorem 29. *Let T be any tree and G any graph with $0<\operatorname{diam}V(G)=\operatorname{diam}T/2$. Then, we have*

$$\delta(G\boxtimes T)=\operatorname{diam}V(G)+\frac{1}{4}.$$

Proof. By Theorem 19, we have that $\operatorname{diam}V(G)\leqslant\delta(G\boxtimes T)\leqslant\operatorname{diam}V(G)+1/2$.

Now we show a geodesic bigon B in $G\boxtimes T$ with $\delta(B)=\operatorname{diam}V(G)+1/4$. Define by $n:=\operatorname{diam}V(G)$ and consider $v_1,\dots,v_{n+1}\in V(G)$ with $v_i\sim v_{i+1}$ for $i=1,\dots,n$ and $d_G(v_1,v_{n+1})=n$. Also, consider $w_1,\dots,w_{2n+1}\in V(T)$ with $w_i\sim w_{i+1}$ for $i=1,\dots,2n$ and $d_T(w_1,w_{2n+1})=\operatorname{diam}T=2n$. Denote by a (respectively, b) the midpoint of $[(v_1;w_1),(v_2;w_1)]$ (respectively, $[(v_1;w_{2n+1}),(v_2;w_{2n+1})]$). Let us consider

$$\gamma^*:=[a(v_1;w_1)]\bigcup\left\{\bigcup_{i=1}^{2n}[(v_1;w_i),(v_1;w_{i+1})]\right\}\bigcup[(v_1;w_{2n+1})b]$$

and

$$\gamma' := [a(v_2;w_1)] \bigcup \left\{ \bigcup_{i=1}^{n-1} [(v_{i+1};w_i),(v_{i+2};w_{i+1})] \right\} \bigcup [(v_{n+1};w_n),(v_{n+1};w_{n+1})] \bigcup$$
$$\bigcup [(v_{n+1};w_{n+1}),(v_{n+1};w_{n+2})] \bigcup \left\{ \bigcup_{j=1}^{n-1} [(v_{n+2-j};w_{n+1+j}),(v_{n+1-j};w_{n+2+j})] \right\} \bigcup$$
$$\bigcup [(v_2;w_{2n+1})b].$$

Consider the geodesic bigon $B := \{\gamma^*, \gamma'\}$ in $G \boxtimes T$. Let p be the midpoint of γ' and let p_0 be a point in γ' with $d_{G\boxtimes T}(p_0,p) = 1/4$; then $d_{G\boxtimes T}(p_0,\gamma^*) = n + 1/4$ and $\delta(G \boxtimes T) \geqslant \delta(B) \geqslant n + 1/4$.

Hence, by Theorem 25 we have $\delta(G \boxtimes T) \in \{n + 1/4\ ,\ n + 1/2\}$. Seeking for a contradiction assume that $\delta(G \boxtimes T) = n + 1/2$. Then there are a geodesic triangle $\triangle = \{x, y, z\}$ in $G\boxtimes T$ and $p \in [xy]$ with $d_{G\boxtimes T}(p, [yz] \cup [zx]) = n+1/2$. By Theorem 26 we can assume that $\triangle$ is a cycle with $x, y, z \in J(G\boxtimes T)$. By Theorem 8, $\operatorname{diam}(G\boxtimes T) = 2n+1$ and we conclude that $L([xy]) = 2n+1$ and p is the midpoint of $[xy]$. Since $\operatorname{diam} V(G\boxtimes T) = 2n$, we have that x, y are midpoints of edges in $G \boxtimes T$, and so, p is a vertex of $G \boxtimes T$. We can write $[xy] \cap V(G \boxtimes T) = \{(a_1;b_1),(a_2;b_2),\dots,(a_{2n+1};b_{2n+1})\}$ with $a_1,\dots,a_{2n+1} \in V(G)$, $(a_i;b_i) \sim (a_{i+1};b_{i+1})$ for $i = 1,\dots,2n$ and $d_T(b_1,b_{2n+1}) = 2n$. Thus, $p = (a_{n+1};b_{n+1})$ and $p \in V(G \boxtimes \{b_{n+1}\})$. Since T is a tree we have that $([yz] \cup [zx]) \cap (G \boxtimes \{b_{n+1}\}) \neq \emptyset$; in particular, $d_{G\boxtimes T}(p, [yz] \cup [zx]) \leqslant \operatorname{diam} V(G)$. This is the contradiction we were looking for, and then $\delta(G \boxtimes T) = \operatorname{diam} V(G) + 1/4$. □

The following lemma will be useful.

Lemma 30. *Let C_m be a cycle graph and G any graph with* $\operatorname{diam} V(G) < \operatorname{diam} V(C_m)$. *Let $\gamma = [xy]$ be a geodesic in $G\boxtimes C_m$ such that $x, y \in J(G\boxtimes C_m)$. Then, $L(P_{C_m}(\gamma)) \leqslant m/2$ where P_{C_m} is the projection on C_m.*

Proof. If $\operatorname{diam} V(G) = 0$, it is a trivial case. Assume now that $\operatorname{diam} V(G) > 0$.

If $L(\gamma) \leqslant m/2$, then we have the result since $L(P_{C_m}(\gamma)) \leqslant L(\gamma)$. Assume that $L(\gamma) > m/2$. Seeking for a contradiction, assume that $L(P_{C_m}(\gamma)) > m/2$.

Assume that m is even (the case m odd is similar). Since $x, y \in J(G \boxtimes C_m)$ and $L(P_{C_m}(\gamma)) > m/2$, there are $x', y' \in \gamma \cap J(G \boxtimes C_m)$ such that $d_{C_m}(P_{C_m}(x'), P_{C_m}(y')) = (m+1)/2$. Without loss of generality we can assume that $x' \in V(G \boxtimes C_m)$ and $y' \notin V(G \boxtimes C_m)$. Let $A, A_1, A_2 \in V(G)$ and $B, B_1, B_2 \in V(C_m)$ such that $x' = (A;B)$ and $y' \in [(A_1;B_1),(A_2;B_2)]$. Since $d_{C_m}(P_{C_m}(x'), P_{C_m}(y')) = (m+1)/2$, without loss of generality we can assume that $d_{C_m}(B, B_1)+1 = d_{C_m}(B, B_2) = m/2$. Since $\operatorname{diam} V(C_m) > \operatorname{diam} V(G)$, by Lemma 2 we have $d_{G\boxtimes C_m}((A;B),(A_1;B_1)) = m/2-1$; thus, $d_{G\boxtimes C_m}(x', y') \leqslant (m-1)/2$. This is the contradiction we were looking for. □

The following theorem provides the exact value of the hyperbolicity constant of the strong product of a cycle C_m and any graph G with $\operatorname{diam} V(G) \leqslant \operatorname{diam} V(C_m)/2$. This result is interesting by itself and, furthermore, it will be useful in order to prove the last theorems of this paper.

Theorem 31. *Let C_m be a cycle graph and G any graph with* $\operatorname{diam} V(G) \leqslant \operatorname{diam} V(C_m)/2$. *Then, we have*

$$\delta(G \boxtimes C_m) = \begin{cases} \lfloor m/2 \rfloor/2 + 1/4, & \text{if } \operatorname{diam} V(G) = \operatorname{diam} V(C_m)/2, \\ m/4, & \text{if } \operatorname{diam} V(G) < \operatorname{diam} V(C_m)/2. \end{cases} \tag{14}$$

Proof. If $\operatorname{diam} V(G) = 0$, then the equality is trivial. Assume now that $\operatorname{diam} V(G) > 0$. Let $V(C_m) = \{w_1, \dots, w_m\}$ where $w_i \sim w_{i+1}$ for $i = 1, \dots, m-1$. Let P_{C_m} be the projection on C_m.

First, we prove that $\delta(G \boxtimes C_m) < (\lfloor m/2 \rfloor + 1)/2$. Seeking for a contradiction, assume that there are a geodesic triangle $T = \{x, y, z\}$ in $G \boxtimes C_m$ and a point $p \in \gamma := [xy]$ with $d_{G\boxtimes C_m}(p, [yz] \cup [zx]) = (\lfloor m/2 \rfloor + 1)/2 = \operatorname{diam}(G \boxtimes C_m)/2$. Then $L(\gamma) = \operatorname{diam}(G \boxtimes C_m)$ and $d_{G\boxtimes C_m}(p, [yz] \cup [zx]) = \operatorname{diam}(G \boxtimes C_m)/2$, and we conclude that p is the midpoint of γ. By Theorem 26, we can assume that T is a cycle with $x, y, z \in J(G \boxtimes C_m)$. Since $\operatorname{diam} V(G \boxtimes C_m) = \operatorname{diam}(G \boxtimes C_m) - 1$, by Theorem 8 we have that x, y are midpoints of edges in $G \boxtimes C_m$. Let V_x (respectively, V_y) be the closest vertex to x (respectively, y) in γ. Let V_x' (respectively, V_y') be the closest vertex to x (respectively, y) in $[xz]$ (respectively, $[yz]$). By Lemma 2, we have $d_{G\boxtimes C_m}(V_x, V_y) = d_{C_m}(P_{C_m}(V_x), P_{C_m}(V_y)) = \lfloor m/2 \rfloor$. Therefore, since $\operatorname{diam} V(G) \leqslant \operatorname{diam} V(C_m)/2$ we have $d_{C_m}(P_{C_m}(V_x), P_{C_m}(p)) = d_{C_m}(P_{C_m}(p), P_{C_m}(V_y)) = \lfloor m/2 \rfloor/2$. By Lemma 30 we have $L(P_{C_m}(\gamma)) \leqslant m/2$; since $2\big(\lfloor m/2 \rfloor/2 + 1/2\big) > m/2$ we have either $P_{C_m}(V_x) = P_{C_m}(x) = P_{C_m}(V_x')$ or $P_{C_m}(V_y) = P_{C_m}(y) = P_{C_m}(V_y')$. So, we have

$$d_{G\boxtimes C_m}(p, [xz] \cup [yz]) \leqslant d_{G\boxtimes C_m}(p, \{V_x', V_y'\}) \leqslant \lfloor m/2 \rfloor/2 \leqslant m/4.$$

This is the contradiction we were looking for, and we have $\delta(G \boxtimes C_m) < (\lfloor m/2 \rfloor + 1)/2$. So, by Theorem 25 we have $\delta(G \boxtimes C_m) \leqslant \lfloor m/2 \rfloor/2 + 1/4$.

Assume now that $\lfloor m/2 \rfloor = 2 \operatorname{diam} V(G)$. If m is odd (i.e., $m = 4k+1$), then Theorem 15 (a) gives $\delta(G \boxtimes C_m) \geqslant m/4 = \lfloor m/2 \rfloor/2 + 1/4$. So, (14) holds. Assume that m in even (i.e., $m = 4k$). Now we show a geodesic bigon B in $G \boxtimes C_m$ with $\delta(B) = \lfloor m/2 \rfloor/2 + 1/4 = k + 1/4$. Note that $k = \operatorname{diam} V(G)$ and consider $v_1, \dots, v_{k+1} \in V(G)$ with $v_i \sim v_{i+1}$ for $i = 1, \dots, k$ and $d_G(v_1, v_{k+1}) = k$. Denote by a (respectively, b) the midpoint of $[(v_1; w_1), (v_2; w_1)]$ (respectively, $[(v_1; w_{2k+1}), (v_2; w_{2k+1})]$). Let us consider

$$\gamma^* := [a(v_1; w_1)] \bigcup \left\{ \bigcup_{i=1}^{2k} [(v_1; w_i), (v_1; w_{i+1})] \right\} \bigcup [(v_1; w_{2k+1})b]$$

and

$$\gamma' := [a(v_2; w_1)] \bigcup \left\{ \bigcup_{i=1}^{k-1} [(v_{i+1}; w_i), (v_{i+2}; w_{i+1})] \right\} \bigcup [(v_{k+1}; w_k), (v_{k+1}; w_{k+1})] \bigcup$$

$$\bigcup [(v_{k+1}; w_{k+1}), (v_{k+1}; w_{k+2})] \bigcup \left\{ \bigcup_{j=1}^{k-1} [(v_{k+2-j}; w_{k+1+j}), (v_{k+1-j}; w_{k+2+j})] \right\} \bigcup$$

$$\bigcup [(v_2; w_{2k+1}) b].$$

Then $B := \{\gamma^*, \gamma'\}$ is a geodesic bigon in $G \boxtimes C_m$ with $\delta(B) = k + 1/4 = \lfloor m/2 \rfloor/2 + 1/4$.

Finally, assume that $\lfloor m/2 \rfloor > 2 \operatorname{diam} V(G)$. By Theorem 15 (a) it suffices to prove $\delta(G \boxtimes C_m) \leqslant m/4$. If m is odd, then $\lfloor m/2 \rfloor/2 + 1/4 = m/4$ and (14) holds.

Assume that m is even, then $\operatorname{diam} V(G) \leqslant m/4 - 1/2$. Fix any geodesic triangle $T = \{x, y, z\}$ in $G \boxtimes C_m$ and $p \in [xy]$. By Remark 27, we can assume that T is a cycle, $x, y, z \in J(G \boxtimes C_m)$ and p satisfies $d_G(p, V(G)) \in \{0, 1/4, 1/2\}$. If $d_{G\boxtimes C_m}(p, \{x, y\}) \leqslant m/4$, then $d_{G\boxtimes C_m}(p, [yz] \cup [zx]) \leqslant m/4$. Assume that $d_{G\boxtimes C_m}(p, \{x, y\}) > m/4$; since $x, y \in J(G \boxtimes C_m)$ and $d_G(p, V(G)) \in \{0, 1/4, 1/2\}$, we have $d_{G\boxtimes C_m}(p, \{x, y\}) \geqslant m/4 + 1/4$. We have $L([xy]) > m/2$. Let V_x (respectively, V_y) be the closest vertex to x (respectively, y) in $[xy]$; then $d_{G\boxtimes C_m}(p, \{V_x, V_y\}) \geqslant m/4 - 1/4$. Let V_x' (respectively, V_y') be the closest vertex to x (respectively, y) in $[xz]$ (respectively, $[yz]$). Since m is even and $x, y \in J(G \boxtimes C_m)$ we have $d_{G\boxtimes C_m}(V_x, V_y) \geqslant m/2$ and we conclude $d_{G\boxtimes C_m}(V_x, V_y) = m/2$. By Lemma 2 we have $d_{G\boxtimes C_m}(V_x, V_y) = d_{C_m}(P_{C_m}(V_x), P_{C_m}(V_y)) = m/2$; by Lemma 30 we conclude $L\big(P_{C_m}([xy])\big) = m/2$. Since $m/2 = \lfloor m/2 \rfloor > \operatorname{diam} V(G)$, we have $P_{C_m}(V_x) = P_{C_m}(x) = P_{C_m}(V_x')$ and $P_{C_m}(V_y) = P_{C_m}(y) = P_{C_m}(V_y')$. Since $d_{G\boxtimes C_m}(p, \{V_x, V_y\}) \leqslant d_{G\boxtimes C_m}(V_x, V_y)/2 = m/4$, without loss of generality we can assume that $d_{G\boxtimes C_m}(p, \{V_x, V_y\}) = d_{G\boxtimes C_m}(p, V_x) \leqslant m/4$. Let V_p be the closest vertex to p in $[xp]$. Since $d_{G\boxtimes C_m}(p, V_x) \geqslant m/4 - 1/4 > m/4 - 1/2 \geqslant \operatorname{diam} V(G)$, we have $\operatorname{diam} V(G) \geqslant d_{G\boxtimes C_m}(V_p, V_x) = d_{C_m}(P_{C_m}(V_p), P_{C_m}(V_x)) = d_{C_m}(P_{C_m}(V_p), P_{C_m}(V_x'))$ and we conclude $d_{G\boxtimes C_m}(V_p, V_x) = d_{G\boxtimes C_m}(V_p, V_x')$ and $d_{G\boxtimes C_m}(p, [xz] \cup [yz]) \leqslant d_{G\boxtimes C_m}(p, V_x') \leqslant d_{G\boxtimes C_m}(p, V_x) \leqslant m/4$. Then $\delta(G \boxtimes C_m) \leqslant m/4$. $\square$

As a consequence of Theorems 19, 28, 29 and 31 we obtain the precise values of the hyperbolicity constants of the following families of graphs.

Theorem 32. *Let T_1, T_2 be two trees with $\operatorname{diam} T_1 \leqslant \operatorname{diam} T_2$. Then*

$$\delta(T_1 \boxtimes T_2) = \begin{cases} 0, & \text{if } \operatorname{diam} T_1 = 0, \\ \operatorname{diam} T_1 + 1/2, & \text{if } 0 < \operatorname{diam} T_1 < (\operatorname{diam} T_2)/2, \\ \operatorname{diam} T_1 + 1/4, & \text{if } 0 < \operatorname{diam} T_1 = (\operatorname{diam} T_2)/2, \\ (\operatorname{diam} T_2 + 1)/2, & \text{if } \operatorname{diam} T_1 > (\operatorname{diam} T_2)/2. \end{cases}$$

Corollary 33. *Let P_n, P_m be two path graphs with $2 \leqslant n \leqslant m$. Then*

$$\delta(P_n \boxtimes P_m) = \begin{cases} m/2, & \text{if } m - 1 < 2(n-1), \\ n - 3/4, & \text{if } m - 1 = 2(n-1), \\ n - 1/2, & \text{if } m - 1 > 2(n-1). \end{cases}$$

Theorem 34. *Let C_n, C_m be two cycle graphs with $3 \leqslant n \leqslant m$. Then*

$$\delta(C_n \boxtimes C_m) = \begin{cases} \lfloor m/2 \rfloor /2 + 1/2, & \text{if } \lfloor m/2 \rfloor < 2\lfloor n/2 \rfloor, \\ \lfloor m/2 \rfloor /2 + 1/4, & \text{if } \lfloor m/2 \rfloor = 2\lfloor n/2 \rfloor, \\ m/4, & \text{if } \lfloor m/2 \rfloor > 2\lfloor n/2 \rfloor. \end{cases}$$

Theorem 35. *For every $m \geqslant 2, n \geqslant 3$,*

$$\delta(C_n \boxtimes P_m) = \begin{cases} \lfloor n/2 \rfloor + 1/2, & \text{if } \lfloor n/2 \rfloor < (m-1)/2, \\ \lfloor n/2 \rfloor + 1/4, & \text{if } \lfloor n/2 \rfloor = (m-1)/2, \\ m/2, & \text{if } (m-1)/2 < \lfloor n/2 \rfloor \leqslant (m-1), \\ \big(\lfloor n/2 \rfloor + 1\big)/2, & \text{if } m-1 < \lfloor n/2 \rfloor < 2(m-1), \\ \lfloor n/2 \rfloor /2 + 1/4, & \text{if } \lfloor n/2 \rfloor = 2(m-1), \\ n/4, & \text{if } \lfloor n/2 \rfloor > 2(m-1). \end{cases}$$

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