A bipartite graph with non-unimodal independent set sequence

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Abstract

We show that the independent set sequence of a bipartite graph need not be unimodal.

1 Introduction

For a graph $G = (V, E)$ and an integer $t \geqslant 0$, let $i_t(G)$ denote the number of independent sets of size t in G. (Recall that an independent set is a set of vertices spanning no edges.) The *independent set sequence* of G is the sequence $i(G) = (i_t(G))_{t=0}^{\alpha(G)}$, where $\alpha(G)$ is the size of a largest independent set in G.

It was conjectured by Levit and Mandrescu [LM06] that for any bipartite graph G, $i(G)$ is unimodal; that is, that there is a k for which

$$i_0(G) \leqslant i_1(G) \leqslant \cdots \leqslant i_k(G) \geqslant i_{k+1}(G) \geqslant \cdots \geqslant i_{\alpha(G)}(G).$$

Evidence in favor of this was given by Levit and Mandrescu [LM06] and by Galvin (in [Gal12], which got us interested in the problem, and in [Gal11]).

In this note, we disprove the conjecture:

Theorem 1. *There are bipartite graphs G for which $i(G)$ is not unimodal.*

Still open and very interesting is the possibility, first suggested by Alavi, Malde, Schwenk and Erdős [AMSE87], that trees and forests have unimodal independent set sequences. See also [Sta89] for a general survey of unimodality and the stronger notion of log-concavity.

2 Counterexample

Given positive integers a and $b > a$, let $G = G(a, b) = (V, E)$ with: $V = V_1 \cup V_2 \cup V_3$, where V_1, V_2, V_3 are disjoint; $|V_1| = b - a$ and $|V_2| = |V_3| = a$; and E consists of a complete bipartite graph between V_1 and V_2 and a perfect matching between V_2 and V_3.

Lemma 2. *For every $t \geqslant 0$, $i_t(G) = (2^t - 1)\binom{a}{t} + \binom{b}{t}$.*

Proof. Each independent set in G is a subset of either $V_1 \cup V_3$ or $V_2 \cup V_3$. Among independent sets of size t, the number of the first type is $\binom{b}{t}$, the number of the second type is $2^t\binom{a}{t}$, and the number that are of both types (that is, that are subsets of V_3) is $\binom{a}{t}$. $\square$

We now assert that $i(G)$ is not unimodal if a is large and (say) $b = \lfloor a \log_2 3 \rfloor$. In this case, the expressions $\binom{b}{t}$ and $2^t\binom{a}{t}$ are maximized at $t_1 = b/2$ and $t_2 = 2a/3 + O(1)$ respectively (the overlap $\binom{a}{t}$ is negligible), with each maximum on the order of $3^a/\sqrt{a}$. On the other hand, each expression is $o(3^a/\sqrt{a})$ if t is at least $\omega(\sqrt{a})$ from the maximizing value. In particular, $i_t(G)$ is much smaller for $t = (t_1 + t_2)/2$ than for $t \in \{t_1, t_2\}$, so $i(G)$ is not unimodal.

For a concrete example, we may take $a = 95$ and $b = 151$, for which explicit calculation gives

$$i_{70}(G) = 18987441601605235984576411514620264336031 5069,$$
$$i_{71}(G) = 187958904435447560369145399619337946363249075,$$
$$i_{72}(G) = 188299580501161488791208803278091384597416875.$$

In fact, we find that 95 is the minimum value of a for which the above construction produces a counterexample.

3 Remarks

The construction above can be generalized to show that (for bipartite G) $i(G)$ can have arbitrarily many local maxima. Given (positive) integers k and $a, a_1, \ldots, a_k$, let $G = G(a, a_1, \ldots, a_k) = (A \cup B, E)$ be the bipartite graph where: $A = \cup_{i=0}^{k} A_i$ and $B = \cup_{j=1}^{k} B_j$, with all A_i's and B_j's disjoint; $|A_0| = |B_1| = |B_2| = \cdots = |B_k| = a$ and $|A_i| = a_i$ for $i > 0$; and E consists of a perfect matching between A_0 and B_j for each $j > 0$, together with a complete bipartite graph between A_i and B_j for all (i, j) with $j \leqslant i$. Then for $a, a_1, \ldots, a_k$ large with all the $k + 1$ expressions $2^{a_1 + \cdots + a_i}(1 + 2^{k-i})^a$ roughly equal, an analysis similar to the one above shows that $i(G)$ has $k + 1$ local maxima.

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