A New Statistic on the Hyperoctahedral Groups

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Abstract

We introduce a new statistic on the hyperoctahedral groups (Coxeter groups of type B), and give a conjectural formula for its signed distributions over arbitrary descent classes. The statistic is analogous to the classical Coxeter length function, and features a parity condition. For descent classes which are singletons the conjectured formula gives the Poincaré polynomials of the varieties of symmetric matrices of fixed rank.

For several descent classes we prove the conjectural formula. For this we construct suitable supporting sets for the relevant generating functions. We prove cancellations on the complements of these supporting sets using suitably defined sign reversing involutions.

Keywords: Hyperoctahedral groups, signed permutation statistics, sign reversing involutions, descent sets, generating functions

1 Introduction

There is an extensive literature concerned with identities for generating functions for S_n, the symmetric group of degree n. These are typically (multi-variable) polynomials obtained by summing the values of integer-valued functions, or *statistics*, on the Coxeter

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group S_n. Sometimes the sums are twisted with the non-trivial linear character of S_n. Occasionally, one can prove more refined versions where the sums are restricted to descent classes. Recently, there has been an interest in finding generalisations, or suitable analogues, of such results for the hyperoctahedral groups; see for example [8, 1, 2, 4]. The *hyperoctahedral group* B_n is the group of permutations w of the set $[\pm n]_0 := \{-n, \dots, n\}$ such that $w(-j) = -w(j)$ for all $j \in [\pm n]_0$.

In the present paper we study generating functions involving a new statistic $L : B_n \to \mathbb{N}_0$, defined as follows. For $w \in B_n$ we set

$$L(w) = \frac{1}{2}\#\{(i,j) \in [\pm n]_0^2 \mid i < j,\ w(i) > w(j),\ i \not\equiv j \bmod (2)\}. \tag{1}$$

To state our results we introduce some further notation. Let $\mathbb{N}$ denote the set of positive integers, and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. For $n \in \mathbb{N}$, let $[n] = \{1, 2, \dots, n\}$ and $[n]_0 = \{0\} \cup [n]$. We write $(\underline{n})_X$ or $(\underline{n})$ for the polynomial $1 - X^n \in \mathbb{Z}[X]$, where X is an indeterminate. We set $(\underline{0}) = 1$ and write $(\underline{n})_X!$ or $(\underline{n})!$ for $(\underline{1})(\underline{2})\cdots(\underline{n})$. For a real number x, we write $\lfloor x \rfloor$ for the largest integer less than or equal to x. Let $I = \{i_1, \dots, i_l\}_< \subseteq [n-1]_0$, that is $i_1 < \dots < i_l$. We put $i_1 = \min(I \cup \{n\})$ and $i_{l+1} = n$, respectively. Let $S = \{s_0, \dots, s_{n-1}\}$ be the set of Coxeter generators for B_n described in [3, Section 8.1] (see also Section 2) and let $l : B_n \to \mathbb{N}_0$ denote the (Coxeter) length function on B_n with respect to S. Define the *quotient* (or *descent class*)

$$B_n^I = \{w \in B_n \mid D(w) \subseteq I^{\mathrm{c}}\},$$

where $D(w) := \{i \in [n-1]_0 \mid l(ws_i) < l(w)\}$ denotes the (right) *descent set* of w and I^{c} denotes the complement $[n-1]_0 \setminus I$; cf. [3, Sections 2.4 and 8.1]. Thus $w \in B_n^{I^{\mathrm{c}}}$ if and only if $D(w) \subseteq I$. For $n \in \mathbb{N}$ and $I = \{i_1, \dots, i_l\}_< \subseteq [n-1]_0$ we define the polynomials

$$f_{n,I}(X) = \frac{(\underline{n})!}{(\underline{i_1})!} \prod_{r=1}^{l} \prod_{\sigma=1}^{\lfloor (i_{r+1}-i_r)/2 \rfloor} (\underline{2\sigma})^{-1} \in \mathbb{Z}[X].$$

In [10] we stated the following conjecture:

Conjecture 1. [10, Conjecture 1.6] For $n \in \mathbb{N}$ and $I = \{i_1, \dots, i_l\}_< \subseteq [n-1]_0$,

$$\sum_{w \in B_n^{I^{\mathrm{c}}}} (-1)^{l(w)} X^{L(w)} = f_{n,I}(X). \tag{2}$$

For instance, if $I = [n-1]_0$ then $B_n^{I^{\mathrm{c}}} = B_n$, and formula (2) reads

$$\sum_{w \in B_n} (-1)^{l(w)} X^{L(w)} = (\underline{n})!.$$

Our main result is the following.

Theorem 2. *Conjecture 1 holds in the following cases:*

1. $n \in \mathbb{N}$ *and* $I = \{0\}$,
2. $n \in \mathbb{N}$ *and* $I = [n-1]_0$,
3. $n \in 2\mathbb{N}$ *and* $I \subseteq [n-1]_0 \cap 2\mathbb{Z}$.

The three parts of Theorem 2 are proved in Sections 3-5, namely Propositions 9, 12, and 25. Our methods are based on defining supporting sets for the sums in question, and sign reversing involutions on their complements which preserve their intersections with the descent classes $B_n^{I^c}$ and leave L invariant. The sets $B_n^{I^c}$ in (2) may thus be replaced by their intersections with the supporting sets; the contributions of the other elements to the sums cancel out. On the supporting sets the statistic L behaves better than on the whole of $B_n^{I^c}$: in Section 5 we establish, for instance, two additivity results for L with respect to certain parabolic factorisations.

For one-element sets $I = \{i\}$, where $i \in [n-1]_0$, the polynomials $f_{n,\{i\}}$ are closely related to the Poincaré polynomials of the varieties of symmetric $n \times n$ matrices over $\mathbb{F}_q$ of rank $n-i$. Indeed, it is well known that, for all prime powers q,

$$\#\{x \in \mathrm{Mat}_n(\mathbb{F}_q) \mid x = x^{\mathrm{t}}, \, \mathrm{rk}(x) = n-i\} = q^{\binom{n+1}{2}-\binom{i+1}{2}} f_{n,\{i\}}\,(q^{-1});$$

see, for instance, [6, Lemma 10.3.1] and compare [10, Lemma 3.1 (3.4)]. It is interesting whether – at least in these cases – Conjecture 1 reflects cohomological properties of the varieties of symmetric matrices of fixed rank.

The restriction of L to S_n agrees with the function L defined in [7, Definition 5.1]. In fact, Conjecture 1 may be seen as a type-B-analogue of [7, Conjecture C]. The polynomials in this conjecture encode the numbers of non-degenerate flags in finite vector spaces equipped with a non-degenerate quadratic form.

The results in the current paper are mainly motivated by our work [10] on representation zeta functions of nilpotent groups. In the remainder of the introduction we describe this connection briefly. Let G be a finitely generated, torsion-free nilpotent group. The *representation zeta function* of G is the Dirichlet generating series

$$\zeta_G(s) := \sum_{n=1}^{\infty} \widetilde{r}_n(G) n^{-s},$$

where s is a complex variable, and $\widetilde{r}_n(G)$ denotes the number of n-dimensional irreducible complex representations of G, up to twisting by 1-dimensional representations. In [10, Theorem C], the representation zeta functions are explicitly computed for three infinite families of groups of nilpotency class 2, namely $F_{2n+\eta}(\mathcal{O})$, $G_{2n}(\mathcal{O})$, $H_{2n}(\mathcal{O})$, where $n \in \mathbb{N}$, $\eta \in \{0,1\}$, and $\mathcal{O}$ is the ring of integers in an arbitrary number field. When $2n+\eta = 2n = 2$ these groups all coincide with the Heisenberg group of 3×3 upper unitriangular matrices over $\mathcal{O}$. Let $\mathbf{G}$ denote any of the group schemes $F_{2n+\eta}$, G_{2n} or H_{2n}. It can be shown that $\zeta_{\mathbf{G}(\mathcal{O})}(s)$ has an Euler product

$$\zeta_{\mathbf{G}(\mathcal{O})}(s) = \prod_{\mathfrak{p}} \zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s),$$

where $\mathfrak{p}$ runs through the non-zero prime ideals of $\mathcal{O}$ and $\mathcal{O}_\mathfrak{p}$ denotes the completion of $\mathcal{O}$ at $\mathfrak{p}$, and that each local factor $\zeta_{\mathbf{G}(\mathcal{O}_\mathfrak{p})}(s)$ is a rational function in q^{-s}, where $q = |\mathcal{O}/\mathfrak{p}|$ is the residue field cardinality at $\mathfrak{p}$. In fact, these properties hold much more generally; see [10, Proposition 2.2 and Corollary 2.19]. In [10] we showed that the local zeta functions $\zeta_{\mathbf{G}(\mathcal{O}_\mathfrak{p})}(s)$ are related to q-series and statistics on hyperoctahedral groups. More precisely, [10, Theorem C] states that there exist a family of polynomials $(f_{\mathbf{G},I}(X))_{I\subseteq[n-1]_0}$ in $\mathbb{Z}[X]$ and integers $(a(\mathbf{G},i))_{i\in[n-1]_0}$ such that, for all $\mathfrak{p}$,

$$\zeta_{\mathbf{G}(\mathcal{O}_\mathfrak{p})}(s) = \sum_{I\subseteq[n-1]_0} f_{\mathbf{G},I}(q^{-1}) \prod_{i\in I} \frac{q^{a(\mathbf{G},i)-(n-i)s}}{1-q^{a(\mathbf{G},i)-(n-i)s}}. \tag{3}$$

The polynomials $f_{\mathbf{G},I}(X)$ turn out to have a combinatorial interpretation: in [10, Proposition 4.6] we showed that, for $I \subseteq [n-1]_0$,

$$f_{F_{2n+\eta},I}(X) = \sum_{w\in B_n^{I^c}} (-1)^{\mathrm{neg}(w)} X^{2l(w)+(2\eta-1)\,\mathrm{neg}(w)}, \tag{4}$$

$$f_{G_{2n},I}(X) = \sum_{w\in B_n^{I^c}} (-1)^{\mathrm{neg}(w)} X^{l(w)}. \tag{5}$$

Here $\mathrm{neg}(w) := \#\{i \in [n] \mid w(i) < 0\}$ for $w \in B_n$. Key to the equations (4) and (5) are formulae for the joint distributions of the statistics neg and l on descent sets of B_n which were given by V. Reiner; cf. [10, Lemma 4.5]. For the group schemes H_{2n}, we know that

$$f_{H_{2n},I}(X) = f_{n,I}(X)$$

(cf. [10, Theorem C]) and Conjecture 1 is a conjectural analogue of (4) and (5). Identities like (2), (4), and (5) often have interesting consequences for zeta functions of the form (3). Provided the statistics involved satisfy suitable invariance conditions, such identities may, for instance, facilitate proofs that the corresponding zeta functions satisfy certain functional equations; see [7, Theorem B].

2 Signed permutations, chessboard elements and supporting sets

Throughout, we keep the notation introduced in Section 1. Let W be a Coxeter group with Coxeter generating set S. For $I \subseteq S$, we denote by $W_I = \langle s_i \mid i \in I\rangle$ the corresponding standard parabolic subgroup of W. We also introduce the quotient $W^I := \{w \in W \mid D(w) \subseteq I^c\}$. It is well known that every element $w \in W$ has a unique factorisation (or "parabolic decomposition")

$$w = w^I w_I, \quad \text{where } w^I \in W^I \text{and } w_I \in W_I. \tag{6}$$

The elements of W^I are the unique representatives of the cosets in W/W_I of shortest length. The Coxeter length function l on W is additive with respect to this factorisation, that is

$$l(w) = l(w^I) + l(w_I); \tag{7}$$

see [5, Section 1.10].

Let now, specifically, W be the hyperoctahedral group B_n. This Coxeter group has a concrete combinatorial description, which we now recall; cf. [3, Section 8.1]. The group B_n has a faithful representation which identifies it with the group of "signed permutation matrices", that is, monomial $n \times n$ matrices with non-zero entries in $\{-1, 1\}$, acting on standard basis column vectors and their negatives. For $w \in B_n$ we use the "window notation" $w = [a_1, \ldots, a_n]$ to mean that, for $i \in [n]$, $w(i) = a_i \in [\pm n]_0$. In this notation, define

$$\begin{aligned} &s_i = [1, \ldots, i-1, i+1, i, i+2, \ldots, n] \text{ for } i \in [n-1] \text{ and} \\ &s_0 = [-1, 2, \ldots, n]. \end{aligned}$$

The set $S := \{s_0, s_1, \ldots, s_{n-1}\}$ is a set of Coxeter generators for B_n. The Coxeter length function with respect to S may be described in terms of certain statistics on B_n. For $w \in B_n$, define

$$\begin{aligned} \operatorname{inv}(w) &= \#\{(i,j) \in [n]^2 \mid i < j,\ w(i) > w(j)\}, \\ \operatorname{neg}(w) &= \#\{i \in [n] \mid w(i) < 0\}, \\ \operatorname{nsp}(w) &= \#\{\{i,j\} \subseteq [n] \mid i \neq j,\ w(i) + w(j) < 0\}. \end{aligned}$$

It is well known (see [3, Proposition 8.1.1]) that

$$l(w) = \operatorname{inv}(w) + \operatorname{neg}(w) + \operatorname{nsp}(w). \tag{8}$$

The descent set $D(w)$ of an element $w \in B_n$ may be characterised as follows:

$$D(w) = \{i \in [n-1]_0 \mid w(i) > w(i+1)\}.$$

We identify the parabolic subgroup $(B_n)_{[n-1]} = \{w \in B_n \mid \operatorname{neg}(w) = 0\}$ with the symmetric group S_n, with standard Coxeter generating set $\{s_1, \ldots, s_n\}$. In the combinatorial description given above, this identifies S_n with the group of $n \times n$ permutation matrices. We will freely switch between viewing elements of B_n as permutations of $[\pm n]_0$ or as signed permutation matrices, as appropriate. Given a Coxeter group W with Coxeter generating set S, we usually just write l for the associated Coxeter length function. Only in case of ambiguity will we use a subscript to indicate the relevant Coxeter group.

Let $M \in \operatorname{Mat}(r \times s; \mathbb{Z})$. If M has exactly one non-zero entry in column $j \in [s]$ we write

$$i_M(j) := i(j) \in [r]$$

for the unique integer i such that $M_{ij} \neq 0$; informally, $i(j)$ indicates the row of M which contains the non-zero entry in column j. Similarly, if M has exactly one non-zero entry in row $i \in [r]$ we write

$$j_M(i) := j(i) \in [s]$$

for the number of the column of M which contains the non-zero entry in row i. In particular, if $w \in B_n$ then $i_w(j) = |w(j)|$ and $j_w(i) = |w^{-1}(i)|$.

We call elements of the quotient $B_n^{[n-1]}$ *ascending*. An element $w \in B_n$ is ascending if and only if $w(1) < w(2) < \cdots < w(n)$. Such an element is determined by its *row pattern*, that is, by the function

$$\rho_w : [n] \longrightarrow \{\pm 1\}, \qquad \rho_w(i) = w_{i,j(i)}, \tag{9}$$

defined for all $w \in B_n$.

Let $n \in \mathbb{N}$ and $I = \{i_1, \ldots, i_l\}_< \subseteq [n-1]_0$. Our first step towards proving Theorem 2 is to show that the sum in (2) is supported on relatively small and manageable subsets of $B_n^{I^c}$ which we now define.

Definition 3. Set

$$\begin{aligned}
\mathcal{C}_{n,0} &= \{(w_{ij}) \in B_n \mid w_{ij} \neq 0 \Longrightarrow i+j \equiv 0 \bmod (2)\},\\
\mathcal{C}_{n,1} &= \{(w_{ij}) \in B_n \mid w_{ij} \neq 0 \Longrightarrow i+j \equiv 1 \bmod (2)\},\\
\mathcal{C}_n &= \mathcal{C}_{n,0} \cup \mathcal{C}_{n,1}.
\end{aligned}$$

We call $\mathcal{C}_n$ the group of *chessboard elements* and $\mathcal{C}_{n,0}$ the subgroup of *even chessboard elements*. Clearly $\mathcal{C}_n$ contains $\mathcal{C}_{n,0}$ as a subgroup of index 2. The name comes from imagining a signed permutation matrix $w \in B_n$ printed on an $n \times n$ "chessboard" made up from white and black squares. The element w is then a chessboard element exactly if all the non-zero entries of w occupy squares of the same colour. Chessboard elements were introduced in [7] for the symmetric group S_n. Definition 3 is an extension of [7, Definition 5.3] to the group B_n.

Let $w = (w_{ij}) \in \mathcal{C}_{n,0}$ and $m_1 = \lfloor \frac{n+1}{2} \rfloor$, $m_2 = \lfloor \frac{n}{2} \rfloor$. Let $w_1 = (w_{2a+1,2b+1})$, where $0 \leqslant a, b \leqslant m_1$, and $w_2 = (w_{2a,2b})$, where $1 \leqslant a, b \leqslant m_2$. Then $w_1 \in B_{m_1}$ and $w_2 \in B_{m_2}$. This defines a group isomorphism

$$\sigma_0 : \mathcal{C}_{n,0} \longrightarrow B_{m_1} \times B_{m_2}, \quad w \longmapsto (w_1, w_2).$$

More generally, let $w \in B_n$ and define

$$\begin{aligned}
w_1 &= (w_{i(2a-1),2a-1}) \in B_{m_1}, & 1 \leqslant a \leqslant m_1,\\
w_2 &= (w_{i(2a),2a}) \in B_{m_2}, & 1 \leqslant a \leqslant m_2.
\end{aligned}$$

Informally, w_1 is the submatrix of w obtained by selecting the odd-numbered columns of w together with the corresponding rows of w, and w_2 is obtained analogously by selecting the even-numbered columns. We obtain a map of sets

$$\sigma : B_n \longrightarrow B_{m_1} \times B_{m_2}, \quad w \longmapsto (w_1, w_2), \tag{10}$$

whose restriction to $\mathcal{C}_{n,0}$ agrees with σ_0. Given $w_1 \in B_{m_1}$ and $w_2 \in B_{m_2}$ we write $w_1 * w_2 := \sigma_0^{-1}(w_1, w_2) \in \mathcal{C}_{n,0}$ for the unique even chessboard element in the fibre $\sigma^{-1}(w_1, w_2)$.

Our next aim is to give a combinatorial description of the statistic L, akin to the formula (8) for the Coxeter length function on B_n. To this end, we introduce the following statistics.

Definition 4. Let $r, s \in \mathbb{N}$ and $M = (M_{ij}) \in \mathrm{Mat}(r \times s, \mathbb{Z})$. Let $\mathcal{S} \subseteq [s]$ denote the set of indices of columns of M which contain a unique non-zero entry. Define

$$\begin{aligned}
a(M) &= \#\{j \in \mathcal{S} \mid M_{i(j),j} = -1,\ j \not\equiv 0 \bmod (2)\},\\
b(M) &= \#\{(j, j') \in \mathcal{S}^2 \mid j < j',\ i(j) > i(j'),\ j \not\equiv j' \bmod (2)\},\\
c(M) &= \#\{(j, j') \in \mathcal{S}^2 \mid M_{i(j'),j'} = -1,\ j < j',\ i(j) < i(j'),\ j \not\equiv j' \bmod (2)\}.
\end{aligned}$$

By viewing elements of B_n as signed permutation matrices, these formulae define, in particular, functions a, b and c on the hyperoctahedral groups.

Example 5. Consider $w = [1, -4, -3, 2] \in B_4$. Then $a(w) = c(w) = 1$ and $b(w) = 2$.

The following characterisation of L will be used throughout the paper.

Lemma 6. *Let $w \in B_n$ and $\sigma(w) = (w_1, w_2)$. Then*

$$\begin{aligned}
L(w) &= a(w) + b(w) + 2c(w) \text{ and} &(11)\\
L(w) &= \mathrm{neg}(w_1) + (\mathrm{inv} + \mathrm{nsp})(w) - (\mathrm{inv} + \mathrm{nsp})(w_1) - (\mathrm{inv} + \mathrm{nsp})(w_2) &(12)\\
&= \mathrm{neg}(w_1) + l(w) - l(w_1) - l(w_2).
\end{aligned}$$

Proof. To prove (11), set

$$\mathfrak{N}_n = \{(i, j) \in [\pm n]_0^2 \mid |i| < |j|,\ i < j,\ i \not\equiv j \bmod (2)\}$$

and

$$\mathfrak{M}_n(w) = \{(i, j) \in \mathfrak{N}_n \mid w(i) > w(j)\}.$$

By definition, $L(w) = \#\mathfrak{M}_n(w)$. Let $(i, j) \in \mathfrak{N}_n$. If $|w(i)| > |w(j)|$ then exactly one of (i, j) and $(-i, j)$ are in $\mathfrak{M}_n(w)$. (Note that in this case $i \neq 0$.) Thus $b(w) = \#\{(i, j) \in \mathfrak{M}_n(w) \mid |w(i)| > |w(j)|\}$. If $|w(i)| < |w(j)|$ we distinguish further by the sign of $w(j)$. If $w(j) < 0$ then both (i, j) and $(-i, j)$ are in $\mathfrak{M}_n(w)$. Note that $(i, j) = (-i, j)$ if and only if $i = 0$, in which case j is odd. If $w(j) > 0$ then neither (i, j) nor $(-i, j)$ are in $\mathfrak{M}_n(w)$. Thus $a(w)+2c(w) = \#\{(i, j) \in \mathfrak{M}_n(w) \mid |w(i)| < |w(j)|\}$ and $L(w) = a(w)+b(w)+2c(w)$ as claimed.

We now prove (12). First, it is clear that $a(w) = \mathrm{neg}(w_1)$. Omitting the parity conditions in the definitions of the functions b and c given in Definition 4 yields

$$\begin{aligned}
\bar{b}(M) &:= \#\{(j, j') \in \mathcal{S}^2 \mid j < j',\ i(j) > i(j')\},\\
\bar{c}(M) &:= \#\{(j, j') \in \mathcal{S}^2 \mid M_{i(j'),j'} = -1,\ j < j',\ i(j) < i(j')\}.
\end{aligned}$$

We claim that $\bar{b}+2\bar{c} = \mathrm{inv} + \mathrm{nsp}$ on B_n. To show this, we make the following observations. The function $\bar{b} + 2\bar{c}$ counts certain column pairs (j, j'), depending only on the 2×2 submatrices determined by (j, j'). The same is true for the function $\mathrm{inv} + \mathrm{nsp}$. To establish the claim thus amounts to checking it on B_2. A simple calculation confirms it there, and thus $\bar{b} + 2\bar{c} = \mathrm{inv} + \mathrm{nsp}$ on B_n. We further observe that

$$b(w) = \bar{b}(w) - \bar{b}(w_1) - \bar{b}(w_2) \quad \text{and} \quad c(w) = \bar{c}(w) - \bar{c}(w_1) - \bar{c}(w_2).$$

Using (11) this yields

$$\begin{aligned}L(w) &= a(w) + b(w) + 2c(w)\\ &= a(w) + \bar{b}(w) - \bar{b}(w_1) - \bar{b}(w_2) + 2(\bar{c}(w) - \bar{c}(w_1) - \bar{c}(w_2))\\ &= \operatorname{neg}(w_1) + (\operatorname{inv} + \operatorname{nsp})(w) - (\operatorname{inv} + \operatorname{nsp})(w_1) - (\operatorname{inv} + \operatorname{nsp})(w_2).\end{aligned}$$

Using, finally, the facts that $l = \operatorname{inv} + \operatorname{nsp} + \operatorname{neg}$ (see (8)) and $\operatorname{neg}(w) = \operatorname{neg}(w_1) + \operatorname{neg}(w_2)$, we obtain the second equality in (12). □

The unique longest element of B_n is $w_0 = [-1, -2, \dots, -n]$, of length $l(w_0) = n^2$. It is well known that the Coxeter length function l on B_n is well-behaved under multiplication by w_0. More precisely, the equalities

$$l(ww_0) = l(w_0w) = l(w_0) - l(w)$$

hold for all $w \in B_n$; cf. [5, Section 1.8]. At least in this respect the statistic L behaves analogously.

Corollary 7. *Let $w_0 \in B_n$ be the longest element. Then, for all $w \in B_n$, we have*

$$L(ww_0) = L(w_0w) = L(w_0) - L(w).$$

Moreover, the trivial element in B_n is the only element $w \in B_n$ with $L(w) = 0$, and hence w_0 is the unique element in B_n on which L attains its maximum $\binom{n+1}{2}$.

Proof. Let $w \in B_n$. Note that $w_0 = -\operatorname{Id}_n$, where Id_n is the $n \times n$ identity matrix, so $ww_0 = w_0w = -w$. Obviously $\operatorname{neg}(w_0) = n$, and so $\operatorname{neg}(ww_0) = n - \operatorname{neg}(w) = \operatorname{neg}(w_0) - \operatorname{neg}(w)$. Since $l = \operatorname{inv} + \operatorname{nsp} + \operatorname{neg}$, we thus have

$$(\operatorname{inv} + \operatorname{nsp})(-w) = (\operatorname{inv} + \operatorname{nsp})(w_0) - (\operatorname{inv} + \operatorname{nsp})(w) = n^2 - n - (\operatorname{inv} + \operatorname{nsp})(w). \quad (13)$$

Let $\sigma(w) = (w_1, w_2) \in B_{m_1} \times B_{m_2}$, where $m_1 = \lfloor \frac{n+1}{2} \rfloor$ and $m_2 = \lfloor \frac{n}{2} \rfloor$. Using Lemma 6 (12) together with (13) and the fact that $n = m_1 + m_2$, we then obtain

$$\begin{aligned}L(ww_0) &= L(-w)\\ &= \operatorname{neg}(-w_1) + (\operatorname{inv} + \operatorname{nsp})(-w) - (\operatorname{inv} + \operatorname{nsp})(-w_1) - (\operatorname{inv} + \operatorname{nsp})(-w_2)\\ &= m_1 - \operatorname{neg}(w_1) + n^2 - n - (\operatorname{inv} + \operatorname{nsp})(w) - (m_1^2 - m_1 - (\operatorname{inv} + \operatorname{nsp})(w_1))\\ &\quad - (m_2^2 - m_2 - (\operatorname{inv} + \operatorname{nsp})(w_2))\\ &= m_1 + n^2 - n - m_1^2 + m_1 - m_2^2 + m_2 - L(w) = m_1(2m_2 + 1) - L(w).\end{aligned}$$

Using Lemma 6 (11) it is easy to see that $L(w_0) = \binom{n+1}{2}$. Clearly $m_1(2m_2 + 1) = \binom{n+1}{2}$, so $L(ww_0) = L(w_0w) = L(w_0) - L(w)$, as asserted. This immediately implies that $L(w_0) = \binom{n+1}{2}$ is the maximal value attained by L. To see that w_0 is the unique element on which L attains its maximum, it suffices to show that $L(w) = 0$ implies $w = 1$. Assume thus that $L(w) = 0$, for some $w \in B_n$. By Lemma 6 (11), this implies that $a(w) = b(w) = c(w) = 0$. Let $j \in [n-1]$. Then $b(w) = 0$ implies that $i(j) < i(j+1)$, and $c(w) = 0$ then implies that $i(j+1) = i(j) + 1$. Since this is true for all $j \in [n-1]$, we have either $w = 1$ or $w = s_0$. But $a(s_0) = 1$, so we must have $w = 1$. □

As mentioned previously, our approach to proving Conjecture 1 is to show that the sum in (2) is supported on certain proper subsets of $B_n^{I^c}$. The following is our first result in this direction, and says that the sum is supported on the even chessboard elements in $B_n^{I^c}$. Key to its proof is the construction of a suitable sign reversing involution. For any subset $X \subseteq B_n$ and $I \subseteq [n-1]_0$, we set

$$X^I := X \cap B_n^I.$$

Lemma 8. *For $n \in \mathbb{N}$ and $I \subseteq [n-1]_0$,*

$$\sum_{w \in B_n^{I^c}} (-1)^{l(w)} X^{L(w)} = \sum_{w \in \mathcal{C}_{n,0}^{I^c}} (-1)^{l(w)} X^{L(w)}.$$

Proof. Let $w = (w_{ij}) \in B_n \setminus \mathcal{C}_{n,0}$. Thus there exists $i \in [n-1]_0$ such that $j(i) \equiv j(i+1) \bmod (2)$. Let i be minimal with this property and set $w^* := s_i w$. Lemma 6 (11) implies that $L(w) = L(w^*)$. Moreover, $l(w) = l(w^*) \pm 1$. Since $|j(i) - j(i+1)| \geqslant 2$, we have $D(w) = D(w^*)$. Note that $(w^*)^* = w$ and $w \neq w^*$. Every element $w \in B_n^{I^c} \setminus \mathcal{C}_{n,0}$ may thus be paired up with a unique, distinct element $w^* \in B_n^{I^c} \setminus \mathcal{C}_{n,0}$, such that $(-1)^{l(w)} L(w) + (-1)^{l(w^*)} L(w^*) = 0$. This implies the assertion. $\square$

3 The case $I = \{0\}$

In this section we prove Conjecture 1 in the case where $I = \{0\}$, that is Case (1) of Theorem 2. In this case, the sum in (2) runs over $B_n^{[n-1]}$, that is, ascending matrices. Let $\tilde{n} := 2[\frac{n-1}{2}] + 1$ be the largest odd integer less than or equal to n. Then, by definition,

$$f_{n,\{0\}}(X) = \frac{(\underline{n})!}{\prod_{\sigma=1}^{\lfloor n/2 \rfloor} (\underline{2\sigma})} = (\underline{1})(\underline{3}) \cdots (\underline{\tilde{n}}).$$

Proposition 9. *Conjecture 1 holds for $I = \{0\}$, that is*

$$\sum_{w \in B_n^{[n-1]}} (-1)^{l(w)} X^{L(w)} = (\underline{1})(\underline{3}) \ldots (\underline{\tilde{n}}).$$

Proof. By Lemma 8, it is enough to prove the assertion where the sum runs over $\mathcal{C}_{n,0}^{[n-1]}$, that is ascending even chessboard elements. Assume first that n is odd, and that Proposition 9 is true for n. Since n is odd, we have $\tilde{n} = \widetilde{n+1} = n$. In this case restriction of to $[\pm n]_0$ yields a one-to-one correspondence between elements of $\mathcal{C}_{n+1,0}^{[n]}$ and elements of $\mathcal{C}_{n,0}^{[n-1]}$. Indeed, if $w \in \mathcal{C}_{n+1,0}^{[n]}$ then $w_{n+1,n+1} = 1$. Moreover, it is clear that L and l are preserved under this correspondence. Hence

$$\sum_{w \in \mathcal{C}_{n+1,0}^{[n]}} (-1)^{l(w)} X^{L(w)} = \sum_{w \in \mathcal{C}_{n,0}^{[n-1]}} (-1)^{l(w)} X^{L(w)} = (\underline{1})(\underline{3}) \cdots (\underline{\tilde{n}}) = (\underline{1})(\underline{3}) \cdots (\underline{\widetilde{n+1}}).$$

Hence, if Proposition 9 is true for all odd n then it is also true for all even n.

We now prove Proposition 9 for odd n by induction in steps of two. For $n = 1$ we have $f_{1,\{0\}}(X) = 1 - X$, and $B_1 = \{1, -1\} = B_1^{\varnothing}$, so

$$\sum_{w \in B_1^{\varnothing}} (-1)^{l(w)} X^{L(w)} = (-1)^{l(1)} X^{L(1)} + (-1)^{l(-1)} X^{L(-1)} = 1 - X.$$

Assume now that n is odd and that Proposition 9 holds for n. We show how every element in $\mathcal{C}_{n+2,0}^{[n+1]}$ is obtained from one in $\mathcal{C}_{n,0}^{[n-1]}$ in exactly one of two ways. Let $w \in \mathcal{C}_{n,0}^{[n-1]}$. Then we may associate to w two elements in $\mathcal{C}_{n+2,0}^{[n+1]}$, namely

$$w^+ := \begin{pmatrix} w & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad \text{and} \quad w^- := \begin{pmatrix} & & w \\ & -1 & \\ -1 & & \end{pmatrix}.$$

We claim that all elements in $\mathcal{C}_{n+2,0}^{[n+1]}$ are of this form. To see this, let $v \in \mathcal{C}_{n+2,0}^{[n+1]}$. If $v_{n+2,j(n+2)} = 1$ then $j(n+2) = n+2$, since v is ascending. Deleting row $n+2$ and column $n+2$ leaves an element $v' \in \mathcal{C}_{n+1,0}^{[n]}$. Since $v \in \mathcal{C}_{n+2,0}^{[n+1]}$, we must have $v'_{n+1,n+1} = 1$, and so v is of the form w^+. If $v_{n+2,j(n+2)} = -1$ then $j(n+2) = 1$, since v is ascending. Deleting row $n+2$ and column 1 leaves an element $v' \in \mathcal{C}_{n+1,1}^{[n]}$. Hence $w_{n+1,2} = -1$ and v is of the form w^-.

By Lemma 6 (11), we have $L(w^+) = L(w)$ and $L(w^-) = n + 2 + L(w)$. Moreover, $l(w^+) = l(w)$, and

$$l(w^-) = (\mathrm{neg} + \mathrm{nsp})(w^-) = 2 + \mathrm{neg}(w) + 2n + 1 + \mathrm{nsp}(w) \equiv 1 + l(w) \bmod (2).$$

By the induction hypothesis, we obtain

$$\begin{aligned}
\sum_{w \in \mathcal{C}_{n+2,0}^{[n+1]}} (-1)^{l(w)} X^{L(w)} &= \sum_{w \in \mathcal{C}_{n,0}^{[n-1]}} \left((-1)^{l(w^+)} X^{L(w^+)} + (-1)^{l(w^-)} X^{L(w^-)} \right) \\
&= \sum_{w \in \mathcal{C}_{n,0}^{[n-1]}} \left((-1)^{l(w)} X^{L(w)} + (-1)^{1+l(w)} X^{n+2+L(w)} \right) \\
&= \sum_{w \in \mathcal{C}_{n,0}^{[n-1]}} (-1)^{l(w)} X^{L(w)} (1 - X^{n+2}) \\
&= (\underline{1})(\underline{3}) \cdots (\underline{n})(1 - X^{n+2}) = (\underline{1})(\underline{3}) \cdots (\underline{n})(\widetilde{\underline{n+2}}).
\end{aligned}$$

$\square$

We record, without further proof, a corollary of the proof of Proposition 9 on the structure of ascending even chessboard elements.

Lemma 10. *Let $w \in \mathcal{C}_{n,0}^{[n-1]}$ be an ascending even chessboard element and $j \in \lfloor \frac{n}{2} \rfloor$. Then $i(2j) - i(2j-1)$ is odd. Furthermore, the following hold:*

1. *If $w(2j) > 0$ then $i(2j) - i(2j-1) > 0$ and $w^{-1}(e) < 0$ for all $e \in \mathbb{N}$ such that $i(2j-1) < e < i(2j)$. Moreover, $w(2j-1) > 0$ unless possibly if $i(2j-1) = 1$.*

2. *If $w(2j) < 0$ then $i(2j-1) - i(2j) = 1$.*

Informally, an ascending even chessboard element is built up from pairs of adjacent columns satisfying one of the following:

(1) Both columns typically contain positive entries, "sandwiching" an even number of consecutive rows of w, all containing negative entries.

(2) Both columns contain negative entries in adjacent rows of w.

In particular, ascending even chessboard elements have no *odd sandwich* in the sense of Definition 15.

4 The case $I = [n-1]_0$

In this section we prove Conjecture 1 in the case where $I = [n-1]_0$, that is Case (2) of Theorem 2. In this case, we have $B_n^{I^c} = B_n$ and $f_{n,[n-1]_0}(X) = \frac{(\underline{n})!}{\prod_{\sigma=1}^{0}(\underline{2\sigma})} = (\underline{n})!$. By Lemma 8, the sum defining $f_{n,[n-1]_0}(X)$ is supported on even chessboard matrices, that is

$$\sum_{w \in B_n} (-1)^{l(w)} X^{L(w)} = \sum_{w \in \mathcal{C}_{n,0}} (-1)^{l(w)} X^{L(w)}.$$

We now show that the latter sum is supported on diagonal elements. More precisely, let

$$\mathcal{D}_n = \{(w_{ij}) \in \mathcal{C}_{n,0} \mid w_{ij} = 0 \text{ if } i \neq j\}$$

denote the subgroup of $\mathcal{C}_{n,0}$ consisting of diagonal elements.

Lemma 11.

$$\sum_{w \in \mathcal{C}_{n,0}} (-1)^{l(w)} X^{L(w)} = \sum_{w \in \mathcal{D}_n} (-1)^{l(w)} X^{L(w)}.$$

Proof. Observe that $w \in \mathcal{C}_{n,0} \setminus \mathcal{D}_n$ if and only if there exists $i \in [n-2]_0$ such that either $j(i+1) < \min\{j(i), j(i+2)\}$ or $j(i+1) > \max\{(j(i), j(i+2)\}$. Let $w \in \mathcal{C}_{n,0} \setminus \mathcal{D}_n$ and let i be minimal with respect to this property. Define $w^\circ := s_{i+1}s_is_{i+1}w \in \mathcal{C}_{n,0} \setminus \mathcal{D}_n$. Informally, w° is obtained from w by interchanging rows i and $i+2$ if i is positive, and by changing the sign in row 2 if $i = 0$. Clearly $l(w^\circ) \equiv l(w) + 1 \bmod (2)$. Using Lemma 6 it is easy to see that $L(w^\circ) = L(w)$. Note that $(w^\circ)^\circ = w$ and $w \neq w^\circ$. Every element $w \in \mathcal{C}_{n,0} \setminus \mathcal{D}_n$ may thus be paired up with a unique, distinct element $w^\circ \in \mathcal{C}_{n,0} \setminus \mathcal{D}_n$ such that $(-1)^{l(w)}L(w) + (-1)^{l(w^\circ)}L(w^\circ) = 0$. This implies the assertion. $\square$

Proposition 12. *Conjecture 1 holds for $I = [n-1]_0$, that is*

$$\sum_{w \in B_n} (-1)^{l(w)} X^{L(w)} = (\underline{n})!.$$

Proof. The proof is by induction on n. The assertion holds trivially for $n = 1$. Assume now that the assertion is true for some $n - 1 \geqslant 1$. Given $v \in \mathcal{D}_{n-1}$ we define

$$v^+ := \begin{pmatrix} v & \\ & 1 \end{pmatrix} \in \mathcal{D}_n \quad \text{and} \quad v^- := \begin{pmatrix} v & \\ & -1 \end{pmatrix} \in \mathcal{D}_n.$$

Using the formula $l = \mathrm{inv} + \mathrm{neg} + \mathrm{nsp}$ (cf. (8)) and Lemma 6 (11) we see that

$$\begin{aligned} l(v^+) &= l(v), & l(v^-) &= l(v) + 2n - 1, \\ L(v^+) &= L(v), & L(v^-) &= L(v) + n. \end{aligned}$$

Hence, by Lemma 8, Lemma 11 and the induction hypothesis, we obtain

$$\begin{aligned} \sum_{w \in B_n} (-1)^{l(w)} X^{L(w)} &= \sum_{w \in \mathcal{C}_{n,0}} (-1)^{l(w)} X^{L(w)} = \sum_{w \in \mathcal{D}_n} (-1)^{l(w)} X^{L(w)} \\ &= \sum_{v \in \mathcal{D}_{n-1}} \left((-1)^{l(v^+)} X^{L(v^+)} + (-1)^{l(v^-)} X^{L(v^-)} \right) \\ &= \sum_{v \in \mathcal{D}_{n-1}} \left((-1)^{l(v)} X^{L(v)} + (-1)^{l(v)+2n-1} X^{L(v)+n} \right) \\ &= \sum_{v \in \mathcal{D}_{n-1}} (-1)^{l(v)} X^{L(v)} (1 - X^n) = (\underline{n-1})!(1 - X^n) = (\underline{n})!. \end{aligned}$$

□

5 The case n even and I even

In this section we push further the ideas that led to the proof of Lemma 8. There we proved that the relevant sums over $B_n^{I^c}$ are supported over chessboard matrices $\mathcal{C}_{n,0}^{I^c}$. In the proof we described a sign reversing involution $*$ on $B_n \setminus \mathcal{C}_{n,0}$ such that $D(w) = D(w^*)$, $L(w) = L(w^*)$ and $l(w) \not\equiv l(w^*) \bmod (2)$ for all $w \in B_n \setminus \mathcal{C}_{n,0}$. Consequently, these elements' contributions to the sums in question cancelled each other out. A similar idea was used in the proof of Lemma 11. In the current section we further restrict the "supporting sets" $\mathcal{C}_{n,0}^{I^c}$ and show how, under suitable conditions, elements outside these sets may be cancelled by means of a sign reversing involution; see Definition 17. In Sections 5.2 and 5.3 we establish additivity results for L with respect to two parabolic factorisations. In conjunction, they allow us to establish Conjecture 1 in the case where n is even and $I \subseteq [n-1]_0 \cap 2\mathbb{Z}$, that is Case (3) of Theorem 2, in Proposition 25.

5.1 Parabolic factorisations and supporting sets

Recall that we may factorise any element $w \in B_n$ as $w = w^{[n-1]}w_{[n-1]}$, where $w^{[n-1]} \in B_n^{[n-1]}$ is ascending, and $w_{[n-1]} \in \langle s_1, \ldots, s_{n-1}\rangle \cong S_n$; cf. (6). Let $w \in \mathcal{C}_{n,0}$ be an even chessboard element. Since $\mathcal{C}_n$ is a group containing $\mathcal{C}_{n,0}$ as a subgroup, there are three possibilities for this factorisation of w:

1. $w^{[n-1]}, w_{[n-1]} \in \mathcal{C}_{n,0}$,
2. $w^{[n-1]}, w_{[n-1]} \in \mathcal{C}_{n,1}$,
3. $w^{[n-1]}, w_{[n-1]} \in B_n \setminus \mathcal{C}_n$.

Definition 13. Let

$$\mathcal{E}_n = \{w \in \mathcal{C}_{n,0} \mid w^{[n-1]}, w_{[n-1]} \in \mathcal{C}_{n,0}\}$$

denote the set of even chessboard elements whose factorisation is into even chessboard elements. Similarly, let

$$\mathcal{M}_n = \{w \in \mathcal{C}_{n,0} \mid (w^{[n-1]}, w_{[n-1]} \in \mathcal{C}_{n,0}) \vee (w^{[n-1]}, w_{[n-1]} \in \mathcal{C}_{n,1})\}$$

denote the set of even chessboard elements whose factorisation is into chessboard elements.

Note that $\mathcal{E}_n \subseteq \mathcal{M}_n \subseteq \mathcal{C}_{n,0}$ and that $\mathcal{M}_n = \mathcal{E}_n$ if n is odd.

Some of the key features of the case where n and I are even are recorded in the following lemma. A subset $I = \{i_1, i_2, \ldots, i_l\}_< \subseteq [n-1]_0 \cap 2\mathbb{Z}$ is called *even*. We say that w is of *even descent type* if $I = D(w)$ is even.

Lemma 14. *Let n be even and $w \in \mathcal{C}_n \cap S_n$ be a chessboard element in S_n. Suppose that $D(w)$ is even, and write $\sigma(w) = (w_1, w_2) \in S_{n/2} \times S_{n/2}$. Then $w \in \mathcal{C}_{n,0}$ and $w_1 = w_2 \in S_{n/2}^{(I/2)^c}$. In particular, for all even $I \subseteq [n-1]_0$,*

$$\mathcal{M}_n^{I^c} = \mathcal{E}_n^{I^c}.$$

Moreover, $l_{S_n}(w) = 4l_{S_{n/2}}(w_1)$.

Informally, Lemma 14 states that w is a "block permutation matrix", composed of 2×2 identity matrices.

Proof. Let $w = (w_{ij})$, and recall that $w_{i,j(i)}$ denotes the non-zero entry in the i-th row. Assume first that $w \in \mathcal{C}_{n,1}$. Then $i + j(i)$ is odd for all $i \in [n]$, so $j(1)$ is even. This implies that w has a descent at $j(1) - 1$, which is impossible since $D(w)$ is even. Thus $w \in \mathcal{C}_{n,0}$, and $i + j(i)$ is even for all $i \in [n]$. Suppose that $j(2) \leqslant j(1) - 1$ or $j(2) \geqslant j(1) + 3$. Then $i(j(2) - 1) \geqslant 3$, so there is a descent at $j(2) - 1$, contradicting the assumption that $D(w)$ is even. Thus $j(2) = j(1) + 1$. Continuing the same argument for the $2i + 1$-th and $2i + 2$-th row, for each $i \in [\frac{n}{2} - 1]$, we obtain

$$j(2i + 2) = j(2i + 1) + 1, \text{ for all } i \in [\frac{n}{2} - 1]_0.$$

By definition (10) of the map σ this means that $w_1 = w_2 \in S_{n/2}$.

Furthermore, w has a descent at $2a$ if and only if w_1 has a descent at a, for all $a \in [n-1]$. Hence $w_1 \in S_{n/2}^{(I/2)^c}$. This implies that $w \in \mathcal{M}_n$ if and only if $w \in \mathcal{E}_n$. Indeed, if $w_{[n-1]} \in \mathcal{C}_n$, then we have shown that $w_{[n-1]} \in \mathcal{C}_{n,0}$, and so $w^{[n-1]} \in \mathcal{C}_{n,0}$, and hence $w \in \mathcal{E}_n$. Thus $\mathcal{M}_n^{I^c} = \mathcal{E}_n^{I^c}$ for all even I. The statement about the lengths is clear. $\square$

Definition 15. Let $w = (w_{ij}) \in B_n$. A pair of natural numbers (r, h), where $r \in [n-2]$ and $h \in [n-1]$ is odd, is said to be an *odd sandwich in* w if it satisfies one of the following conditions:

1. $w_{r,j(r)} = w_{r+h+1,j(r+h+1)}$, and $w_{r,j(r)} \neq w_{r+i,j(r+i)}$ for all $i \in [h]$,
2. $r = 1$, $w_{1,j(1)} = w_{1+i,j(1+i)}$ for all $i \in [h]$, and $w_{1,j(1)} \neq w_{1+h+1,j(1+h+1)}$.

We say that w *has an odd sandwich* if there exists an odd sandwich in w.

Recall that $s_0 \in S$ is the Coxeter generator such that, for any $w \in B_n$, the matrix $s_0 w$ is obtained by changing the sign of $w_{1,j(1)}$. Informally speaking, w has an odd sandwich if and only if in either w or $s_0 w$ there exists a row containing a 1, followed by an odd number of consecutive rows containing -1s, followed by a row containing a 1, or if there exists a row containing a -1, followed by an odd number of consecutive rows containing 1s, followed by a row containing a -1.

Lemma 16. *Let $w \in \mathcal{C}_{n,0}$. Then $w \in \mathcal{M}_n$ if and only if w has no odd sandwich.*

Proof. Write $w = (w_{ij}) = w^{[n-1]} w_{[n-1]}$. Then w has the same row pattern as $w^{[n-1]}$; cf. (9). Moreover, $w \in \mathcal{M}_n$ if and only if $w^{[n-1]} \in \mathcal{C}_n$. To prove the lemma, it therefore suffices to prove that for any $v \in B_n^{[n-1]}$ we have $v \in \mathcal{C}_n^{[n-1]}$ if and only if v has no odd sandwich.

Assume that $v \in \mathcal{C}_n^{[n-1]}$ and that v has an odd sandwich (r, h). It is easily seen that the smallest integer $i \in [n-1]$ such that $v_{i,j_v(i)} \neq v_{i+1,j_v(i+1)}$ is odd. Since v is ascending, we have $|j_v(r) - j_v(r+h+1)| = 1$. But since h is odd, we have $r + j_v(r) \not\equiv r + h + 1 + j_v(r+h+1) \bmod (2)$, and so $v \notin \mathcal{C}_n$; contradiction. Thus $v \in \mathcal{C}_n^{n-1}$ implies that v does not have an odd sandwich.

Conversely, assume that $v \in B_n^{[n-1]} \backslash \mathcal{C}_n$. This means that there exists an integer $j \in [n]$, such that $i_v(j) + j \not\equiv i_v(j+1) + j + 1 \bmod (2)$. (Informally, the non-zero entries in columns j and $j+1$ are on chessboard squares of different colours.) In particular, $h := |i_v(j) - i_v(j+1)| - 1$ is odd. Let $r := \min\{i_v(j), i_v(j+1)\}$, so that $r + h + 1 = \max\{i_v(j), i_v(j+1)\}$. If $v_{r,j(r)} = v_{r+h+1,j(r+h+1)}$ then, because v is ascending, $v_{r,j(r)} \neq v_{r+s,j(r+s)}$, for all $s \in [h]$, so v has an odd sandwich (r, h). If $v_{r,j(r)} \neq v_{r+h+1,j(r+h+1)}$ then, again because v is ascending, $r = 1$, and so v has an odd sandwich $(1, h)$. In either case $v \in B_n^{[n-1]} \setminus \mathcal{C}_n$ implies that v has an odd sandwich. $\square$

Let $w \in \mathcal{C}_{n,0} \setminus \mathcal{M}_n$. By Lemma 16 this means that w has an odd sandwich. Let (r, h) be the *topmost odd sandwich* in w, that is the unique odd sandwich in w such that if (r', h') is another odd sandwich in w, then $r \leqslant r'$. In the following we define an element $w^\vee \in \mathcal{C}_{n,0} \setminus \mathcal{M}_n$ with the property that $L(w) = L(w^\vee)$, the positive parts of the descent sets $D(w)$ and $D(w^\vee)$ agree and the parities of $l(w)$ and $l(w^\vee)$ differ. For this end, we factorise $w = w^{[n-1]} w_{[n-1]}$ with $w^{[n-1]} \in B_n^{[n-1]}$ and $w_{[n-1]} \in (B_n)_{[n-1]} \cong S_n$. Since $w^{[n-1]}$ is ascending, the non-zero entries in rows r and $r+h+1$ must lie in adjacent columns; in other words, if $j := j_{w^{[n-1]}}(r)$ and $j' := j_{w^{[n-1]}}(r+h+1)$ then $|j - j'| = 1$. Set $\mu = \min\{j, j'\} \in [n]$.

Definition 17. Given $w \in \mathcal{C}_{n,0} \setminus \mathcal{M}_n$ with topmost odd sandwich (r, h) and μ as above. Set
$$w^\vee = w^{[n-1]} s_\mu w_{[n-1]} \in \mathcal{C}_{n,0} \setminus \mathcal{M}_n.$$

Informally, $w^\vee$ is obtained from $w = w^{[n-1]} w_{[n-1]}$ from transposing columns μ and $\mu + 1$ in $w^{[n-1]}$ or, equivalently, transposing rows μ and $\mu + 1$ in $w_{[n-1]}$. The element $w^\vee$ may also be thought of as obtained from w by interchanging columns r and $r + h + 1$, deliminating the topmost odd sandwich in w. Before we prove that the involution $w \mapsto w^\vee$ on $\mathcal{C}_{n,0} \setminus \mathcal{M}_n$ has the desired properties, we consider an example.

Example 18. For $n = 3$, let
$$w = \begin{pmatrix} & & -1 \\ & -1 & \\ 1 & & \end{pmatrix} = \begin{pmatrix} & -1 & \\ -1 & & \\ & & 1 \end{pmatrix} \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix} = w^{[2]} w_{[2]} \in \mathcal{C}_{3,0}^{\{0,2\}} \setminus \mathcal{M}_3,$$
with $l(w) = 5$, $L(w) = 3$ and $D(w) = \{1\}$. The unique – and therefore topmost – odd sandwich in w is $(r, h) = (1, 1)$, involving the first and last row. Clearly $\mu = \min\{2, 3\} = 2$, and thus
$$w^\vee = \begin{pmatrix} & -1 & \\ -1 & & \\ & & 1 \end{pmatrix} s_2 \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix} = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix},$$
with $l(w^\vee) = 4$, $L(w^\vee) = 3$ and $D(w^\vee) = \{0, 1\}$.

Lemma 19. *Let $w = (w_{ij}) \in \mathcal{C}_{n,0} \setminus \mathcal{M}_n$ with topmost odd sandwich (r, h). Then $D(w) \setminus \{0\} = D(w^\vee) \setminus \{0\}$, and if (r, h) satisfies Definition 15 (1) then $D(w) = D(w^\vee)$. Moreover,*
$$L(w) = L(w^\vee) \quad \text{and} \quad l(w) = l(w^\vee) \pm 1.$$

Proof. We first prove the statements about the descent types. Let $w^\vee = w^{[n-1]} s_\mu w_{[n-1]}$ as above. Since $w_{[n-1]}^{-1}(\mu) \equiv w_{[n-1]}^{-1}(\mu + 1) \bmod (2)$ the non-zero entries of $w_{[n-1]}$ in rows μ and $\mu + 1$ are not in adjacent columns. Thus, transposing rows μ and $\mu + 1$ does not change the descent type of $w_{[n-1]}$, that is, $D(w_{[n-1]}) = D(s_\mu w_{[n-1]})$. For any element $u \in B_n$, we have $D(u) \setminus \{0\} = D(u_{[n-1]})$. Since $(w^\vee)_{[n-1]} = s_\mu w_{[n-1]}$, we get $D(w) \setminus \{0\} = D(w_{[n-1]}) = D(s_\mu w_{[n-1]}) = D(w^\vee) \setminus \{0\}$. If (r, h) satisfies Definition 15 (1) then $w_{r, j(r)} = w_{r+h+1, j(r+h+1)}$ so $0 \in D(w)$ if and only if $0 \in D(w^\vee)$ and thus $D(w) = D(w^\vee)$.

We now prove that $L(w) = L(w^\vee)$. Let $j_{\min} := \min\{j(r), j(r + h + 1)\}$ and $j_{\max} := \max\{j(r), j(r+h+1)\}$. Then $j_{\min} \equiv j_{\max} \bmod (2)$, since $w \in \mathcal{C}_{n,0}$ and $|i(j_{\max}) - i(j_{\min})| =$

$h+1$ is even. We write $w = (w_{ij})$ and $w^\vee = (w^\vee_{ij})$, where $i, j, \in [n]$. Consider the $(h+2)\times n$-submatrices

$$v := (w_{ij}) \text{ and } v^\vee := (w^\vee_{ij}), \text{ where } r \leqslant i \leqslant r+h+1, \quad j \in [n].$$

Recall that v and $v^\vee$ are obtained from one another by interchanging their first and last rows. Using the fact that $L = a + b + 2c$ (cf. Lemma 6 (11)), and noting that w and $w^\vee$ coincide outside of the rows i such that $r \leqslant i \leqslant r+h+1$, we see that in order to prove that $L(w) = L(w^\vee)$, it is sufficient to prove that $(a+b+2c)(v) = (a+b+2c)(v^\vee)$. To prove the latter it is sufficient to show that for any column j in v, the contribution to L from the three columns j,$j_{\min}$, $j_{\max}$ in v is equal to the contribution to L from the three columns j,$j_{\min}$, $j_{\max}$ in $v^\vee$. As $L = a+b+2c$ and $a(v) = a(v^\vee)$, it is enough to consider the contribution to b and c. Let j be a non-zero column in v, that is $j \in [n]$ such that $r \leqslant i(j) \leqslant r+h+1$. Since we only need to consider the contribution to b and c from the columns j,$j_{\min}$, $j_{\max}$, we may assume that $j \not\equiv j_{\min} \bmod (2)$, which is equivalent to $j \not\equiv j_{\max} \bmod (2)$, since $j_{\min} \equiv j_{\max} \bmod (2)$. There are then three possible cases: $j < j_{\min}$, $j_{\min} < j < j_{\max}$, and $j_{\max} < j$, respectively. In the sequel we consider only the case that $w_{r,j(r)} = w_{r+h+1,j(r+h+1)}$ (cf. Definition 15 (1)), omitting similar arguments for the case that $w_{r,j(r)} \neq w_{r+h+1,j(r+h+1)}$ (cf. Definition 15 (2)).

Consider the first case, $j < j_{\min}$. Suppose that $w_{i(j_{\min}),j_{\min}} = w_{i(j_{\max}),j_{\max}} = 1$. If $i(j_{\min}) < i(j_{\max})$, the total contribution to $b+2c$ from the column pairs $(j, j_{\min})$ and $(j, j_{\max})$ is 1. If $i(j_{\min}) > i(j_{\max})$, the total contribution to $b+2c$ from the column pairs $(j, j_{\min})$ and $(j, j_{\max})$ is also 1. Suppose on the other hand that $w_{i(j_{\min}),j_{\min}} = w_{i(j_{\max}),j_{\max}} = -1$. If $i(j_{\min}) < i(j_{\max})$, the total contribution to $b+2c$ from the column pairs $(j, j_{\min})$ and $(j, j_{\max})$ is 3. If $i(j_{\min}) > i(j_{\max})$, the total contribution to $b+2c$ from the column pairs $(j, j_{\min})$ and $(j, j_{\max})$ is also 3. Thus $(a+b+2c)(v) = (a+b+2c)(v^\vee)$ in the first case.

Next, consider the second case, $j_{\min} < j < j_{\max}$. Suppose that $w_{i(j_{\min}),j_{\min}} = w_{i(j_{\max}),j_{\max}} = 1$. If $i(j_{\min}) < i(j_{\max})$, the total contribution to $b+2c$ from the column pairs $(j_{\min}, j)$ and $(j, j_{\max})$ is 2. If $i(j_{\min}) > i(j_{\max})$, the total contribution to $b+2c$ from the column pairs $(j_{\min}, j)$ and $(j, j_{\max})$ is also 2. Suppose on the other hand that $w_{i(j_{\min}),j_{\min}} = w_{i(j_{\max}),j_{\max}} = -1$. If $i(j_{\min}) < i(j_{\max})$, the total contribution to $b+2c$ from the column pairs $(j_{\min}, j)$ and $(j, j_{\max})$ is 2. If $i(j_{\min}) > i(j_{\max})$, the total contribution to $b+2c$ from the column pairs $(j_{\min}, j)$ and $(j, j_{\max})$ is also 2. Thus $(a+b+2c)(v) = (a+b+2c)(v^\vee)$ in the second case.

Finally, consider the third case, $j_{\max} < j$. Suppose that $w_{i(j_{\min}),j_{\min}} = w_{i(j_{\max}),j_{\max}} = 1$. If $i(j_{\min}) < i(j_{\max})$, the total contribution to $b+2c$ from the column pairs $(j_{\min}, j)$ and $(j_{\max}, j)$ is 1. If $i(j_{\min}) > i(j_{\max})$, the total contribution to $b+2c$ from the column pairs $(j_{\min}, j)$ and $(j_{\max}, j)$ is also 1. Suppose on the other hand that $w_{i(j_{\min}),j_{\min}} = w_{i(j_{\max}),j_{\max}} = -1$. If $i(j_{\min}) < i(j_{\max})$, the total contribution to $b+2c$ from the column pairs $(j_{\min}, j)$ and $(j_{\max}, j)$ is 2. If $i(j_{\min}) < i(j_{\max})$, the total contribution to $b+2c$ from the column pairs $(j_{\min}, j)$ and $(j_{\max}, j)$ is 1. If $i(j_{\min}) > i(j_{\max})$, the total contribution to $b+2c$ from the column pairs $(j_{\min}, j)$ and $(j_{\max}, j)$ is also 2. Thus $(a+b+2c)(v) = (a+b+2c)(v^\vee)$ in the third case.

To finish the proof of the lemma, recall from (7) that for any $g \in B_n$, and any $s \in S$, we have $l(g) = l(g^{[n-1]}) + l(g_{[n-1]})$, and $l(sg) = l(g) \pm 1$. Thus

$$l(w^\vee) = l(w^{[n-1]}) + l(s_\mu w_{[n-1]}) = l(w^{[n-1]}) + l(w_{[n-1]}) \pm 1 = l(w) \pm 1.$$

□

Corollary 20. *Let $n \in \mathbb{N}$ and $I \subseteq [n-1]_0$. Then*

$$\sum_{w \in B_n^{(I_0)^c}} (-1)^{l(w)} X^{L(w)} = \sum_{w \in \mathcal{M}_n^{(I_0)^c}} (-1)^{l(w)} X^{L(w)}. \tag{14}$$

Assume that either n is odd or both n and $I \subseteq [n-1]_0$ are even. Then

$$\sum_{w \in B_n^{(I_0)^c}} (-1)^{l(w)} X^{L(w)} = \sum_{w \in \mathcal{E}_n^{(I_0)^c}} (-1)^{l(w)} X^{L(w)}.$$

Proof. Without loss of generality we may assume that $0 \in I$, so that $I = I_0$. By Lemma 8 the sum over $B_n^{I^c}$ is supported on $\mathcal{C}_{n,0}^{I^c}$. Lemma 19 asserts that for every $w \in \mathcal{C}_{n,0}^{I^c} \setminus \mathcal{M}_n$ there exists a unique $w^\vee \in \mathcal{C}_{n,0}^{I^c} \setminus \mathcal{M}_n$ such that $(-1)^{l(w)} X^{L(w)} + (-1)^{l(w^\vee)} X^{L(w^\vee)} = 0$. Moreover, $D(w) \setminus \{0\} = D(w^\vee) \setminus \{0\}$, so $w \in B_n^{I^c}$ if and only if $w^\vee \in B_n^{I^c}$. Hence the sum over $B_n^{I^c}$ is supported on $\mathcal{M}_n^{I^c}$.

When n is even and $I \subseteq [n-1]_0$ is even, Lemma 14 states that $\mathcal{M}_n^{I^c} = \mathcal{E}_n^{I^c}$, whence the second equality. When n is odd, it follows from the first, as $\mathcal{M}_n = \mathcal{E}_n$. □

Remark 21. Example 18 illustrates that the sign reversing involution $\vee$ on $\mathcal{C}_{n,0} \setminus \mathcal{M}_n$ does not, in general, preserve the descent type. This is in contrast to the involution $*$ defined in the proof of Lemma 8. The weaker statement (14) is, however, sufficient for our application in the proof of Proposition 25.

5.2 A first additivity result for L

We now consider how the statistic L behaves with respect to the parabolic factorisation $w = w^{[n-1]} w_{[n-1]}$. For an arbitrary element $w \in \mathcal{C}_{n,0}$, it is not necessarily true that L is additive with respect to this factorisation, that is $L(w) = L(w^{[n-1]}) + L(w_{[n-1]})$. A counter-example is given by

$$w = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = w^{[1]} w_{[1]} \in \mathcal{M}_2 \setminus \mathcal{E}_2,$$

where $L(w) = 2$, $L(w^{[1]}) = 2$ and $L(w_{[1]}) = 1$.

The following result shows that the situation improves when we assume that $w \in \mathcal{E}_n$.

Proposition 22. *Suppose that $w \in \mathcal{E}_n$. Then*

$$L(w) = L(w^{[n-1]}) + L(w_{[n-1]}).$$

Proof. Since $w \in \mathcal{E}_n$, we have $w^{[n-1]}, w_{[n-1]} \in \mathcal{C}_{n,0}$. Let $w = w_1 * w_2$ and $w^{[n-1]} = (w^{[n-1]})_1 * (w^{[n-1]})_2$. We claim that

$$w^{[n-1]} = (w_1)^{[n-1]} * (w_2)^{[n-1]}, \tag{15}$$

that is $(w^{[n-1]})_1 = (w_1)^{[n-1]}$ and $(w^{[n-1]})_2 = (w_2)^{[n-1]}$. Indeed, the ascending matrix $w^{[n-1]}$ is obtained from w by a permutation of columns, and since both $w^{[n-1]}$ and w are chessboard elements, each column of w is moved an even amount to obtain the corresponding column of $w^{[n-1]}$. Clearly, $w^{[n-1]}$ has the same row pattern as w; cf. (9). For any $v \in \mathcal{C}_{n,0}$, with $v = v_1 * v_2$, every non-zero entry at (i, j) is either equal to the entry at $(\frac{i+1}{2}, \frac{j+1}{2})$ in v_1, if i (and therefore j) is odd, or is equal to the entry at $(\frac{i}{2}, \frac{j}{2})$ in v_2, if i (and therefore j) is even. The row pattern of $(w^{[n-1]})_1$ is therefore the same as that of w_1, and the row pattern of $(w^{[n-1]})_2$ is the same as that of w_2. Any descent in $(w^{[n-1]})_1$ or $(w^{[n-1]})_2$ would give rise to a descent in $w^{[n-1]}$, so the matrices $(w^{[n-1]})_1$ and $(w^{[n-1]})_2$ must be ascending. Thus $(w^{[n-1]})_1 = (w_1)^{[n-1]}$ and $(w^{[n-1]})_2 = (w_2)^{[n-1]}$, establishing (15).

In a similar way, we let $w_{[n-1]} = (w_{[n-1]})_1 * (w_{[n-1]})_2$, and we claim that

$$w_{[n-1]} = (w_1)_{[n-1]} * (w_2)_{[n-1]}, \tag{16}$$

that is $(w_{[n-1]})_1 = (w_1)_{[n-1]}$ and $(w_{[n-1]})_2 = (w_2)_{[n-1]}$. Indeed, σ_0 is a homomorphism, and so

$$\begin{aligned}(w_1, w_2) = \sigma_0(w) = \sigma_0(w^{[n-1]}w_{[n-1]}) &= \sigma_0(w^{[n-1]})\sigma_0(w_{[n-1]}) \\ &= ((w^{[n-1]})_1(w_{[n-1]})_1, (w^{[n-1]})_2(w_{[n-1]})_2),\end{aligned}$$

and thus $w_1 = (w^{[n-1]})_1(w_{[n-1]})_1$ and $w_2 = (w^{[n-1]})_2(w_{[n-1]})_2$. Using (15) we obtain

$$w_1 = (w_1)^{[n-1]}(w_1)_{[n-1]} = (w^{[n-1]})_1(w_1)_{[n-1]} = (w^{[n-1]})_1(w_{[n-1]})_1,$$

whence $(w_{[n-1]})_1 = (w_1)_{[n-1]}$. The equality $(w_{[n-1]})_2 = (w_2)_{[n-1]}$ is proved in the same way. This proves (16).

Recall that $l(w) = l(w^{[n-1]}) + l(w_{[n-1]})$; see (7). Since $\mathrm{inv} + \mathrm{nsp} = l - \mathrm{neg}$ (see (8)) and $\mathrm{neg}(w) = \mathrm{neg}(w^{[n-1]}) + \mathrm{neg}(w_{[n-1]}) = \mathrm{neg}(w^{[n-1]})$, this implies that

$$(\mathrm{inv} + \mathrm{nsp})(w) = (\mathrm{inv} + \mathrm{nsp})(w^{[n-1]}) + (\mathrm{inv} + \mathrm{nsp})(w_{[n-1]}).$$

Lemma 6 (12) and the equalities (15) and (16) now imply that

$$\begin{aligned}&L(w^{[n-1]}) + L(w_{[n-1]}) = \\ &\quad \mathrm{neg}((w_1)^{[n-1]}) + (\mathrm{inv} + \mathrm{nsp})(w^{[n-1]}) - (\mathrm{inv} + \mathrm{nsp})((w_1)^{[n-1]}) - (\mathrm{inv} + \mathrm{nsp})((w_2)^{[n-1]}) \\ &+ \mathrm{neg}((w_1)_{[n-1]}) + (\mathrm{inv} + \mathrm{nsp})(w_{[n-1]}) - (\mathrm{inv} + \mathrm{nsp})((w_1)_{[n-1]}) - (\mathrm{inv} + \mathrm{nsp})((w_2)_{[n-1]}) \\ &\qquad = \mathrm{neg}(w_1) + (\mathrm{inv} + \mathrm{nsp})(w) - (\mathrm{inv} + \mathrm{nsp})(w_1) - (\mathrm{inv} + \mathrm{nsp})(w_2) = L(w).\end{aligned}$$

□

5.3 A second additivity result for L

We now consider how the statistic L behaves with respect to parabolic factorisations of the form $w = w^{[i-1]_0} w_{[i-1]_0}$, where $i \in [n-1]$. Even if $w \in \mathcal{E}_n$, it is not necessarily true that $L(w) = L(w^{[i-1]_0}) + L(w_{[i-1]_0})$. A counter-example is given by $i = 2$ and $w = [-5, 2, 1, -4, 3] \in \mathcal{E}_5$. Here $D(w) = \{0, 2, 3\}$ and $L(w) = 7$. But $L(w^{\{0,1\}}) = L([2, 5, 1, -4, 3]) = 6$ and $L(w_{\{0,1\}}) = L([-2, 1, 3, 4, 5]) = 2$.

The following result establishes additivity of L under this kind of parabolic factorisation under additional conditions.

Proposition 23. *Suppose that n is even and $w \in \mathcal{E}_n$ has even descent type $D(w)$. Let $e \in [n-1]$ be an even integer such that $e \leqslant \min\{(D(w) \cup \{n\}) \setminus \{0\}\}$, that is $w(1) < \cdots < w(e)$. Then*

$$L(w) = L(w^{[e-1]_0}) + L(w_{[e-1]_0}).$$

Proof. Write the factorisation $w = w^{[e-1]_0} w_{[e-1]_0}$ as

$$w = \left(\begin{array}{c|c} A & M \end{array}\right) = \left(\begin{array}{c|c} B & M \end{array}\right) \left(\begin{array}{c|c} \overline{A} & 0 \\ \hline 0 & \mathrm{Id}_{n-e} \end{array}\right) = w^{[e-1]_0} w_{[e-1]_0},$$

where $A \in \mathrm{Mat}(n \times e, \mathbb{Z})$ comprises the first e columns of w, $M \in \mathrm{Mat}(n \times (n-e), \mathbb{Z})$ comprises the last $n-e$ columns and Id_{n-e} denotes the identity matrix of size $n-e$. We now describe the matrices B and $\overline{A}$. Define

$$f : [e] \longrightarrow [e], \quad \kappa \mapsto \#\{(r, s) \in [n] \times [e] \mid w_{rs} \neq 0 \wedge r \leqslant i(\kappa)\}.$$

Informally speaking, f enumerates the rows in A containing a non-zero entry, so that for $\kappa \in [e]$, the non-zero entry of w in column κ lies in the $f(\kappa)$-th non-zero row in A. Since each column of A contains exactly one non-zero entry, the function f is a bijection. Given this definition, B is the $n \times i$-matrix whose $(i_w(j), f(j))$-entry is 1 for $j \in [e]$, and all other entries zero, and $\overline{A}$ is the $i \times i$ ascending matrix whose $(f(j), j)$-entry is $w_{i_w(j),j}$.

Recall the formula $L(w) = a(w) + b(w) + 2c(w)$ given in Lemma 6 (11). Using the assumptions that n and $D(w)$ are even, we will show that the functions a,b and c are each additive over the factorisation $w = w^{[e-1]_0} w_{[e-1]_0}$. Clearly $a(w) = a(A) + a(M)$, $a(w_{[e-1]_0}) = a(\overline{A}) = a(A)$ and $a(w^{[e-1]_0}) = a(M)$, so a is additive. It is easy to verify that $b(w_{[e-1]_0}) = b(\overline{A}) = b(A)$, $c(w_{[e-1]_0}) = c(\overline{A}) = c(A)$ and that $b(B) = c(B) = 0$. Hence the respective additivity of b and c is equivalent to

$$b(w) - b(A) - b(M) = b(w^{[e-1]_0}) - b(B) - b(M) \tag{17}$$

and

$$c(w) - c(A) - c(M) = c(w^{[e-1]_0}) - c(B) - c(M). \tag{18}$$

These two equations can be interpreted in the following way. Let $V \in \mathrm{Mat}(n, \mathbb{Z})$ be a matrix with at most one non-zero entry in each column, such as w, $w^{[e-1]_0}$ or $w_{[e-1]_0}$. Suppose that V has the form

$$V = (V_1 \mid V_2)\,,$$

where $V_1 \in \mathrm{Mat}(n \times e, \mathbb{Z})$ consists of the first e columns of V, and $V_2 \in \mathrm{Mat}(n \times (n-e), \mathbb{Z})$ consists of the remaining $n-e$ columns. Then, by Definition 4,

$$b(V) - b(V_1) - b(V_2) = \#\{(j_1, j_2) \in [e] \times [n-e] \mid i_V(j_1) > i_V(j_2),\ j_1 \not\equiv j_2 \bmod (2)\}$$

and

$$\begin{aligned} &c(V) - c(V_1) - c(V_2) \\ &\qquad = \#\{(j_1, j_2) \in [e] \times [n-e] \mid V_{i_V(j_2), j_2} = -1,\ i_V(j_1) < i_V(j_2),\ j_1 \not\equiv j_2 \bmod (2)\} \end{aligned}$$

Informally, the value $b(V) - b(V_1) - b(V_2)$ is equal to the contribution to b given by column pairs (j_1, j_2) such that j_1 denotes a column of V_1 and j_2 denotes a column of V_2. Similar considerations hold for the function c.

To prove the equations (17) and (18) it therefore suffices to establish a bijection $\varphi : [e] \to [e]$, inducing a bijection between the columns of A and the columns of B such that

$$j \equiv \varphi(j) \bmod (2), \tag{19}$$

$$i_w(j) > i_w(k) \Longleftrightarrow i_B(\varphi(j)) > i_w(k) \text{ for all } j \in [e],\ e < k \leqslant n. \tag{20}$$

We consider w as obtained from the ascending matrix $w^{[n-1]}$ by column permutations, given by $w_{[n-1]}$. Since both n and $D(w)$ are even, Lemma 14 implies that $w_{[n-1]} = (w_{[n-1]})_1 * (w_{[n-1]})_1$. This implies that w is obtained from $w^{[n-1]}$ by permuting pairs of adjacent columns of $w^{[n-1]}$, indexed by pairs of the form $(2j-1, 2j)$, for $j \in [n/2]$. Note that $w^{[n-1]} \in \mathcal{C}_{n,0}$. We may therefore apply Lemma 10 to the column pairs of $w^{[n-1]}$, and any statement about these column pairs remains true for the column pairs of the submatrix A of w. Assume that $w^{[n-1]}$ is non-trivial; otherwise, there is nothing to prove. To define the bijection φ, we consider a pair $(2j-1, 2j)$ for $j \in [e/2]$. We distinguish two cases:

Case $w(2j) > 0$. Here Lemma 10 implies that $f(2j) \equiv 0 \bmod (2)$ and $f(2j-1) \equiv 1 \bmod (2)$, and in this case we set

$$\varphi(2j-1) = f(2j-1), \quad \varphi(2j) = f(2j).$$

Case $w(2j) < 0$. Here Lemma 10 implies that $w(2j-1) = w(2j) - 1$ and thus $f(2j-1) = f(2j) + 1$. Therefore, if $f(2j-1) \equiv 1 \bmod (2)$ then $f(2j) \equiv 0 \bmod (2)$, and in this case we set

$$\varphi(2j-1) = f(2j-1), \quad \varphi(2j) = f(2j).$$

On the other hand, if $f(2j-1) \not\equiv 1 \bmod (2)$ then $f(2j) \not\equiv 0 \bmod (2)$. In other words, in this case we have $f(2j-1) \equiv 0 \bmod (2)$ and $f(2j) \equiv 1 \bmod (2)$, and we set

$$\varphi(2j-1) = f(2j), \quad \varphi(2j) = f(2j-1).$$

Note that this last case is the only one where φ does not agree with f.

By definition the bijection φ satisfies condition (19). Moreover, in the cases where $\varphi(j) = f(j)$ we have $i_B(\varphi(j)) = i_B(f(j)) = i_w(j)$, since, as noted previously, the non-zero entry in column $f(j)$ in the matrix B lies in row $i_w(j)$. Thus, condition (20) is satisfied whenever $\varphi(j) = f(j)$. Finally, in the case where $(2j-1, 2j)$ is a column pair such that $\varphi(2j-1) = f(2j)$ and $\varphi(2j) = f(2j-1)$, we have $f(2j-1) = f(2j)+1$, so

$$i_B(\varphi(2j)) + 1 = i_B(f(2j-1)) + 1 = i_w(2j-1) + 1 = i_w(2j) = i_B(f(2j)) = i_B(\varphi(2j-1)).$$

Thus, for k such that $e < k \leqslant n$, we have

$$\begin{aligned} i_w(2j-1) > i_w(k) &\Longleftrightarrow i_w(2j) > i_w(k) \\ \Longleftrightarrow i_B(\varphi(2j-1)) > i_w(k) &\Longleftrightarrow i_B(\varphi(2j)) > i_w(k). \end{aligned}$$

Therefore condition (20) is satisfied also in this case.

We have thus established the existence of a bijection φ with the required properties, and this finishes the proof. □

5.4 Proof of Case (3) of Theorem 2

For $a, b \in \mathbb{N}_0$ such that $a \geqslant b$, the *X-binomial coefficient* is defined as

$$\binom{a}{b}_X = \frac{(\underline{a})!}{(\underline{a-b})!(\underline{b})!} \in \mathbb{Z}[X].$$

More generally, for $n \in \mathbb{N}$ and $I = \{i_1, \dots, i_l\}_< \subseteq [n-1]_0$, the *$X$-multinomial coefficient* is

$$\binom{n}{I}_X = \binom{n}{i_l}_X \binom{i_l}{i_{l-1}}_X \cdots \binom{i_2}{i_1}_X \in \mathbb{Z}[X].$$

It is well known that

$$\sum_{w \in S_n^{I^c}} X^{l(w)} = \binom{n}{I}_X; \tag{21}$$

see, for instance, [9, Proposition 1.3.17].

Lemma 24. *Suppose that n and $I \subseteq [n-1]_0$ are even. Then*

$$\sum_{w \in S_n^{I^c}} (-1)^{l(w)} X^{L(w)} = \binom{n/2}{I/2}_{X^2}.$$

Proof. By arguing as in the proof of Lemma 8, we may argue that the sum is supported on the set $S_n^{I^c} \cap \mathcal{C}_{n,0}$. Let $w \in S_n^{I^c} \cap \mathcal{C}_{n,0}$ and write $w = w_1 * w_2$. By Lemma 14 we have $w_1 = w_2 \in S_{n/2}^{(I/2)^c}$ and $l_{S_n}(w) = 4 l_{S_{n/2}}(w_1)$. By Lemma 6 (12) we have $L(w) = l(w) - 2l(w_1)$.

(Here and in the sequel we suppress subscripts in the notation for various Coxeter length functions.) Using (21) we obtain

$$\begin{aligned}\sum_{w\in S_n^{I^{\mathrm{c}}}}(-1)^{l(w)}X^{L(w)} &= \sum_{w\in S_n^{I^{\mathrm{c}}}\cap\mathcal{C}_{n,0}}(-1)^{l(w)}X^{l(w)-2l(w_1)}\\ &= \sum_{w_1\in S_{n/2}^{(I/2)^{\mathrm{c}}}}(-1)^{4l(w_1)}X^{4l(w_1)-2l(w_1)}\\ &= \sum_{w_1\in S_{n/2}^{(I/2)^{\mathrm{c}}}}X^{2l(w_1)} = \binom{n/2}{I/2}_{X^2}.\end{aligned}$$

□

Proposition 25. *Conjecture 1 holds when both n and $I\subseteq[n-1]_0$ are even, that is, in this case*

$$\sum_{w\in B_n^{I^{\mathrm{c}}}}(-1)^{l(w)}X^{L(w)} = \binom{n/2}{I/2}_{X^2}\frac{(\underline{1})(\underline{3})\cdots(\underline{n-1})}{(\underline{1})(\underline{3})\cdots(\underline{i_1-1})}.$$

Proof. By Corollary 20, we have

$$\sum_{w\in B_n^{(I_0)^{\mathrm{c}}}}(-1)^{l(w)}X^{L(w)} = \sum_{w\in\mathcal{E}_n^{(I_0)^{\mathrm{c}}}}(-1)^{l(w)}X^{L(w)}.$$

Recall that $i_1=\min(I\cup\{n\})$. By the two additivity results for L established in Propositions 22 and 23, this may be written as

$$\begin{aligned}\sum_{w\in B_n^{(I_0)^{\mathrm{c}}}}(-1)^{l(w)}X^{L(w)} &= \left(\sum_{w\in B_n^{[n-1]}}(-1)^{l(w)}X^{L(w)}\right)\left(\sum_{w\in S_n^{I^{\mathrm{c}}}}(-1)^{l(w)}X^{L(w)}\right)\\ &= \left(\sum_{w\in B_n^{I^{\mathrm{c}}}}(-1)^{l(w)}X^{L(w)}\right)\left(\sum_{w\in B_{i_1}^{[i_1-1]}}(-1)^{l(w)}X^{L(w)}\right).\end{aligned}$$

The proposition now follows from Proposition 9 (twice) and Lemma 24. □

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