Repeated columns and an old chestnut

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Abstract

Let \( t \geq 1 \) be a given integer. Let \( \mathcal{F} \) be a family of subsets of \([m] = \{1, 2, \ldots, m\}\). Assume that for every pair of disjoint sets \( S, T \subset [m] \) with \( |S| = |T| = k \), there do not exist \( 2t \) sets in \( \mathcal{F} \) where \( t \) subsets of \( \mathcal{F} \) contain \( S \) and are disjoint from \( T \) and \( t \) subsets of \( \mathcal{F} \) contain \( T \) and are disjoint from \( S \). We show that \( |\mathcal{F}| \) is \( O(m^k) \).

Our main new ingredient is allowing, during the inductive proof, multisets of subsets of \([m] \) where the multiplicity of a given set is bounded by \( t - 1 \). We use a strong stability result of Anstee and Keevash. This is further evidence for a conjecture of Anstee and Sali. These problems can be stated in the language of matrices. Let \( t \cdot M \) denote \( t \) copies of the matrix \( M \) concatenated together. We have established the conjecture for those configurations \( t \cdot F \) for any \( k \times 2 \) \((0,1)\)-matrix \( F \).

Keywords: extremal set theory, extremal hypergraphs, \((0,1)\)-matrices, multiset, forbidden configurations, trace, subhypergraph.

1 Introduction

We will be considering a problem in extremal hypergraphs that can be phrased as how many edges a hypergraph on \( m \) vertices can have when there is a forbidden subhypergraph. There are a variety of ways to define this problem (we could, but do not, restrict to (simple) \( k \)-uniform hypergraphs). We can encode a hypergraph on \( m \) vertices as an \( m \)-rowed \((0,1)\)-matrix where the \( i \)th column is the incidence vector of the \( i \)th hyperedge.

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A hypergraph is simple if there are no repeated edges. We define a matrix to be simple if it is a (0,1)-matrix with no repeated columns. We will use the language of matrices in this paper.

Let $M$ be an $m$-rowed (0,1)-matrix. Some notation about repeated columns is needed. For an $m \times 1$ (0,1)-column $\alpha$, we define $\mu(\alpha, M)$ as the multiplicity of column $\alpha$ in a matrix $M$. We consider matrices of bounded column multiplicity. We define a matrix $A$ to be $t$-simple if it is a (0,1)-matrix and every column $\alpha$ of $A$ has $\mu(\alpha, A) \leq t$. Simple matrices are 1-simple.

For a given matrix $M$, let $\text{supp}(M)$ denote the maximal simple $m$-rowed submatrix of $M$, so that if $\mu(\alpha, M) \geq 1$ then $\mu(\alpha, \text{supp}(M)) = 1$. The matrices below are a 3-simple matrix $M$ and its support $\text{supp}(M)$.

$$M = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \quad \text{supp}(M) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

For two (0,1)-matrices $F$ and $A$, we say that $F$ is a configuration in $A$, and write $F \prec A$ if there is a row and column permutation of $F$ which is a submatrix of $A$. Let $F$ denote a finite set of (0,1)-matrices. Let $\text{Avoid}(m, F, t)$ denote all $m$-rowed $t$-simple matrices $A$ for which $F \not\prec A$ for all $F \in F$. We are most interested in cases with $|F| = 1$ [5]. We do not require any $F \in F$ to be simple which is quite different from usual forbidden subhypergraph problems. Let $\|A\|$ denote the number of columns of $A$.

Our extremal function of interest is

$$\text{forb}(m, F) = \max_A \{\|A\| : A \in \text{Avoid}(m, F, 1)\}.$$  

We find it helpful to also define

$$\text{forb}(m, F, t) = \max_A \{\|A\| : A \in \text{Avoid}(m, F, t)\}.$$  

If $A \in \text{Avoid}(m, F, t)$ then $\text{supp}(A) \in \text{Avoid}(m, F, 1)$ and $\|A\| \leq t \cdot \|\text{supp}(A)\|$. We obtain

$$\text{forb}(m, F) \leq \text{forb}(m, F, t) \leq t \cdot \text{forb}(m, F),$$  

so that the asymptotic growth of $\text{forb}(m, F)$ is the same as that of $\text{forb}(m, F, t)$ for fixed $t$.

We have an important conjecture about $\text{forb}(m, F)$. We use the notation $[M \mid N]$ to denote the matrix obtained from concatenating the two matrices $M$ and $N$. We use the notation $k \cdot M$ to denote the matrix $[M \mid M \mid \cdots \mid M]$ consisting of $k$ copies of $M$ concatenated together. Let $I_k$ denote the $k \times k$ identity matrix and let $I_k^\complement$ denote the (0,1)-complement of $I_k$. Let $T_k$ denote the $k \times k$ triangular (0,1)-matrix with the $(i, j)$ entry being 1 if and only if $i \leq j$. For an $m_1 \times n_1$ matrix $X$ and an $m_2 \times n_2$ matrix $Y$, we define the 2-fold product $X \times Y$ as the $(m_1 + m_2) \times n_1 n_2$ matrix where each column consisting of a column of $X$ placed on a column of $Y$ and this is done in all possible ways. This extends to $p$-fold products.
**Definition 1.** Let $X(F)$ be the smallest $p$ so that $F \prec A_1 \times A_2 \times \cdots \times A_p$ for every choice of $A_i$ as either $I_{m/p}$, $I^c_{m/p}$ or $T_{m/p}$ for sufficiently large $m$.

Alternatively, assuming $F \not\prec I$ or $F \not\prec I^c$ or $F \not\prec T$, then $X(F) - 1$ is the largest choice of $p$ so that $F \not\prec A_1 \times A_2 \times \cdots \times A_p$ for some choices of $A_i$ as either $I_{m/p}$, $I^c_{m/p}$ or $T_{m/p}$. We note that if $A_1 \times A_2 \times \cdots \times A_p \in \text{Avoid}(m, F)$, then $\text{forb}(m, F)$ is $\Omega(m^p)$.

Details are in [5]. We are assuming $m$ is large and divisible by $p$, in particular that $m \geq (k + 1)(k\ell + 1)$ so that $m/p \geq k\ell + 1$. Divisibility by $p$ does not affect the asymptotic growth, thus $\text{forb}(m, F)$ is $\Omega(m^{X(F)-1})$ using an appropriate $(X(F)-1)$-fold product.

**Conjecture 2.** [4] Let $F$ be given. Then $\text{forb}(m, F) = \Theta(m^{X(F)-1})$. □

The conjecture was known to be true for all 3-rowed $F$ [4] and all $k \times 2$ $F$ [3]. Section 3 shows how Theorem 3 establishes the conjecture for matrices $t \cdot F$ when $F$ is a $k \times 2$ matrix. It is of interest to generalize Conjecture 2 to $\text{forb}(m, F)$ where $|F| > 1$ but we know example of $F$ where the modified form of the conjecture fails (see [5]).

We define $F_{e,f,g,h}$ as the $(e + f + g + h) \times 2$ matrix consisting of $e$ rows $[11]$, $f$ rows $[10]$, $g$ rows $[01]$ and $h$ rows $[00]$. Let $1_f$ denote the $(e + f) \times 1$ vector of $1$’s on top of $f$ 0’s so that $F_{e,f,g,h} = [1_e + f 0_{g+h} | 1_e 0_f 1_g 0_h]$. We let $1_e$ denote the $e \times 1$ vector of $e$ 1’s and $0_f$ denote the $f \times 1$ vector of $f$ 0’s. Our main result is the following which had foiled many previous attempts.

**Theorem 3.** Let $t \geq 2$ be given. Then $\text{forb}(m, t \cdot F_{0,k,k,0})$ is $\Theta(m^k)$.

The forbidden configuration $t \cdot F_{0,k,k,0}$ in the language of sets, consists of two disjoint $k$-sets $S, T$, and a family of $t$ sets containing $S$ but disjoint from $T$, and the other family of another $t$ sets containing $T$ but disjoint from $S$. This theorem echoes our statement in the abstract.

The result for $t = 2$ and $k = 2$ was proven in [1] and many details worked out for $t = 2$ and $k > 2$ by the first author and Peter Keevash. The extension for $t > 2$, $k = 2$ had been open since then [5]. The proof for $t > 2$, $k = 2$ is in Section 2. The proof for $t > 2$, $k > 2$ is in Section 3. Matrices $F_6(t), F_7(t)$ were given in [5] as 4-rowed forbidden configurations (with some columns of multiplicity $t$) for which Conjecture 2 predicts $\text{forb}(m, F_6(t))$ and $\text{forb}(m, F_7(t))$ are $O(m^2)$. Note that $t \cdot F_{0,2,2,0} \prec F_6(t)$ and $t \cdot F_{0,2,2,0} \prec F_7(t)$ and so Theorem 3 is a step towards these bounds which would establish Conjecture 2 for all 4-rowed $F$. Our proof use a new induction given in Section 2 that considers $t$-simple matrices as well as a strong stability result Lemma 10. We offer some additional applications in Section 4.

## 2 New Induction

We consider a new form of the standard induction for forbidden configurations [5]. Let $F$ be a matrix with maximum column multiplicity $t$. Thus $F \prec t \cdot \text{supp}(F)$. Let
\[ A \in \text{Avoid}(m, F, t - 1). \text{ Assume } \|A\| = \text{forb}(m, \mathcal{F}, t - 1). \text{ Given a row } r \text{ we permute rows and columns of } A \text{ to obtain} \]
\[
A = \text{ row } r \to \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & \cdots & 1 \\ G & C & C & C & D \end{bmatrix}.
\]

Now \( \mu(\alpha, G) \leq t - 1 \) and \( \mu(\alpha, H) \leq t - 1 \). For those \( \alpha \) for which \( \mu(\alpha, [GH]) \geq t \), let \( C \) be formed with \( \mu(\alpha, C) = \min\{\mu(\alpha, G), \mu(\alpha, H)\} \). We rewrite our decomposition of \( A \) as follows:
\[
A = \text{ row } r \to \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & \cdots & 1 \\ B & C & C & C & D \end{bmatrix}.
\]

Then we deduce that \([BCD]\) and \( C \) are both \((t - 1)\)-simple. The former follows from \( \mu(\alpha, [BCD]) = \mu(\alpha, G) + \mu(\alpha, H) - \min\{\mu(\alpha, G), \mu(\alpha, H)\} \leq t - 1 \). We have that \( F \not\prec [BCD] \) for \( F \in \mathcal{F} \). Also for any \( F' \prec C \) then \([01] \times F' \prec A \) so we define
\[
\mathcal{G} = \{F' : \text{ for } F \in \mathcal{F}, F \not\prec [01] \times F' \text{ and } F \not\prec [01] \times F'' \text{ for all } F'' \prec F', F'' \neq F\}. \tag{4}
\]

Basically, \( \mathcal{G} \) is the family after removing redundancy from all configurations \( F' \) that are obtained by removing one row from any \( F \) in \( \mathcal{F} \).

Also since each column \( \alpha \) of \( C \) has \( \mu(\alpha, [GH]) \geq t \), we deduce that \( \text{supp}(F) \not\prec C \) for each \( F \in \mathcal{F} \). Our induction on \( m \) becomes:
\[
\text{forb}(m, \mathcal{F}, t - 1) = \|A\| = \|[BCD]\| + \|C\| \\
\leq \text{forb}(m - 1, \mathcal{F}, t - 1) + (t - 1) \cdot \text{forb}(m - 1, \mathcal{G} \cup \{\text{supp}(F) : F \in \mathcal{F}\}). \tag{5}
\]

**Lemma 4.** Let \( H \) be a given simple matrix satisfying \( \text{forb}(m, H) \) is \( O(m^t) \). Then \( \text{forb}(m, t \cdot H) \) is \( O(m^{t+1}) \).

Proof. We use the induction (5) where \( F = t \cdot H \) and \( H = \text{supp}(F) \). Induction on \( m \) yields the desired bound. \( \square \)

**Proof of Theorem 3 for \( k = 2 \):** We will use induction on \( m \) to show \( \text{forb}(m, t \cdot F_{0,2,2,0}, t) \) is \( O(m^2) \). The maximum multiplicity of a column in \( t \cdot F_{0,2,2,0} \) is \( t \) and \( F_{0,2,2,0} = \text{supp}(t \cdot F_{0,2,2,0}) \). Also \( t \cdot F_{0,2,2,0} \prec [01] \times (t \cdot F_{0,2,1,0}) \). Let \( A \in \text{Avoid}(m, t \cdot F_{0,2,2,0}, t - 1) \) with \( \|A\| = \text{forb}(m, t \cdot F_{0,2,2,0}, t - 1) \). Apply (5). We have
\[
\text{forb}(m, t \cdot F_{0,2,2,0}, t - 1) = \|A\| = \|[BCD]\| + \|C\| \\
\leq \text{forb}(m - 1, t \cdot F_{0,2,2,0}, t - 1) + (t - 1) \cdot \text{forb}(m - 1, \{F_{0,2,2,0}, t \cdot F_{0,2,1,0}\}).
\]

We apply Lemma 5 with induction on \( m \) to deduce that \( \text{forb}(m, t \cdot F_{0,2,2,0}, t - 1) \) is \( O(m^2) \). Then by (1), \( \text{forb}(m, t \cdot F_{0,2,2,0}) \) is also \( O(m^2) \). \( \square \)

Theorem 3 was proven for \( t = k = 2 \) in [1] using induction in the spirit of (5) \(((t - 1)\text{-simple matrices are simple})\) and Lemma 5 for \( t = 2 \).
Lemma 5. We have that \( \text{forb}(m, \{F_{0,2,2,0}, t \cdot F_{0,2,1,0}\}) \) is \( O(m) \).

**Proof.** Let \( A \in \text{Avoid}(m, \{F_{0,2,2,0}, t \cdot F_{0,2,1,0}\}) \). Avoiding \( F_{0,2,2,0} \) creates structure: Let \( X_i \) denote the columns of \( A \) of column sum \( i \). Let \( J_{a \times b} \) denote the \( a \times b \) matrix of 1’s and let \( 0_{a \times b} \) denote the \( a \times b \) matrix of 0’s. Now \( F_{0,2,2,0} \not\subset X_i \) and so for \( \|X_i\| \geq 3 \), we may deduce that there is a partition of the rows \([m]\) into \( A_i \cup B_i \cup C_i \). Let \( x_i = |X_i| \). After suitable row and column permutations, we have \( X_i \) as follows:

\[
\text{type 1: } X_i = \begin{cases}
A_i \{ \begin{bmatrix} I_{x_i} \\ J_{(i-1) \times x_i} \\ 0_{(m-x_i-i+1) \times x_i} \end{bmatrix} \} & \text{or type 2: } X_i = \begin{cases}
A_i \{ \begin{bmatrix} I_{x_i}^T \\ J_{(i-x_i+1) \times x_i} \\ 0_{(m-i-1) \times x_i} \end{bmatrix} \}
\end{cases}
\end{cases}
\]

We will say \( i \) is of type \( j \) (\( j = 1 \) or \( j = 2 \)) if the columns of sum \( i \) are of type \( j \). These are the sunflowers (for type 1) and inverse sunflowers (type 2) of [7] where for type 1 the petals are \( A_i \) with center \( B_i \).

Let \( T(1) = \{i : i \text{ is of type 1 and } \|X_i\| \geq t + 2\} \). We wish to show for that \( B_i \subset B_j \) for \( i, j \in T(1) \) and \( i < j \). Assume \( p \in B_i \setminus B_j \). Given that \( |B_i| < |B_j| \), there are two rows \( r, s \in B_j \setminus B_i \). Then we find a copy of \( t \cdot F_{0,2,1,0} \) in rows \( p, r, s \) of \([X_i] \) (we would not choose the possible column of \( X_i \) that has a 1 in row \( r \) and the column of \( X_i \) that has a 1 in row \( s \) ), a contradiction showing no such \( p \) exists and hence \( B_i \subset B_j \).

We form a matrix \( Y_1 \) from those \( X_i \) with \( i \in T(1) \). We have \( \|Y_1\| = \sum_{i \in T(1)} \|X_i\| = \sum_{i \in T(1)} |A_i| \). Assume \( \sum_{i \in T(1)} |A_i| > (t + 1)m \). Then there is some row \( p \) and \( (t + 2) \)-set \( \{s(1), s(2), \ldots, s(t + 2)\} \) with \( p \in A_i \) for all \( i \in \{s(1), s(2), \ldots, s(t + 2)\} \). Assume \( s(1) < s(2) < \cdots < s(t + 2) \). We have \( B_{s(1)} \subset B_{s(2)} \subset \cdots \subset B_{s(t+2)} \). We may choose \( r, s \in B_{s(t+2)} \setminus B_{s(t)} \) so that \( r, s \in A_{s(i)} \cup C_{s(i)} \) for \( i = 1, 2, \ldots, t \). We find a copy of \( t \cdot F_{0,2,1,0} \) in rows \( p, r, s \) as follows. We take one column from each \( X_{s(i)} \) for \( j = 1, 2, \ldots, t \) and \( t \) columns from the \( X_{s(t+2)} \). We conclude that \( \|Y_1\| \leq (t + 1)m \). Similarly the matrix \( Y_2 \) formed from those \( X_i \) such that \( i \) is of type 2 and \( \|X_i\| \geq t + 2 \) has \( \|Y_2\| \leq (t + 1)m \). Now \( Y_1 \) and \( Y_2 \) represent all columns of \( A \) with the exception of columns of sum \( i \) with \( \|X_i\| \leq t + 1 \) and so we conclude \( \|A\| \leq \|Y_1\| + \|Y_2\| + (t + 1)(m - 1) + 2 \). Thus \( \|A\| \) is \( O(m) \).

\[ \square \]

3 More evidence for the Conjecture

This section first explores the Conjecture 2 for \( t \cdot F \) when \( F \) is \( k \times 2 \). The section concludes with the proof of Theorem 3 for \( k > 2 \). The following verifies Conjecture 2 for all \( k \times 2 \) \( F \). Note that any \( k \times 2 \) matrix \( F \) can be written as \( F_{a,b,c,d} \) (\( b \geq c \)) under proper row and column permutations. Since \( \text{forb}(m, F) \) is invariant under taking \((0,1)\)-complement, we can further assume \( a \geq d \). The case of \( t = 1 \) was solved in [3] by the following theorem.

**Theorem 6.** [3] Suppose \( a \geq d \) and \( b \geq c \). Then \( \text{forb}(m, F_{a,b,c,d}) \) is \( \Theta(m^{a+b-1}) \) if either \( b > c \) or \( a, b \geq 1 \). Also \( \text{forb}(m, F_{a,0,0,d}) \) is \( \Theta(m^a) \) and \( \text{forb}(m, F_{0,b,0}) \) is \( \Theta(m^b) \). \[ \square \]
Note that Conjecture 2 is verified if there is a product construction avoiding $F$ yielding the same asymptotic growth as an upper bound on $\text{forb}(m,F)$. The $k$-fold product $I_{m/k} \times I_{m/k} \times \cdots \times I_{m/k} \in \text{Avoid}(m,t \cdot F_{0,k,k,0})$ has $\Theta(m^k)$ columns. Thus Theorem 3 verifies the conjecture for $t \cdot F_{0,k,k,0}$. The following results verify the conjecture for $t \cdot F$ for the remaining $k \times 2 F$.

**Theorem 7.** For $b > c$ or $a,b \geq 1$ then $\text{forb}(m,t \cdot F_{a,b,c,d})$ is $\Theta(m^{a+b})$.

*Proof.* The upper bound follows from $\text{forb}(m,F_{a,b,c,d})$ being $\Theta(m^{a+b-1})$ and then applying Lemma 4. The lower bound follows from $2 \cdot 1_{a+b} \prec t \cdot F_{a,b,c,d}$ so that the $(a+b)$-fold product $I_{m/(a+b)} \times I_{m/(a+b)} \times \cdots \times I_{m/(a+b)} \in \text{Avoid}(m,F_{a,b,c,d})$ and hence $\text{forb}(m,t \cdot F_{a,b,c,d})$ is $\Omega(m^{a+b})$.

**Theorem 8.** Let $a \geq d$ be given. Then $\text{forb}(m,t \cdot F_{a,0,0,d})$ is $\Theta(m^a)$.

*Proof.* This follows using Lemma 9 repeatedly and also $\text{forb}(m,t \cdot F_{a,0,0,0})$ is $O(m^a)$ using Theorem 14. The $a$-fold product $I_{m/a} \times I_{m/a} \times \cdots \times I_{m/a} \in \text{Avoid}(m,t \cdot F_{a,0,0,d})$.

The following result can be found in the survey on forbidden configurations [5]

**Lemma 9.** Assume $\text{forb}(m,F)$ is $O(m^\ell)$. Then $\text{forb}(m, [1 \ 0] \times F)$ is $O(m^{\ell+1})$.

Here is the summary of results on $\text{forb}(m,t \cdot F_{a,b,c,d})$ ($a \geq d$ and $b \geq c$), which verify Conjecture 2 for all $k \times 2 F$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>Configuration</th>
<th>result</th>
<th>reference</th>
<th>Lower bound construction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 1$</td>
<td>$F_{a,b,c,d}$ ($b &gt; c$ or $a,b \geq 1$)</td>
<td>$\Theta(m^{a+b-1})$</td>
<td>[3]</td>
<td>$I \times I \times \cdots \times I \times I$</td>
</tr>
<tr>
<td>$F_{a,0,0,d}$</td>
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<td>$I \times I \times \cdots \times I \times I$</td>
<td></td>
</tr>
<tr>
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<td>$\Theta(m^b)$</td>
<td>[3]</td>
<td>$I \times I \times \cdots \times I \times T$</td>
<td></td>
</tr>
<tr>
<td>$t \geq 2$</td>
<td>$t \cdot F_{a,b,c,d}$ ($b &gt; c$ or $a,b \geq 1$)</td>
<td>$\Theta(m^{a+b})$</td>
<td>Lemma 4</td>
<td>$I \times I \times \cdots \times I \times I$</td>
</tr>
<tr>
<td>$t \cdot F_{a,0,0,d}$</td>
<td>$\Theta(m^a)$</td>
<td>Lemma 4</td>
<td>$I \times I \times \cdots \times I \times I$</td>
<td></td>
</tr>
<tr>
<td>$t \cdot F_{0,b,b,0}$</td>
<td>$\Theta(m^b)$</td>
<td>Theorem 3</td>
<td>$I \times I \times \cdots \times I \times T$</td>
<td></td>
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</tbody>
</table>

Table 1: All cases of $\text{forb}(m,t \cdot F_{a,b,c,d})$ with $a \geq d$ and $b \geq c$.

We note that the bound for $\text{forb}(m,t \cdot F_{a,0,0,d})$ can be readily established by a pigeonhole argument. We return to Theorem 3 and first obtain some useful lemmas. Let
$X_i \in \text{Avoid}(m, F_{0,k,k,0})$ with all column sums $i$. We define $X_i$ to be of type $(a, b)$ if $a, b \geq 0$ are integers with $a + b = k - 1$ and there is a partition $C_i \cup D_i = [m]$ with $|D_i| + a - b = i$ such that any column $\alpha$ of $X_i$ has exactly $a$ 1’s in rows $C_i$ and exactly $b$ 0’s in rows $D_i$. We are able to use this structure in view of the following ‘strong stability’ result:

**Lemma 10.** [3] Let $Y_i \in \text{Avoid}(m, F_{0,k,k,0})$ with all column sums $i$. Assume $\|Y_i\| \geq (6(k-1))^{5k+2}m^{k-2}$. Then there is an $m$-rowed submatrix $X_i$ of $Y_i$ and a pair of integers $a, b \geq 0$ with $a + b = k - 1$ such that $X_i$ is of type $(a, b)$ and where $\|Y_i\| - \|X_i\| \leq m^{k-3}$.

**Lemma 11.** Let $X_i \in \text{Avoid}(m, F_{0,k,k,0})$ have all columns of sum $i$ and assume $X_i$ is of type $(a, b)$ with $a, b \geq 1$ with $a + b = k - 1$. Let $C_i \cup D_i = [m]$ be the associated partition of the rows. We form a bipartite graph $G_i = (V_i, E_i)$ with $V_i = (C_i) \cup (D_i)$ where we have $(C, D) \in E_i$ if there is a column of $X_i$ with a 1’s in rows $C$ and $D \setminus D$ and $b$ 0’s in rows $D$ and $C \setminus C$. Assume $|E_i| \geq 2km^{k-2}$. Then there is subgraph $G'_i = (V'_i, E'_i)$ of $G_i$ with $|E'| \geq \frac{1}{2}|E_i|$ such that for every pair $C \in (C_i)$ and $D \in (D_i)$ with $(C, D) \in E'$ we have

$$d_{G'_i}(C) \geq (b + 1/2)m^{b-1}, \quad d_{G'_i}(D) \geq (a + 1/2)m^{a-1}. \quad (6)$$

**Proof.** Simply delete vertices $C \in (C_i)$ with $d_G(C) < (b + 1/2)m^{b-1}$ and vertices $D \in (D_i)$ with $d_G(D) < (a + 1/2)m^{a-1}$ and continue deleting vertices until conditions (6) are satisfied for any remaining vertices of $G'$. This will delete a maximum of $(b + 1/2)m^{b-1} |C_i| + (a + 1/2)m^{a-1} |D_i| < km^{k-2}$ edges which deletes less than half the edges of $G$. \hfill \Box

**Lemma 12.** Let $k$ be given. Then $\text{forb}(m, \{F_{0,k,k,0}, t \cdot F_{0,k,k-1,0}\}) = O(m^{k-1})$.

**Proof.** Let $A \in \text{Avoid}(m, \{F_{0,k,k,0}, t \cdot F_{0,k,k-1,0}\})$. Let $Y_i$ denote the columns of $A$ of column sum $i$. For all $i$ for which $|Y_i| < (6(k-1))^{5k+2}m^{k-2}$, delete the columns of $Y_i$ from $A$. This may delete $(6(k-1))^{5k+2}m^{k-1}$ columns. For $i$ with $|Y_i| \geq (6(k-1))^{5k+2}m^{k-2}$, apply Lemma 10 and obtain $X_i$ with $|X_i| \geq (6(k-1))^{5k+2}m^{k-2} - m^{k-3}$.

We consider a choice $a, b$ with $a + b = k - 1$. Let $T(a, b) = \{i : X_i$ is of type $(a, b)\}$. We will show that $\sum_{i \in T(a,b)} |X_i| \leq (tk)m^{k-1}$.

**Case 1.** $a, b \geq 1$.

Create $G_i$ as described in Lemma 11 to obtain $G'_i$ for each $i \in T(a, b)$. Now if $\sum_{i \in T(a,b)} |E'_i| > (t + 1)m^{a+b}$, then there will be some edge $(C, D) \in E'_i$ for at least $t + 2$ choices $i \in T(a,b)$. Let those choices be $s(1), s(2), \ldots, s(t+2)$ where $s(1) < s(2) < \cdots < s(t+2)$. We wish to show that $X_{s(i)}$ has $t \cdot F_{0,k-1,0,0}$ on rows $C \cup D$.

<table>
<thead>
<tr>
<th>rows $C$</th>
<th>1</th>
<th>11</th>
<th>111</th>
<th>1111</th>
<th>00</th>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>0</td>
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<td></td>
</tr>
<tr>
<td>rows $D$</td>
<td>0</td>
<td>11</td>
<td>111</td>
<td>1111</td>
<td>00</td>
<td>000</td>
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</tr>
</tbody>
</table>
For a given set $D \in \binom{D_{s(i)}}{b}$, we compute $|\{H \in \binom{D_{s(i)}}{b} : H \cap D \neq \emptyset\}| \leq \sum_{j=1}^{b} \binom{b}{j} \binom{D_{s(i)} \setminus D}{b-j} < bm^{b-1}$.

Now if $d_{G'}(C) \geq (b + 1/2)m^{b-1}$ and $(C, D) \in E_{s(i)}'$ then there are at least $t$ edges $(C, H) \in E_{s(i)}'$ with $H \cap D = \emptyset$. We are using $(b + 1/2)m^{b-1} > bm^{b-1} + t + 2$ which is true for $m$ large enough and so asymptotics are unaffected. Thus we have $t$ columns of $X_{s(1)}$ with $1_{k-1}$ on rows $C \cup D$ and, because these columns have a 1’s on rows $C \subseteq C_{s(1)}$, these columns are 0’s on the remaining rows of $C_{s(1)} \setminus C$.

Similarly, because $d_{G'}(D) \geq (a + 1/2)m^{a-1}$ there will be $t + 2$ edges $(K, D) \in E_{s(i)}$ with $K \cap C = \emptyset$ and so there are $t$ columns of $X_{s(t+2)}$ with $0_{k-1}$ on rows $C \cup D$ and, because these columns have 0’s on rows $D$, these columns are 1’s on rows of $D_{s(t+2)} \setminus D$.

We choose $k$ rows in $Z = D_{s(t+2)} \setminus D_{s(1)}$ so that $Z \subseteq C_{s(1)}$. We deduce that in the chosen $t$ columns of $X_{s(1)}$ we have $0_k$ in rows $Z$ since $Z \subseteq C_{s(1)} \setminus C$ and the columns have $1_{k-1}$ in rows $C \cup D$. In the chosen $t$ columns of $X_{s(t+2)}$ we have $1_k$ in rows $Z$ since $Z \subseteq D_{s(t+2)} \setminus D$ and the columns have $0_{k-1}$ in rows $C \cup D$. This yields $t \cdot F_{0, k, k-1, 0}$, a contradiction. Thus $\sum_{i \in \text{Type}(a, b)} |E'_i| \leq (t + 1)m^{k-1}$. This concludes Case 1.

**Case 2.** $a = k - 1, b = 0$ or $a = 0, b = k - 1$.

We proceed similarly. We need only consider $a = k - 1, b = 0$ since the case $a = 0, b = k - 1$ is just the (0,1)-complement. For $i \in T(k - 1, 0)$, $X_i$ has partition $C_i \cup D_i = [m]$ and columns of $X_i$ have 1’s on exactly $k - 1$ rows of $C_i$ and all 1’s on rows $D_i$. Assume $\sum_{i \in T(k - 1, 0)} |X_i| \geq (tk)m^{k-1}$. Then there are $tk$ choices $s(1), s(2), \ldots, s(tk) \in T(k - 1, 0)$ where $s(1) < s(2) < \cdots < s(tk)$ such that, for some $C \in \binom{C_{s(i)}}{k-1}$, each $X_{s(i)}$ has a column with 1’s in rows $C \cup D_{s(i)}$ and 0’s in rows $C_{s(i)} \setminus C$. We wish to find $t \cdot F_{0, k - 1, 0, 0}$ in $A$ in rows $C$ as follows using one column from each of $X_{s(i)}$ for $i = 1, 2, \ldots, t$ and $t$ columns from $X_{s(tk)}$.

$$
\begin{array}{cccccccc}
\text{rows } C & & & & & & & \\
& 1 & 1 & 1 & 1 & 0 & \cdots & 0 \\
& 1 & 1 & \ldots & 1 & 1 & 0 & \cdots & 0 \\
& 1 & 1 & 1 & 1 & 0 & \cdots & 0 \\
X_{s(1)} & X_{s(2)} & X_{s(t)} & X_{s(tk)} & X_{s(tk)} &
\end{array}
$$

Given our choice $C \in \binom{C_{s(tk)}}{k-1}$, we compute that $|\{K \in \binom{C_{s(tk)}}{k-1} : K \cap C \neq \emptyset\}| \leq km^{k-2}$. Thus with $|X_{s(tk)}| \geq km^{k-2}$, there will be $t$ choices $K_1, K_2, \ldots, K_t$ disjoint from $C$ and hence one column of $X_{s(tk)}$ for each $i = 1, 2, \ldots, t$ with $0_{k-1}$ on rows of $K_i \subseteq C_{s(tk)} \setminus C$ and 0’s on $C_{s(tk)} \setminus K_i$ and hence $0_{k-1}$ on rows $C$.

We will show below that we can choose $D \subset D_{s(tk)} \setminus \bigcup_{i=1}^{t} D_{s(i)}$ with $|D| = k$. Then we can find $t \cdot F_{0, k, k-1, 0}$ as follows. We have one column in $X_{s(i)}$ for each $i = 1, 2, \ldots, t$ which is $1_{k-1}$ on rows $C$ and $0_k$ on rows $D$ (since $D \subset C_{s(i)} \setminus C$ for each $i = 1, 2, \ldots, t$). The $t$ columns of $X_{s(tk)}$ we have selected have $0_{k-1}$ on rows $C$ and 1’s on $D_{s(tk)}$ where $D \subseteq D_{s(tk)}$ and hence $1_k$ on rows $D$. These $2t$ columns yield $t \cdot F_{0, k, k-1, 0}$ in $[X_{s(1)} | X_{s(2)} \cdots | X_{s(t)} | X_{s(tk)}]$. 

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To show that $D$ can be chosen we first show that $D_{s(i)} \setminus D_{s(j)} \leq k - 2$ for $s(i) < s(j)$. Assume the contrary, $D_{s(i)} \setminus D_{s(j)} \geq k - 1$ for $s(i) < s(j)$. We choose $C' \subseteq D_{s(i)} \setminus D_{s(j)}$ with $|C'| = k - 1$. Given $s(j) > s(i)$, then $D_{s(j)} \setminus D_{s(i)} \geq k$ and so we may choose $D' \subseteq D_{s(j)} \setminus D_{s(i)}$ with $|D'| = k$. Now $C' \subset C_{s(j)}$ and $D' \subset C_{s(i)}$. The number of possible columns of $X_{s(j)}$ with at least one 1 on the rows $C'$ is at most $m^{k-2}$ and with $|X_{s(j)}| \geq m^{k-1} + t$, we find $t$ columns of $X_{s(j)}$ with 0’s on rows $C'$ and necessarily with 1’s on rows $D'$. The number of possible columns of $X_{s(i)}$ with at least one 1 on the rows of $D'$ is $|D'|m^{k-2} < m^{k-1}$. Given $|X_{s(i)}| \geq m^{k-1} + t$, we find $t$ columns of $X_{s(i)}$ with 0’s on rows $D'$ and necessarily with 1’s on rows $C'$. This yields $t \cdot F_{0,k,k-1,0}$ in $[X_{s(i)} \setminus X_{s(j)}]$, a contradiction. Thus $D_{s(i)} \setminus D_{s(j)} \leq k - 2$ for $s(i) < s(j)$. We may now conclude that $|D_{s(k)} \cup_{i=1}^{t} D_{s(i)}| \geq k$ and so a choice for $D$ exists. We conclude $\sum_{i \in T(k-1,0)} |X_i| \leq (tk)m^{k-1}$. This concludes Case 2.

There are $k + 1$ choices for type $(a, b)$ and so
\[
\sum_{i=0}^{m} |X_i| \leq \sum_{j=0}^{k} \left( \sum_{i \in T(j,k-1-j)} |X_i| \right) \leq (k + 1)(2tk)m^{k-1}
\]
and so $\|A\| \leq (2tk(k+1))m^{k-1} + (6(k-1))^{5k+2}m^{k-2}$ which is $O(m^{k-1})$. 

\[\square\]

**Proof of Theorem 3 for $k \geq 3$:** We use (5) so that
forb($m, t \cdot F_{0,k,k,0}, t-1$) \leq forb($m-1, t \cdot F_{0,k,k,0}, t-1$) + $(t-1)$ forb($m, \{F_{0,k,k,0}, t \cdot F_{0,k,k-1,0}\}$).

Induction on $m$ and Lemma 12 yields the bound. 

\[\square\]

## 4 Some applications of the Induction

Let $K_k$ denote the $k \times 2^k$ of all possible (0,1)-columns on $k$ rows. The following is the fundamental result about forbidden configurations.

**Theorem 13.** [Sauer [10], Perles and Shelah [11], Vapnik and Chervonenkis [12]] We have that
\[
\text{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}.
\]
Thus forb($m, K_k$) is $\Theta(m^{k-1})$.

We can apply this result as follows.

**Theorem 14.** [8] Let $F$ be a given $k \times \ell$ (0,1)-matrix. Then forb($m, F$) is $O(m^k)$.

**Proof.** Let $t$ be the maximum multiplicity of a column in $F$ (of course $t \leq \ell$). Then $F \preceq t \cdot K_k$ and so supp($F$) $\preceq K_k$. Now Lemma 4 combined with Theorem 13 yields the result. 

\[\square\]

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Interestingly this yields the exact result for \( \text{forb}(m, 2 \cdot K_k) \) [9]. A more precise result of Anstee and Füredi [2] for \( \text{forb}(m, t \cdot K_k) \) has the leading term being bounded by \( \frac{t+k-1}{k+1} \binom{m}{k} \) for \( t \geq 2 \). The following surprising result was obtained by Balogh and Bollobás.

**Theorem 15.** [6] Let \( k \) be given. There is a constant \( c_k \) with \( \text{forb}(m, \{I_k, I^c_k, T_k\}) = c_k \).

This yields the following.

**Theorem 16.** Let \( t, k \geq 2 \) be given. Then \( \text{forb}(m, \{t \cdot I_k, t \cdot I^c_k, t \cdot T_k\}) \) is \( \Theta(m) \).

**Proof.** Apply Lemma 4. The matrix \( I_m \in \text{Avoid}(m, \{t \cdot I_k, t \cdot I^c_k, t \cdot T_k\}) \) shows that \( \text{forb}(m, \{t \cdot I_k, t \cdot I^c_k, t \cdot T_k\}) \) is \( \Theta(m) \). \( \square \)

Lemma 4 is interesting for those \( H \) for which \( \text{forb}(m, H) \) is \( O(m^\ell) \) and the number of rows in \( H \) is bigger than \( \ell \) (see [5] for examples). It is not expected that this will resolve any boundary cases, namely those \( F \) for which \( \text{forb}(m, [F | \alpha]) \) is bigger than \( \text{forb}(m, F) \) by a linear factor (or more) for all choices \( \alpha \) which are either not present in \( F \) or occur at most once in \( F \). The previously mentioned \( F_6(t) \) and \( F_7(t) \) have quite complicated structure and the induction (5) does not appear to work directly.

**References**
