On sumsets of multisets in $\mathbb{Z}_p^m$

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Abstract
For a sequence $A$ of given length $n$ contained in $\mathbb{Z}_p^2$ we study how many distinct subsums $A$ must have when $A$ is not “wasteful” by containing too many elements in same subgroup. Martin, Peilloux and Wong have made a conjecture for a sharp lower bound and established it when $n$ is not too large whereas Peng has previously established the conjecture for large $n$. In this note we build on these earlier works and add an elementary argument leading to the conjecture for every $n$.

Martin, Peilloux and Wong also made a more general conjecture for sequences in $\mathbb{Z}_p^m$. Here we show that the special case $n = mp - 1$ of this conjecture implies the whole conjecture and that the conjecture is equivalent to a strong version of the additive basis conjecture of Jaeger, Linial, Payan and Tarsi.

1 Introduction
For a sequence $A$ contained in an abelian group $G$ we write $\sum A$ for the set of all subsums of $A$, that is, for $A = (a_1, \ldots, a_n)$,

$$\sum A = \left\{ \sum_{i \in I} a_i : I \subseteq \{1, \ldots, n\} \right\} .$$

Note that $\sum A$ always contains 0, the sum of an empty sequence. As the order of the elements of $A$ is not relevant here, we will from now on think of $A$ as a multiset. For a set or multiset $B$, we write $|B|$ for the cardinality of $B$, counted with multiplicity, and $\#B$ for the cardinality of $B$ counted without multiplicity.

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Here we are interested in the relationship between \(|A|\) and \(#\sum A\). As pointed out for instance in [3, Lemma 1.3], in case \(G = \mathbb{Z}_p\) one gets the following result easily by multiple applications of the Cauchy-Davenport inequality (see [6, Theorem 5.4]).

**Lemma 1.** Let \(p \in \mathbb{P}\) and let \(A\) be a multiset contained in \(\mathbb{Z}_p^*\). Then

\[
# \sum A \geq \min\{p, |A| + 1\}.
\]

This lower bound is sharp as \(A\) may consist of \(|A|\) copies of a single element.

Let us now consider the case \(G = \mathbb{Z}_2^p\). In this case one might not get a better lower bound than the above if much of \(A\) is contained in a single subgroup. In particular it is “wasteful” for \(A\) to contain more than \(p - 1\) elements from any subgroup since by Lemma 1 already \(p - 1\) elements guarantee that \(\sum A\) contains the whole subgroup. In light of this we make the following definition (following [3]).

**Definition 2.** A multiset \(A\) contained in \(\mathbb{Z}_2^p\) is called valid if \(0 \not\in A\) and every non-trivial subgroup of \(\mathbb{Z}_2^p\) contains at most \(p - 1\) points of \(A\) (counting multiplicity).

For a valid multiset \(A\) in \(\mathbb{Z}_2^p\) with at most \(p - 1\) elements, one has again the sharp lower bound \(#\sum A \geq |A| + 1\). On the other hand, for large multisets Peng [4] has shown the following.

**Theorem 3.** Let \(p \in \mathbb{P}\) and let \(A\) be a valid multiset contained in \(\mathbb{Z}_2^p\) with \(|A| \geq 2p - 1\). Then \(\sum A = \mathbb{Z}_2^p\).

Hence we can concentrate on the case \(p \leq |A| \leq 2p - 2\). Martin, Peilloux and Wong [3] have made the following conjecture.

**Conjecture 4.** Let \(p \in \mathbb{P}\), let \(k\) be a non-negative integer, and let \(A\) be a valid multiset contained in \(\mathbb{Z}_2^p\) with \(|A| = p + k\). If \(k \leq p - 3\), then \(#\sum A \geq (k + 2)p\) and if \(k = p - 2\), then \(#\sum A \geq p^2 - 1\).

If true, this conjecture would be sharp as pointed out in [3]: First, for \(k \leq p - 3\), the multiset \(A\) may consist of \(p - 1\) copies of \((1,0)\) and \(k + 1\) copies of \((0,1)\), so that \(\sum A = \mathbb{Z}_p \times \{0, \ldots, k + 1\}\). Second, for \(k = p - 2\), \(A\) may consist of \(p - 2\) copies of \((1,0)\) and one copy of each \((i,1), 0 \leq i \leq p - 1\), so that \(\sum A = \mathbb{Z}_p^2 \setminus \{(p - 1,0)\}\).

Martin, Peilloux and Wong [3] proved the conjecture when

\[
k \leq \max\{1, \sqrt{p/(2 \log p + 1)} - 1\}.
\]

Here we will prove the conjecture for every \(k\).

**Theorem 5.** *Conjecture 4 holds.*

Martin, Peilloux and Wong [3] also generalised Conjecture 4 to \(\mathbb{Z}_p^m\) for \(m \geq 2\). They again want to avoid “wasteful” sets and thus only consider “valid” sets. To easily define validity in this setting, for a subgroup \(H\) of \(\mathbb{Z}_p^m\), we write \(\dim H = d\) where \(d\) is the integer for which \(H\) is isomorphic to \(\mathbb{Z}_p^d\).
Definition 6. Let $m \geq 2$. A multiset $A$ contained in $\mathbb{Z}_p^m$ is called valid if $0 \not\in A$ and every non-trivial subgroup $H$ of $\mathbb{Z}_p^m$ contains fewer than $p \cdot \dim H$ points of $A$ (counting multiplicity).

Taking $H = \mathbb{Z}_p^m$ one sees that every valid multiset has size at most $mp - 1$. On the other hand, there are valid multisets of this size, see [3, Example 4.2]. Furthermore in case $m = 2$ the definition of validity agrees with Definition 2 in the interesting case $|A| \leq 2p - 1$. Martin, Peilloux and Wong [3] made the following conjecture.

Conjecture 7. Let $p$ be an odd prime, let $m \geq 2$ be a positive integer, and let $A$ be a valid multiset contained in $\mathbb{Z}_p^m$ with $|A| = qp + k$, where $q \geq 1$ and $0 \leq k \leq p - 1$.

(a) If $0 \leq k \leq p - 3$, then $\# \sum A \geq (k + 2)p^q$;
(b) If $k = p - 2$, then $\# \sum A \geq p^{q+1} - 1$.
(c) If $k = p - 1$, then $\# \sum A \geq p^{q+1}$.

Again the definition of validity is such that, assuming Conjecture 7, it would be “wasteful” for a multiset to be non-valid. Also, if the conjecture is true, it gives the best possible lower bounds, see [3, Discussion after Conjecture 4.3].

Notice in particular the following special case of the conjecture.

Conjecture 8. Let $p$ be an odd prime, let $m$ be a positive integer, and let $A$ be a valid multiset contained in $\mathbb{Z}_p^m$ with $|A| = mp - 1$. Then $\sum A = \mathbb{Z}_p^m$.

In Section 4 we will show that the methods used in the proof of Theorem 5 can be adapted to show the following theorem.


Hence a special case generalising Peng’s result (Theorem 3) implies the whole conjecture. Peng has actually generalised his result to $\mathbb{Z}_p^m$ in [5] but he considers a much wider class of multisets than the valid sets here, so the result in [5] is not helpful here.

Let us close the introduction by discussing the additive basis conjecture of Jaeger, Linial, Payan and Tarsi [2]. We need the following definition from [1].

Definition 10. For a prime $p$ and a positive integer $m$, let $f(p, m)$ denote the minimal integer $t$ such that, for any $t$ bases $B_1, \ldots, B_t$ of $\mathbb{Z}_p^m$ one has

$$\sum_{i=1}^t \left( \bigcup_{i=1}^t B_i \right) = \mathbb{Z}_p^m,$$

where the union is let to be a multiset.
For instance by splitting the set $A$ of size $2p - 2$ below Conjecture 4 into $p - 1$ bases of $\mathbb{Z}_p^2$, one sees that for $p \geq 3$ and $m \geq 2$, $f(p, m) \geq p$. Jaeger, Linial, Payan and Tarsi [2] conjectured that $f(p, m)$ can be bounded from above by a function of $p$ alone and suggested that perhaps even $f(p, m) = p$. They showed that the conjecture has implications to group connectivity of graphs. Alon, Linial and Meshulam [1] showed that $f(p, m) \leq (p - 1) \log m + p - 2$, a bound which depends mildly on $m$.

We make the following related conjecture.

**Conjecture 11.** If $B_1, B_2, \ldots, B_{p-1}$ are bases of $\mathbb{Z}_p^m$ and $A \subset \mathbb{Z}_p^m$ is a (linearly) independent set of size $m - 1$, then

$$\sum_{A \cup \bigcup_{i=1}^{p-1} B_i} = \mathbb{Z}_p^m,$$

where these unions are as multisets.

Clearly this conjecture in particular implies $f(p, m) \leq p$, so that the following theorem which we will prove in Section 4 shows that the conjecture of Martin, Peilloux and Wong actually implies the strongest possible form of the additive basis conjecture.

**Theorem 12.** Conjecture 11 is equivalent to Conjecture 8.

### 2 Auxiliary results

As in [3], we will take advantage of direct sum representations of $\mathbb{Z}_p^m$. Recall that a group $G$ is an internal direct sum of subgroups $H$ and $K$ iff $H \cap K = \{e\}$ and $H + K = G$. As usual, we write in this case $G = H \oplus K$. In particular there exists a projection homomorphism $\pi_H : G \to H$ that is the identity in $H$ and vanishes in $K$.

The following lemma shows that one can deduce information about $\# \sum A$ by studying a subgroup and a projection.

**Lemma 13.** Let $G = H \oplus K$, and let $C$ be a multiset contained in $G$. Let $D = C \cap H$, let $F = C \setminus D$, and let $E = \pi_K(F)$. Then

$$\# \sum C \geq \# \sum D \cdot \# \sum E.$$

**Proof.** This is [3, Lemma 2.8], but we give a short proof for completeness. Let $y \in \sum E$. Then by definition of $E$, $x + y \in \sum F$ for some $x \in H$. Furthermore

$$x + y + \sum D \subseteq (x + y + H) \cap \left(\sum F + \sum D\right) = (y + H) \cap \sum C.$$

Hence, for each $y \in \sum E \subseteq K$, the coset $y + H$ contains at least $\# \sum D$ points of $\sum C$, and the claim follows since these cosets are disjoint. $\square$

Let us now cite Theorem 3 as Peng states and proves it (see [4, Theorem 2]) as it actually tells us something about non-valid sets as well.
Lemma 14. Let \( p \in \mathbb{P} \) and let \( A \) be a multiset of size \( 2p - 1 \) contained in \( \mathbb{Z}_p^2 \). Assume that \( 0 \notin A \) and each non-trivial subgroup of \( \mathbb{Z}_p^2 \) contains at most \( p \) elements of \( A \). Then \( \sum A = \mathbb{Z}_p^2 \).

Actually Lemma 14 is no stronger than Theorem 3 but follows from it, see Lemma 18. Lemma 14 lets us prove the case \( k = p - 2 \) of Conjecture 4 easily.

Lemma 15. Let \( p \in \mathbb{P} \) and let \( A \) be a valid multiset contained in \( \mathbb{Z}_p^2 \) with \( |A| = 2p - 2 \). Then \( \# \sum A \geq p^2 - 1 \).

Proof. Assume, contrary to the claim, that there are two distinct points \( z, w \in \mathbb{Z}_p^2 \setminus \sum A \). Let \( B \) be the multiset \( A \) joined by \( z - w \). This multiset satisfies the hypothesis of Lemma 14 but \( z \notin \sum A + \{0, z - w\} = \sum B \), a contradiction.

The following simple lemma will be the main tool in our inductive argument.

Lemma 16. Let \( G \) be an abelian group and let \( A \subseteq G \). Then for every \( m \geq 2 \),
\[
\#(A + \{0, z, 2z, \ldots, mz\}) - \#(A + \{0, z\}) \leq (m - 1)(\#(A + \{0, z\}) - \#A).
\]

Proof. Here
\[
\#(A + \{0, z, 2z, \ldots, mz\}) = \# \left( \bigcup_{i=0}^{m} (A + iz) \right) = \# \left( A \cup \bigcup_{i=1}^{m} ((A + iz) \setminus (A + (i - 1)z)) \right) \leq \#A + \sum_{i=1}^{m} \#((A + iz) \setminus (A + (i - 1)z)) = \#A + m \cdot \#((A + z) \setminus A),
\]
and the claim follows after a rearrangement.

For the proof of Theorem 12 we need the following direct consequence of the matroid union theorem (see for instance [7, Theorem 2 in Section 8.4]).

Lemma 17. Let \( V \) be a vector space and let \( A \) be a multiset contained in \( V \). If \(|U \cap A| \leq k \cdot \dim U \) for every subspace \( U \leq V \), then \( A \) may be partitioned into \( k \) sets \( A_1, \ldots, A_k \) where every \( A_i \) is linearly independent.

3 Proof of Theorem 5

Let \( A \) be a valid multiset of size \( p + k \) contained in \( \mathbb{Z}_p^2 \). As the case \( k = p - 2 \) was handled in Lemma 15, we can assume that \( 0 \leq k \leq p - 3 \). For \( z \in A \), write \( A_z = A \cap \langle z \rangle \) and
\( A_z^c = A \setminus A_z \). We will induct on \( k \) but let us first handle the case \( |A_z| \geq k + 1 \) for some \( z \in A \) as in [3]. In this case \( |A_z^c| = |A| - |A_z| \leq p - 1 \), and by Lemmas 13 and 1

\[
\# \sum A \geq (|A_z| + 1)(|A_z^c| + 1) = (|A_z| + 1)(|A| - |A_z| + 1) = |A_z|(|A| - |A_z|) + |A| + 1
\]

which attains its minimum when \( |A_z| \) is minimal or maximal. For both \( |A_z| = k + 1 \) and \( |A_z| = p - 1 \), the right hand side is \((k + 2)p\) and the claim follows.

Hence we can assume from now on that, for every \( z \in A, |A_z| \leq k \). Notice that as in [3] this in particular resolves the case \( k = 0 \).

At this point our proof diverges from that in [3], where the authors modified the set \( A \) to contain more elements in some subgroup by replacing \( 2l \) points \( x_i, z - x_i \in A, i = 1, \ldots, l \) by \( l \) copies of \( z \). Here we instead set up an induction on \( k \) (recall that \(|A| = p + k\)). As we already handled the case \( k = 0 \), we can proceed directly to the induction step.

Assume, contrary to the claim, that \( \# \sum A \leq (k + 2)p - 1 \). Notice that, for every \( z \in A \),

\[
\sum A = \sum (A \setminus \{z\}) + \{0, z\},
\]

and here by the induction hypothesis \( \# \sum (A \setminus \{z\}) \geq (k + 1)p \). Hence

\[
\# \left( \sum (A \setminus \{z\}) + \{0, z\} \right) - \# \sum (A \setminus \{z\}) \leq (k + 2)p - 1 - (k + 1)p = p - 1. \tag{1}
\]

Let \( B \) be the multiset which consists of \( A \) and \( p - k - 2 \) additional copies of \( z \), so that \(|B| = 2p - 2\). Since \(|A \cap \{z\}| \leq k \), \( B \) is valid, so that by Lemma 15, \( \# \sum B \geq p^2 - 1 \). On the other hand, applying Lemma 16 and recalling (1), one gets

\[
\# \sum B = \# \left( \sum (A \setminus \{z\}) + \{0, z, 2z, \ldots, (p - k - 1)z\} \right)
\]

\[
\leq \left( \# \sum (A \setminus \{z\}) + \{0, z\} \right) + (p - k - 2) \left( \# \left( \sum (A \setminus \{z\}) + \{0, z\} \right) - \# \sum (A \setminus \{z\}) \right)
\]

\[
\leq \# \sum A + (p - k - 2)(p - 1) \leq (k + 2)p - 1 + (p - k - 2)(p - 1)
\]

\[
= p^2 - p + k + 1 \leq p^2 - 2
\]

since \( k \leq p - 3 \). Hence we have arrived to a contradiction so one must indeed have \( \# \sum A \geq (k + 2)p \). \( \Box \)

### 4 Proofs of Theorems 9 and 12

To prove Theorem 9, we need a few lemmas. The first lemma shows that a stronger statement follows from Conjecture 8, in particular Lemma 14 follows from Theorem 3.

**Lemma 18.** Conjecture 8 implies the following: Let \( p \) be an odd prime and let \( m \) be a positive integer. Let \( A \) be a multiset contained in \( \mathbb{Z}_p^m \) for which

\[
|A \cap H| \leq p \dim H \tag{2}
\]

for every subgroup \( H \leq \mathbb{Z}_p^m \). If \(|A| \geq mp - 1\), then \( \sum A = \mathbb{Z}_p^m \).
Proof. Let us induct on $m$. Case $m = 1$ follows from Lemma 1, so we can move to the induction step. We can clearly assume that $|A| = mp - 1$. Let $H$ be a maximal subgroup of $\mathbb{Z}_p^m$ for which equality holds in (2) (possibly $H = \{0\}$), and write $\mathbb{Z}_p^m = H \oplus K$. If $\pi_K(A \setminus H)$ were not a valid multiset, there would exist a non-trivial subgroup $K_1 \trianglelefteq K$ such that $|(A \setminus H) \cap (H \oplus K_1)| \geq p \cdot \dim K_1$ and consequently

\[ A \cap (H \oplus K_1) = |A \cap H| + |(A \setminus H) \cap (H \oplus K_1)| \geq p \cdot (\dim H + \dim K_1) = p \cdot (\dim H \oplus K_1), \]

which contradicts the maximality of $H$.

Hence $\pi_K(A \setminus H)$ is a valid multiset contained in $K$ with size

\[ |A| - |A \cap H| = mp - 1 - p \cdot \dim H = p \cdot \dim K - 1, \]

so that $\sum \pi_K(A \setminus H) = K$ by the assumed Conjecture 8. Furthermore $A \cap H$ has size $p \cdot \dim H$ and dimension smaller than $m$, and thus by induction hypothesis $\sum (A \cap H) = H$, and the claim follows from Lemma 13.

Theorem 12 follows now immediately:


The following lemma follows from the previous lemma as Lemma 15 follows from Lemma 14.

Lemma 19. Conjecture 8 implies the following: Let $p$ be an odd prime, let $m$ be a positive integer, and let $A$ be a valid multiset contained in $\mathbb{Z}_p^m$ with $|A| = mp - 2$. Then $\# \sum A \geq p^m - 1$.

The third and fourth lemmas will let us show that we can assume that our multiset $A$ is not too concentrated in any subgroup (recall that also in the proof of Theorem 5 we first showed that we can assume that $|A \cap \langle z \rangle| \leq k$ for every $z \in A$).

Lemma 20. Let $m \geq 2$ and $\mathbb{Z}_p^m = H \oplus K$, where $0 < \dim H < m$. If $A$ is a valid multiset contained in $\mathbb{Z}_p^m$ with $|A \setminus H| \leq p \cdot \dim K - 1$, then there exists a non-trivial subgroup $K' \trianglelefteq \mathbb{Z}_p^m$ such that, writing $\mathbb{Z}_p^m = H' \oplus K'$, $\pi_{K'}(A \setminus H')$ is a valid multiset contained in $K'$.

Proof. If $\pi_K(A \setminus H)$ is valid, the claim follows immediately. Otherwise there is a non-trivial subgroup $K_1 \trianglelefteq K$ such that

\[ |(A \setminus H) \cap (H \oplus K_1)| \geq p \cdot \dim K_1. \]

Let $K_1$ be maximal such subgroup and $K = K_1 \oplus K_2$. The bounds (4) and (3) together imply that $K_1 \trianglelefteq K$ so that $K_2 \neq \{0\}$.
If $\pi_{K_2}(A \setminus (H \oplus K_1))$ is valid, the claim follows with $K' = K_2$ and $H' = H \oplus K_1$. Otherwise there exists a non-trivial subgroup $K_3 \leq K_2$ such that

$$|(A \setminus (H \oplus K_1)) \cap (H \oplus K_1 \oplus K_3)| \geq p \cdot \dim K_3.$$ 

Combining with (4) gives

$$|(A \setminus H) \cap (H \oplus K_1 \oplus K_3)| \geq p \cdot (\dim K_1 + \dim K_3) = p \cdot \dim(K_1 \oplus K_3)$$

which contradicts the maximality of $K_1$. □

Lemma 21. Let $p$ be an odd prime and define $f: \mathbb{Z}_{\geq 0} \to \mathbb{N}$ by putting for each $q \geq 0$ and $0 \leq k \leq p - 1$,

$$f(qp + k) = \begin{cases} 
  k + 1 & \text{if } q = 0 \text{ and } 0 \leq k \leq p - 1; \\
  (k + 2)p^q & \text{if } q \geq 1 \text{ and } 0 \leq k \leq p - 3; \\
  p^{q+1} - 1 & \text{if } q \geq 0 \text{ and } k = p - 2; \\
  p^{q+1} & \text{if } q \geq 0 \text{ and } k = p - 1;
\end{cases}$$

Then for every $n_1, n_2 \in \mathbb{Z}_{\geq 0}$ one has $f(n_1) \cdot f(n_2) \geq f(n_1 + n_2)$.

Proof. Write $n_i = q_ip + k_i$. First note that

$$f(q_1p + p - 2)f(p - 2) = (p^{q_1+1} - 1)(p - 1) \geq (p - 2)(p^{q_1+1} - 1) = f(q_1p + p - 2 + p - 2),$$

so we can assume that if $k_1 = k_2 = p - 2$ then $q_2 \neq 0$. One has

$$\frac{f(qp + k)}{f(qp + k - 1)} = \begin{cases} 
  \frac{k+1}{k} = 1 + \frac{1}{k} & \text{if } q = 0 \text{ and } 0 < k \leq p - 1; \\
  \frac{k+2}{k+1} = 1 + \frac{1}{k+1} & \text{if } q \geq 1 \text{ and } 0 \leq k \leq p - 3; \\
  \frac{p^{q+1} - 1}{p^q(p-1)} = 1 + \frac{1}{p^q(p-1)} & \text{if } q \geq 1 \text{ and } k = p - 2; \\
  \frac{p^{q+1}}{p^{q+1}-1} = 1 + \frac{1}{p^{q+1}-1} & \text{if } q \geq 0 \text{ and } k = p - 1.
\end{cases}$$

From this we see that for every $q_1, q_2 \geq 0$ and $0 \leq k_1 \leq k_2 \leq p - 2$ (with $q_1p + k_1 > 0$ and not $(k_1, k_2, q_2) = (p - 2, p - 2, 0)$) one has

$$\frac{f(q_1p + k_1)}{f(q_1p + k_1 - 1)} \geq \frac{f(q_2p + k_2 + 1)}{f(q_2p + k_2)}$$

$$\iff f(q_1p + k_1)f(q_2p + k_2) \geq f(q_1p + k_1 - 1)f(q_2p + k_2 + 1). \quad (5)$$

Applying (5) repeatedly to $f(n_1)f(n_2)$, we can assume that either $k_1 = p - 1$ or $k_2 = p - 1$, and consequently, by symmetry, that $k_1 = p - 1$. The proof can then be completed by an easy case-by-case check according to the value of $k_2$. □
Proof of Theorem 9. Let \( f \) be as in Lemma 21. Conjecture 7 is equivalent to the claim that for every \( m \geq 1 \) and any valid multiset \( A \) contained in \( \mathbb{Z}_p^m \) one has \( \# \sum A \geq f(|A|) \) (since the latter claim holds if \( m = 1 \) or if \( |A| < p \) by Lemmas 1 and 13).

Let us induct on \( m \). Lemma 1 takes care of the case \( m = 1 \), so we can move to the induction step. Let \( |A| = qp + k \). We will induct also on \( k \) but let us first consider the case that for some non-trivial subgroup \( H \leq \mathbb{Z}_p^m \) one has \( |A \setminus H| \leq p \cdot (m - \dim H) - 1 \).

In this case Lemma 20 implies that there exists a non-trivial subgroup \( H' \leq \mathbb{Z}_p^m \) such that, writing \( \mathbb{Z}_p^m = H' \oplus K' \), \( \pi_{K'}(A \setminus H') \) is a valid multiset contained in \( K' \). Since \( \dim K' \leq m \), by the induction hypothesis

\[
\# \sum \pi_{K'}(A \setminus H') \geq f(|A \setminus H'|) \quad \text{and} \quad \# \sum (A \cap H') \geq f(|A \cap H'|).
\]

Hence by Lemmas 13 and 21

\[
\# \sum A \geq f(|A \setminus H'|) \cdot f(|A \cap H'|) \geq f(|A|)
\]

and the claim follows.

Thus we can assume that

\[
|A \setminus H| \geq p \cdot (m - \dim H) \quad (6)
\]

for every non-trivial subgroup \( H \leq \mathbb{Z}_p^m \). In particular taking \( H = \langle z \rangle \) for some \( z \in \mathbb{Z}_p^m \), we see that we can assume that \( q = m - 1 \), so that \( |A| = (m - 1)p + k \). By this and (6) we can thus assume that for every subgroup \( H \leq \mathbb{Z}_p^m \) one has

\[
|A \cap H| = |A| - |A \setminus H| \leq (m - 1)p + k - p \cdot (m - \dim H) = p \cdot (\dim H - 1) + k. \quad (7)
\]

Taking here \( H = \langle z \rangle \) for some \( z \in A \), we see that we can assume that \( k > 0 \). On the other hand, Lemma 19 lets us assume that \( k \leq p - 3 \).

From now on the proof proceeds almost exactly as the proof of Theorem 5, so let us induct also on \( k \) and assume, contrary to the claim, that \( \# \sum A \leq (k+2)p^{m-1} - 1 \). Recall that, for every \( z \in A \),

\[
\sum A = \sum (A \setminus \{ z \}) + \{ 0, z \},
\]

and here by the induction hypothesis \( \# \sum (A \setminus \{ z \}) \geq (k+1)p^{m-1} \). Hence

\[
\# \left( \sum (A \setminus \{ z \}) + \{ 0, z \} \right) - \# \sum (A \setminus \{ z \}) \leq (k+2)p^{m-1} - 1 - (k+1)p^{m-1} = p^{m-1} - 1. \quad (8)
\]

Let \( B \) be the multiset which consists of \( A \) and \( p - k - 2 \) additional copies of \( z \), so that \( |B| = mp - 2 \). Since (7) holds for every non-trivial subgroup \( H, B \) is valid, so that, by Lemma 19, \( \# \sum B \geq p^m - 1 \). On the other hand, applying Lemma 16 recalling (8), one gets

\[
\# \sum B = \# \left( \sum (A \setminus \{ z \}) + \{ 0, z, 2z, \ldots, (p - k - 1)z \} \right)
\]

\[
\leq \# \sum A + (p - k - 2) \left( \# \sum A - \# \sum (A \setminus \{ z \}) \right)
\]

\[
\leq (k + 2)p^{m-1} - 1 + (p - k - 2)(p^{m-1} - 1) = p^m - p + k + 1 \leq p^m - 2
\]

since \( k \leq p - 3 \).  \( \square \)
The proof actually tells us that if, for some $M \geq 2$, Conjecture 8 holds for every $m \leq M$, then so does Conjecture 7. In particular, as was shown already in Section 3, Theorem 3 implies Theorem 5.

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References