The size of edge-critical uniquely 3-colorable planar graphs

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Abstract
A graph $G$ is uniquely $k$-colorable if the chromatic number of $G$ is $k$ and $G$ has only one $k$-coloring up to permutation of the colors. A uniquely $k$-colorable graph $G$ is edge-critical if $G - e$ is not a uniquely $k$-colorable graph for any edge $e \in E(G)$. In this paper, we prove that if $G$ is an edge-critical uniquely 3-colorable planar graph, then $|E(G)| \leq \frac{8}{3}|V(G)| - \frac{17}{3}$. On the other hand, there exists an infinite family of edge-critical uniquely 3-colorable planar graphs with $n$ vertices and $\frac{9}{4}n - 6$ edges.

Our result gives a first non-trivial upper bound for $|E(G)|$.

Keywords: uniquely colorable; edge-critical; planar graph

1 Introduction

In this paper, we only deal with finite undirected simple graphs, and for any vertex subset $U$ in a graph $G$, $\langle U \rangle$ means the subgraph of $G$ induced by $U$.

A $k$-coloring of a graph $G$ is a map $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that for any $uv \in E(G)$, $c(u) \neq c(v)$. A graph $G$ is $k$-colorable if there exists a $k$-coloring of $G$, and the chromatic number of $G$, denoted by $\chi(G)$, is the minimum number $k$ such that $G$ is $k$-colorable. A graph $G$ is uniquely $k$-colorable if $k = \chi(G)$ and $G$ has only one $k$-coloring up to permutation of the colors, where the coloring is called a unique $k$-coloring (note that if $G$ is uniquely $k$-colorable, then it is clear that $|V(G)| \geq k$ by the definition). In other words, every two $k$-colorings of $G$ produce the same partition of $V(G)$ into $k$ independent subsets (color classes). Then, two $k$-colorings $c$ and $c'$ are said to be distinct, denoted by $c \neq c'$, if they produce two distinct partitions of $V(G)$ into $k$ color classes. Moreover, we denote the set of uniquely $k$-colorable graphs by $UC_k$. For two distinct colors $i, j \in \{1, 2, \ldots, k\}$ in a $k$-coloring $c$ of a graph $G$, define $G_{i,j} = (c^{-1}(i) \cup c^{-1}(j))$. For uniquely $k$-colorable graphs, Harary et al. proved the following theorem.

\begin{thebibliography}{9}
\end{thebibliography}
Theorem 1 (Harary et al. [4]) If \( c : V(G) \to \{1,2,\ldots,k\} \) is a unique \( k \)-coloring of \( G \in UC_k \), then the graph \( G_{i,j} \) is connected for all \( i \neq j \) (\( i, j \in \{1,2,\ldots,k\} \)).

If a graph \( G \) is uniquely 1-colorable, then \( G \) has no edges. Hence, throughout this paper, we only consider \( k \geq 2 \) for any uniquely \( k \)-colorable graphs. Moreover, the following corollary holds by Theorem 1. (For other results and related topics, see [3].)

**Corollary 2** If \( G \in UC_k \) with \( |V(G)| \geq n \), then \( G \) has at least \((k - 1)n - \binom{k}{2}\) edges.

In this paper, we consider the size of edge-critical uniquely \( k \)-colorable planar graphs. For a graph \( G \in UC_k \), \( G \) is **edge-critical** if \( G - e \notin UC_k \) for any edge \( e \in E(G) \). However, since uniquely 5-colorable planar graphs do not exist [2], we only consider edge-critical uniquely \( k \)-colorable planar graphs for \( k \in \{2,3,4\} \).

By Corollary 2, if a uniquely \( k \)-colorable planar graph \( G \) has exactly \((k - 1)|V(G)| - \binom{k}{2}\) edges, then \( G \) is edge-critical. Moreover, it is not difficult to see that any edge-critical uniquely 2-colorable planar graph \( G \) has at most \(|V(G)| - 1 \) edges (that is, \( G \) is a tree). On the other hand, since every planar graph \( G \) has at most \( 3|V(G)| - 6 \) edges by Euler’s formula, we have Table 1. (Following this, we denote the upper bound of the size of any edge-critical uniquely 3-colorable planar graph by \( f(n) \), where \( n \) is the number of vertices.) In Table 1, it is clear that \( f(n) \) is at most \( 3n - 6 \) by the planarity, but this is not a “good” bound.

<table>
<thead>
<tr>
<th>( k )</th>
<th>Lower bound (Corollary 2)</th>
<th>Upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( n - 1 )</td>
<td>( n - 1 )</td>
</tr>
<tr>
<td>3</td>
<td>( 2n - 3 )</td>
<td>( f(n) )</td>
</tr>
<tr>
<td>4</td>
<td>( 3n - 6 )</td>
<td>( 3n - 6 )</td>
</tr>
</tbody>
</table>

Table 1: The upper (or lower) bound of the size of any edge-critical uniquely \( k \)-colorable planar graph \( G \), where \(|V(G)| = n \) and \( k \in \{2,3,4\} \).

In 1977, Aksionov [1] conjectured that \( f(n) = 2n - 3 \). However, in the same year, Mel’nikov and Steinberg [5] disproved the conjecture by constructing a counterexample shown in Figure 1. Hence \( f(n) \) is greater than \( 2n - 3 \). However, we have not yet known any reasonable upper bound.

Our main results are the followings.

**Theorem 3** If \( G \) is an edge-critical uniquely 3-colorable planar graph, then,

\[
|E(G)| \leq \frac{8}{3}|V(G)| - \frac{17}{3}.
\]

**Theorem 4** For any integer \( n \geq 12 \) such that \( n \equiv 0 \) (mod 4), there exists an edge-critical uniquely 3-colorable planar graphs with \( n \) vertices and \( \frac{9}{4}n - 6 \) edges.
By our results, we have the following corollary.

**Corollary 5** For any $n \geq 12$ such that $n \equiv 0 \pmod{4}$, we have

$$\frac{9}{4}n - 6 \leq f(n) \leq \frac{8}{3}n - \frac{17}{3}.$$

In the next section, we prove Theorem 3. In Section 3, we construct an infinite family of edge-critical uniquely 3-colorable planar graphs with $n$ vertices and $\frac{9}{4}n - 6$ edges.

## 2 Proof of Theorem 3

For a plane graph $G$, a *Δ-face cycle* $C = T_1T_2 \ldots T_k$ is a subgraph of $G$ which consists of the vertices and edges of $T_i$’s, where $T_i$ is a triangular face and $T_i$ and $T_{i+1}$ share an edge for any $1 \leq i \leq k$ ($T_{k+1} = T_1$), see Figure 2. Similarly to a Δ-face cycle, we define a *Δ-face path* $P = T_0T_1 \ldots T_{l-1}$, where $T_0$ and $T_{l-1}$ do not share an edge. Note that any 3-coloring of a Δ-face cycle and a Δ-face path is unique.

![Figure 2: A Δ-face cycle C](image)

**Lemma 6** Let $G$ be an edge-critical uniquely 3-colorable plane graph. Then $G$ has no Δ-face cycle.
Therefore, we can obtain the coloring \( c \) of \( G \). Let 
\[
\begin{align*}
\text{Lemma 7} & \quad \text{Let } G \text{ be a plane graph with } n \text{ vertices. If } G \text{ has no } \Delta\text{-face cycle, then } |E(G)| \leq \frac{8}{3}n - \frac{17}{3}, \text{ where this estimation is best possible.} \\
\text{Proof.} & \quad \text{It is well-known that any plane graph can be transformed into a triangulation (which is a plane graph such that each face is bounded by a cycle of length three) only by adding edges preserving the simpleness. Hence we regard } G \text{ as a plane graph obtained from a triangulation } T \text{ by removing } k \text{ edges. (Since } |E(G)| \leq 3n - 6 - k \text{ by } |E(T)| = 3n - 6, \text{ we finally show } k \geq \frac{n - 1}{3}. \\
& \text{We consider the dual graph of } G, \text{ denoted by } G^*. \text{ Let } V_3 \text{ be the set of vertices of degree } 3 \text{ in } G^* \text{ (which is the set of triangular faces in } G) \text{ and let } V_{\geq 4} = V(G^*) \setminus V_3. \text{ We now have } |E(G^*)| = 3n - 6 - k \text{ and } |V_3| \geq 2n - 4 - 2k, \text{ since removing a single edge decreases the number of triangular faces by at most two in } G \text{ and any triangulation with } n \text{ vertices has } 2n - 4 \text{ triangular faces by Euler’s formula. We may suppose that } V_3 \neq \emptyset; \text{ otherwise, the lemma holds since we have } |E(G)| \leq 2n - 4 \text{ by } k \geq n - 2. \text{ Moreover, by the assumption, } \langle V_3 \rangle \text{ is a forest (otherwise, } G \text{ has a } \Delta\text{-face cycle). Then, we let } \langle V_3 \rangle = T_1 \cup T_2 \cup \cdots \cup T_m, \text{ where each } T_i \text{ is a tree. Then we now have}
\[
\sum_{i=1}^{m} |V(T_i)| = |V_3|, \quad |E(\langle V_3 \rangle)| = \sum_{i=1}^{m} |E(T_i)| = \sum_{i=1}^{m} (|V(T_i)| - 1) = |V_3| - m.
\]

Let \( e(V_3, V_{\geq 4}) = E(G^*) \setminus (E(\langle V_3 \rangle) \cup E(\langle V_{\geq 4} \rangle)) \) (see Figure 4, for example). Since each vertex in \( V_3 \) has degree exactly three, we have
\[
|e(V_3, V_{\geq 4})| = 3 \sum_{i=1}^{m} |V(T_i)| - 2 \sum_{i=1}^{m} |E(T_i)| = |V_3| + 2m.
\]
Here, let us count \( |E((V_{\geq 4}))| \). By the above equations, we have

\[
|E((V_{\geq 4}))| = 3n - 6 - k - |E((V_3))| - |e(V_3, V_{\geq 4})| \\
= 3n - 6 - k - (|V_3| - m) - (|V_3| + 2m) \\
= 3n - 6 - k - m - 2|V_3| \\
\leq 3n - 6 - k - m - 2(2n - 4 - 2k) \\
= -n + 2 + 3k - m.
\]

Then, by \( m \geq 1 \) and \( |E((V_{\geq 4}))| \geq 0 \), we have

\[
n - 1 \leq 3k.
\]

Therefore, we have \( k \geq \frac{n - 1}{3} \), and hence, \( |E(G)| \leq 3n - 6 - k \leq \frac{8}{3}n - \frac{17}{3} \).

![Figure 4: The black vertices and white ones are members in \( V_3 \) and \( V_{\geq 4} \), respectively, and the dotted segments are members in \( e(V_3, V_{\geq 4}) \).](image)

Next, we show that the estimation is best possible by constructing an infinite family of plane graphs which have no \( \Delta \)-face cycle with \( n \) vertices and \( \frac{8}{3}n - \frac{17}{3} \) edges as follows:

We prepare two graphs \( R \) and \( X \) shown in Figures 5 and 6, respectively. Then, as shown in Figure 7, we glue \( R \) to \( X \) by identifying \( a, b \) and \( c \) and \( a', b' \) and \( c' \), respectively. Moreover, we repeatedly apply the above operation to the resulting graph (the right hand of Figure 7), but from the second step, we identify \( a, d \) and \( c \) and \( a', b' \) and \( c' \), respectively.

Then, it is easy to check that plane graphs constructed by the above operation have no \( \Delta \)-face cycle with \( n \) vertices and \( \frac{8}{3}n - \frac{17}{3} \) edges. Therefore, the lemma holds.

![Figure 5: A graph \( R \) (7 vertices and 13 edges) Figure 6: A graph \( X \) (6 vertices and 8 edges)](image)

**Proof of Theorem 3.** By combining Lemmas 6 and 7, we have \( |E(G)| \leq \frac{8}{3}|V(G)| - \frac{17}{3} \) for any edge-critical uniquely 3-colorable planar graph \( G \). Therefore, this completes the proof of Theorem 3.
In this paper, we estimate the size of plane graphs with no Δ-face cycle, since such graphs are edge-critical uniquely 3-colorable planar graphs. However, since Lemma 7 is best possible, we have to find another structure of uniquely 3-colorable planar graphs to improve Theorem 3.

3 Proof of Theorem 4

In [5], Mel’nikov and Steinberg stated that an infinite family of edge-critical uniquely 3-colorable planar graphs with n vertices and more than $2n - 3$ edges can be obtained by combining the graph $K$ shown in Figure 8 several times. However, they did not clarify its structure. Hence, we give a construction of an infinite family of edge-critical uniquely 3-colorable planar graphs with $n$ vertices and $\frac{9}{4}n - 6$ edges.

Then we construct a graph $H_k$, as follows:

Prepare $k$ triangles $u_1v_1u_2, u_2v_2u_3, \ldots, u_{k-1}v_{k-1}u_k, u_kv_ku_1$, where $k \geq 5$ is an odd integer and for each $i$, $u_i$ is shared by exactly two triangles $u_{i-1}v_{i-1}u_i$ and $u_iv_iu_{i+1}$ and add edges $u_3u_k, u_5u_k, \ldots, u_{k-4}u_k$ (if $k = 5$, then we do not add $u_1u_5$ since there is a triangle $u_5v_5u_1$). Finally, we add two vertices $x$ joined to $v_1, v_2, \ldots, v_{k-3}, v_{k-2}$ and $y$ joined to $v_{k-2}, v_{k-1}, v_k$. (For example, see Figure 9. If $k = 7$, then $H_k$ is isomorphic to the graph of Figure 1.)
It is easy to see that $H_k$ is planar and $|V(H_k)| = n = 2k + 2 \geq 12$. Moreover, since $\deg(x) = k - 2$, $\deg(y) = 3$ and there exist edges $u_i, k$ ($3 \leq i \leq k - 4$, $i$ : odd), we have

$$|E(H_k)| = 3k + (k - 2) + 3 + \left(\frac{k - 1}{2} - 2\right) = \frac{9}{2}k - \frac{3}{2}.$$ 

Hence, we have $|E(H_k)| = \frac{9}{4}n - 6$ by $n = 2k + 2$. Then, we shall prove the following theorem. By the above equation, this theorem implies Theorem 4.

**Theorem 8** For any odd integer $k \geq 5$, $H_k$ is an edge-critical uniquely 3-colorable planar graph.

**Proof.** We first show that $H_k$ is uniquely 3-colorable. (finally, we have a unique 3-coloring of $H_k$ shown in Figure 9). It is not difficult to see that for each $i$ ($1 \leq i \leq k - 4$), $u_i, u_{i+3}$ and $x$ correspond to $a, b$ and $t$ in the graph $K$ shown in Figure 8, respectively. Hence, we use the following claim. Moreover, since it is not difficult to check that the theorem holds for $H_5$ using the following claim, we suppose $k \geq 7$.

**Claim 9** [5] Any 3-coloring of $K$ assigns different colors to the vertices $a$ and $b$.

Let $c$ be a 3-coloring of $H_k$, and by Claim 9, we now have $c(u_i) \neq c(u_{i+3})$ for each $i$ ($1 \leq i \leq k - 4$). Hence, without loss of generality, we may suppose that $c(u_{k-4}) = 1$ and $c(u_{k-1}) = 2$. Then we have $c(u_k) = 3$ and $c(v_{k-1}) = 1$.

Firstly, we show $c(x) = 2$. If $c(x) = 3$, then we have $c(v_{k-2}) = 1, c(u_{k-2}) = 3, c(u_{k-3}) = 2$ and $c(v_{k-4}) = 3$, but this is a contradiction since an edge $xv_{k-4} \in E(H_k)$. Hence we suppose that $c(x) = 1$. Then we now have $c(v_{k-2}) = 3, c(y) = 2, c(v_k) = 1, c(u_1) = 2, c(v_1) = 3$ and $c(u_2) = 1$. Since $u_3u_k \in E(H_k)$ and $c(u_k) = 3$, we have $c(u_3) = 2$ and then $c(v_3) = 3$ and $c(u_4) = 1$. Then we next consider $c(u_5)$ similarly to $c(u_3)$. By repeating the above argument, we have $c(u_{k-5}) = 1$. However, this is a contradiction since $u_{k-5}u_{k-4} \in E(H_k)$. Therefore, we have $c(x) = 2$.

Then, since $c(u_{k-4}) = 1$ and $c(x) = 2$, we now have $c(v_{k-5}) = c(v_{k-4}) = 3$ and $c(u_{k-5}) = c(u_{k-3}) = 2$. By $u_{k-6}u_k \in E(H_k)$ and $c(u_k) = 3$, we have $c(u_{k-6}) = 1$, and
hence, \(c(v_{k-6}) = c(v_{k-7}) = 3\) and \(c(u_{k-7}) = 2\). By repeating this, we have the following coloring for each \(i\) \((2 \leq i \leq k - 5)\):

\[
c(u_i) = \begin{cases} 1 & \text{if } i \text{ odd} \\ 2 & \text{if } i \text{ even} \end{cases}, \quad c(v_i) = 3.
\]

Finally, since we have \(c(u_1) = 1\) by \(c(u_k) = 3\) and \(c(u_2) = 2\), we can obtain \(c(v_1) = 3, c(v_k) = 2, c(y) = 3, c(v_{k-2}) = 1, c(u_{k-2}) = 3\) and \(c(v_{k-3}) = 1\). Therefore, since the 3-coloring \(c\) is uniquely decided as shown in Figure 9, \(H_k\) is uniquely 3-colorable.

Next, we shall show that \(H_k\) is edge-critical. Let \(c\) be a unique coloring of \(H_k\) (for example, see Figure 9. Observe that \(G_{1,2}\) and \(G_{1,3}\) are trees. Hence, it suffices to prove that for any edge \(e\) which is contained in cycles in \(G_{2,3}\), \(H_k - e \notin UC_3\). In \(G_{2,3}\), as shown in Figure 9 for example, there exists a 4-cycle \(xv_{i-1}u_iv_i\) for each \(i\) \((2 \leq i \leq k - 5, i \text{ even})\). Hence, we need to consider the following two cases. For any edge \(e\), let \(c'\) be a new 3-coloring of \(H_k - e\).

**Case 1.** Remove \(xv_{i-1}\) or \(v_{i-1}u_i\) (see Figure 10)

We re-color the vertices of \(H_k - xv_{i-1}\) (or \(v_{i-1}u_i\)) as follows \((i \leq j \leq k - 5)\):

\[
c'(u_j) = \begin{cases} 2 & \text{if } j \text{ odd} \\ 3 & \text{if } j \text{ even} \end{cases}, \quad c'(v_j) = 1,
\]

\[
c'(u_{k-4}) = c'(u_{k-2}) = c'(v_{k-1}) = 2, \quad c'(u_{k-3}) = c'(u_{k-1}) = c'(y) = 1,
\]

\[
c'(v_{k-3}) = c'(v_{k-2}) = 3.
\]

Moreover, \(v_{i-1}\) can be re-colored by the third color since \(\deg(v_{i-1}) = 2\). In this case, since we do not change the color of \(x\), \(c'\) and \(c\) are distinct 3-colorings.

![Figure 10](image)

Figure 10: A new 3-coloring \(c'\) of \(H_k - xv_{i-1}\) in Case 1 for fixed \(i = 4\)

**Case 2.** Remove \(xv_i\) or \(u_iv_i\) (see Figure 11)
Similarly to Case 1, we re-color the vertices as follows ($1 \leq j \leq i$):
\[
c'(u_j) = \begin{cases} 
  2 & \text{if } j \text{ is odd} \\
  3 & \text{if } j \text{ is even}
\end{cases}, \quad c'(v_j) = 1, \quad c'(v_k) = 1.
\]

Figure 11: A new 3-coloring $c'$ of $H_k - xv_i$ in Case 2 for fixed $i = 4$

Moreover, $v_i$ can be re-colored by the third color since $\deg(v_i) = 2$. In this case, since we do not change the color of $x$, $c'$ and $c$ are distinct 3-colorings.

Therefore, since we can obtain distinct 3-colorings of the graph in both cases, $H_k$ is edge-critical. Hence we complete the proof.

4 Concluding Remarks

In this section, we describe the size of edge-critical uniquely 3-colorable abstract graphs. Similarly to Lemma 6, it is not difficult to see that any edge-critical uniquely 3-colorable graph has no wheel as its subgraphs. In [6], the size of any graph $G$ which has no wheel as its subgraphs is at most $\left\lfloor \frac{|V(G)|^2}{4} \right\rfloor + \left\lfloor \frac{|V(G)| + 1}{4} \right\rfloor$. Hence, the size of any edge-critical uniquely 3-colorable abstract graph $G$ is at most $\left\lfloor \frac{|V(G)|^2}{4} \right\rfloor + \left\lfloor \frac{|V(G)| + 1}{4} \right\rfloor$.

We think that this estimation is not sharp similarly to our main results in this paper. However, since the estimation in [6] is best possible, we have to find a “good” structure to improve the estimation.

References


