Separability of density matrices of graphs for multipartite systems

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Abstract

We investigate separability of Laplacian matrices of graphs when seen as density matrices. This is a family of quantum states with many combinatorial properties. We firstly show that the well-known matrix realignment criterion can be used to test separability of this type of quantum states. The criterion can be interpreted as novel graph-theoretic idea. Then, we prove that the density matrix of the tensor product of N graphs is N-separable. However, the converse is not necessarily true. Additionally, we derive a sufficient condition for N-partite entanglement in star graphs and propose a necessary and sufficient condition for separability of nearest point graphs.

Keywords: density matrices of graphs, Laplacian matrices; separability

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1 Introduction

Quantum entanglement is one of the most fascinating features of quantum theory and has numerous applications in quantum information processing, secure communication and channels [1, 2, 3]. The theory of graphs, as a well-developed mathematical area, has many applications in network systems, algorithms, optimization, and other fields [4].

For exploring combinatorial features of entanglement in mixed quantum states, Braunstein et al. [5] introduced a restricted class of states called density matrices of graphs. These states are just normalized graph Laplacians. It follows that study entanglement properties of Laplacians is a combinatorial problem.

One of the reasons to study such a topic is that theorems about general sets of quantum states may be more straightforward to prove when restricted to Laplacian matrices. On the other side, graph-theoretic results may be reinterpreted in terms of the language of quantum mechanics.

Braunstein et al. [6] described a degree condition to test separability of density matrices of graphs. In [7], the PPT criterion [8] was proved to be necessary and sufficient for separability in $H^2 \otimes H^q$. Further results on the multipartite separability of Laplacians were discussed in [9]. Ref. [10] generalized the study of entanglement properties of mixed density matrices to tripartite states.

Ref. [11] analyze in detail the separability and quantitative characteristics of entanglement for Laplacians. However, it is generally known that for the case of mixed bipartite states, no single practical procedure can be guaranteed to detect entanglement, even when we restrict the analysis to Laplacians. It would be important to see if in this specific case there is indeed a method. It is expected that the method would be purely combinatorial.

In this paper, we study multipartite entanglement properties. By applying graph-theoretic ideas, we are able to significantly expand our ability to distinguish entangled and separable Laplacians. It is well known that the realignment criterion [12, 13, 14, 15] (also called the cross-norm criterion) is a very strong one. The criterion involves only straightforward matrix manipulations and it is easy to apply. We consider realignment for graphs and obtain a simple, computable necessary criterion for separability. This is done in Section 2.

In Section 3, we prove that the density matrix of the tensor product of N graphs is N-separable. While this seems to be an obvious statement, it is not immediately clear why the converse should hold. In fact, we can show that if a density matrix is 4-separable, it does not necessarily mean that it can be written as the tensor product of four graphs. In Section 4, we derive a sufficient condition for the entanglement of star graphs and give a necessary and sufficient condition for the separability of the nearest point graph. Comments and conclusions are given in Section 5. It would be valuable to observe whether the difficulty of detecting entanglement for Laplacians is determined by some graph-theoretic parameter, as in the context of fixed-parameter tractability.

We hope that the results of the paper will contribute to diffuse the study of separability criteria for graphs among mathematicians interested in algebraic graph theory and graph products.

2 The Laplacian Matrices of Graphs

Let $G = (V(G),\ E(G))$ be a *graph* with a vertex set $V(G)$ and an edge set $E(G)$, where $V(G) = \{v_1,\ v_2,\ \cdots,\ v_n\}$ is a non-empty and finite set of *vertices* and $E(G) = \{\{v_i,\ v_j\} : v_i,\ v_j \in V\}$ is a non-empty set of unordered *edges*. A *loop* is an edge of the form $\{v_i,\ v_i\}$. We assume that $E(G)$ does not contain any loops in the sequel. A graph G is said to be on n vertices if $|V(G)| = n$. The *degree* of a vertex $v_i \in V(G)$ is the number of edges adjacent to v_i, we denote it as $d_G(v_i)$.

Definition 1. The density matrix of graph G is defined as the matrix

$$\rho(G) = \frac{1}{d_G} L(G), \tag{1}$$

where $d_G = \sum_{i=1}^{n} d_G(v_i)$ is called the *degree sum*. $L(G)$ is the *combinatorial Laplacian matrix* of graph G defined as

$$L(G) = \Delta(G) - M(G), \tag{2}$$

where $\Delta(G)$ is the *degree matrix* with the diagonal entry $d_G(v_i)$. $M(G)$ is the *adjacency matrix* of graph G with the ij-th entry defined as

$$[M(G)]_{i,j} = \begin{cases} 1, & \text{if } (v_i,\ v_j) \in E(G); \\ 0, & \text{if } (v_i,\ v_j) \notin E(G). \end{cases} \tag{3}$$

Two distinct vertices v_i and v_j are said to be *adjacent*, if $\{v_i,\ v_j\} \in E(G)$. If every pair of vertices are adjacent, the graph is *complete*. K_n denotes the *complete graph* on n vertices.

The density matrix of a graph is a uniform mixture of pure density matrices, that is, for a graph G on n vertices $v_1,\ v_2,\ \cdots,\ v_n$, having s edges $\{v_{i_1},\ v_{j_1}\},\ \{v_{i_2},\ v_{j_2}\},\ \cdots,\ \{v_{i_s},\ v_{j_s}\}$, where $1 \leqslant i_1,\ j_1,\ i_2,\ j_2,\ \cdots,\ i_s,\ j_s \leqslant n$,

$$\rho(G) = \frac{1}{s} \sum_{k=1}^{s} \rho(H_{i_k j_k}),$$

here $H_{i_k j_k}$ is the factor of G such that

$$[M(H_{i_k j_k})]_{u,\ w} = \begin{cases} 1, & \text{if } u = i_k \text{ and } w = j_k \text{ or } w = i_k \text{ and } u = j_k; \\ 0, & \text{otherwise.} \end{cases}$$

The density matrix $\rho(H_{i_k j_k})$ is *pure*.

Definition 2. A *star graph* [10] on n vertices $v_1,\ v_2,\ \cdots,\ v_n$ is the graph whose set of edges is $\{\{v_1,\ v_i\} :\ i = 2,\ 3,\ \cdots,\ n\}$.

Definition 3. A *nearest point graph* [10] is a graph whose vertices are identified with the points of a cuboid and the edges have length 1, $\sqrt{2}$ or $\sqrt{3}$.

Definition 4. The state ρ acting on $H = H_1 \otimes H_2 \otimes \cdots \otimes H_N$ is *separable* if it can be written as

$$\rho = \sum_i p_i \rho_1^i \otimes \rho_2^i \otimes \cdots \otimes \rho_N^i, \tag{4}$$

where ρ_j^i is the density matrix on H_j respectively. Otherwise, the state is *entangled*.

Definition 5. The *tensor product of graphs* $G_1, \ldots, G_N$, denoted by $G_1 \otimes \cdots \otimes G_N$, is the graph whose adjacency matrix is $M(G_1 \otimes \cdots \otimes G_N) = M(G_1) \otimes \cdots \otimes M(G_N)$.

Definition 6. Let G_j be a graph on m_j vertices, $v_1^j, v_2^j, \cdots, v_{m_j}^j$, and n_j edges, $\{v_{c_1^j}^j, v_{d_1^j}^j\}, \cdots, \{v_{c_{n_j}^j}^j, v_{d_{n_j}^j}^j\}$ with $1 \leqslant c_1^j, d_1^j, \cdots, c_{n_j}^j, d_{n_j}^j \leqslant m_j$, $j = 1, 2, \cdots, N$. Suppose Hilbert space H_1 is the space spanned by the orthonormal basis $\{|v_1^1\rangle, |v_2^1\rangle, \cdots, |v_{m_1}^1\rangle\}$ associated to $V(G_1)$; $\cdots$; H_N is the space spanned by the orthonormal basis $\{|v_1^N\rangle, |v_2^N\rangle, \cdots, |v_{m_N}^N\rangle\}$ associated to $V(G_N)$. The density matrix $\rho(G_1 \otimes \cdots \otimes G_N)$ is *N-separable* with respect to Hilbert space $H_1 \otimes \cdots \otimes H_N$ and any graph G on n vertices is *N-separable* with respect to space $C^{m_1} \otimes \cdots \otimes C^{m_N}$, where $n = m_1 m_2 \cdots m_N$.

Definition 7. For an $m \times n$ matrix $A = [a_{ij}]$, where a_{ij} is the matrix entry of A, the vector $vec(A)$ is defined as

$$vec(A) = [a_{11}, \cdots, a_{m1}, a_{12}, \cdots, a_{m2}, \cdots, a_{1n}, \cdots, a_{mn}]^T. \tag{5}$$

Let Z be an $m \times m$ block matrix with block size $n \times n$. The realigned matrix $\widetilde{Z}$ of size $m^2 \times n^2$ is defined as

$$\widetilde{Z} \equiv \begin{pmatrix} vec(Z_{1,1}^T) \\ \vdots \\ vec(Z_{m,1}^T) \\ \vdots \\ vec(Z_{1,m}^T) \\ \vdots \\ vec(Z_{m,m}^T) \end{pmatrix}. \tag{6}$$

If an $m \times n$ bipartite density matrix ρ_{AB} is separable, then for the $m^2 \times n^2$ matrix $\widetilde{\rho_{AB}}$ the Ky Fan norm which is the sum of all the singular values of $\widetilde{\rho_{AB}}$ is less than or equal to 1 [14]. Next applying this result, we present a separability criterion.

For a $pq \times pq$ matrix A, A_{ij} denotes the (i, j)-th element of A. Let f be the canonical bijection between $1, \cdots, p \times 1, \cdots, q$ and $1, \cdots, pq : f(i, j) = (i - 1)q + j$. If $f(i_1, j_1) = k$ and $f(i_2, j_2) = l$, then A_{kl} can be written as $A_{(i_1,j_1)(i_2,j_2)}$.

Theorem 8. *For the graph G with the Laplacian matrix $A = (a_{ij})_{n^2\times n^2}$, the density matrix of the graph G is entangled if*

$$|\sum_{k=1}^{n} d_G(v_k) - N(k_1,k_2)| > 1, \tag{7}$$

for all vertices $v_k = f(k,k) = (k-1)n+k$, $k = 1,\cdots,n$, where $N(k_1,k_2)$ is the number of edges between v_{k_1} and v_{k_2}, $v_{k_1} \neq v_{k_2}$.

Proof. Suppose

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} & \cdots & a_{1n^2-n} & \cdots & a_{1n^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{1n} & \cdots & a_{nn} & \cdots & a_{nn^2-n} & \cdots & a_{nn^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{1n^2-n} & \cdots & a_{nn^2-n} & \cdots & a_{n^2-nn^2-n} & \cdots & a_{n^2-nn^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{1n^2} & \cdots & a_{nn^2} & \cdots & a_{n^2-nn^2} & \cdots & a_{n^2n^2} \end{pmatrix} \tag{8}$$

is the Laplacian matrix of a graph, and the realigned matrix $\widetilde{A}$ is

$$\widetilde{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} & \cdots & a_{1n} & \cdots & a_{nn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{1n^2-n} & \cdots & a_{1n^2} & \cdots & a_{nn^2-n} & \cdots & a_{nn^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{1n^2-n} & \cdots & a_{nn^2-n} & \cdots & a_{1n^2} & \cdots & a_{nn^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{n^2-nn^2-n} & \cdots & a_{n^2-nn^2} & \cdots & a_{n^2-nn^2} & \cdots & a_{n^2n^2} \end{pmatrix}. \tag{9}$$

The diagonal elements of $\widetilde{A}$ are only $N(k_1,k_1)$ or $N(k_1,k_2)$, $k_1,k_2 = 1,\cdots,n$. So if $|\sum_{k=1}^{n} d_G(v_k) - N(k_1,k_2)| > 1$, we have $tr(\widetilde{A}) > 1$, the sum of the eigenvalue of $\widetilde{A}$ is greater than 1. It is obvious that the sum of the singular values of $\widetilde{A}$ is greater than 1, therefore the density matrix of the graph G is entangled. □

3 Separability of Graphs

In Ref. [10] the entanglement of mixed density matrices for tripartite states was discussed. Now, we generalized their results to multipartite quantum systems.

Theorem 9. *The density matrix of the tensor product of N graphs is N-separable.*

Proof. Using the Definitions 1 and 6, we have

$$\rho(G_1) = \frac{1}{n_1}\sum_{p_1=1}^{n_1} \rho(H^{(1)}_{c^1_{p_1} d^1_{p_1}}), \rho(G_2) = \frac{1}{n_2}\sum_{p_2=1}^{n_2} \rho(H^{(2)}_{c^2_{p_2} d^2_{p_2}}), \cdots, \rho(G_N) = \frac{1}{n_N}\sum_{p_N=1}^{n_N} \rho(H^{(N)}_{c^N_{p_N} d^N_{p_N}}).$$

$H^{(i)}$ denote $H^{(i)}_{c^i_{p_i} d^i_{p_i}}$ with $i = 1, \cdots, N$, we have

$$
\begin{aligned}
& \rho(G_1 \otimes G_2 \otimes \cdots \otimes G_N) \\
= & \frac{1}{d_{G_1 \otimes G_2 \otimes \cdots \otimes G_N}} [\Delta(G_1 \otimes G_2 \otimes \cdots \otimes G_N) - M(G_1 \otimes G_2 \otimes \cdots \otimes G_N)] \\
= & \frac{1}{2^N \cdot n_1 n_2 \cdots n_N} \sum_{p_1, p_2, \cdots, p_N = 1}^{n_1, n_2, \cdots, n_N} [\Delta(H^{(1)}) \otimes \Delta(H^{(2)}) \otimes \cdots \otimes \Delta(H^{(N)}) \\
& - M(H^{(1)}) \otimes M(H^{(2)}) \otimes \cdots \otimes M(H^{(N)})] \\
= & \frac{1}{2^{N-1} \cdot n_1 n_2 \cdots n_N} \sum_{p_1, p_2, \cdots, p_N = 1}^{n_1, n_2, \cdots, n_N} [\rho(H^{(1)}) \otimes \rho_+(H^{(2)}) \otimes \cdots \otimes \rho_+(H^{(N)}) \\
& + \rho_+(H^{(1)}) \otimes \rho(H^{(2)}) \otimes \cdots \otimes \rho_+(H^{(N)}) + \cdots + \rho_+(H^{(1)}) \otimes \rho_+(H^{(2)}) \otimes \cdots \otimes \rho(H^{(N)}) \\
& + \cdots + \rho(H^{(1)}) \otimes \rho(H^2) \otimes \cdots \otimes \rho_+(H^{(N)})], \qquad (10)
\end{aligned}
$$

where $\rho_+(H^{(i)}) \overset{\text{def}}{=} \Delta(H^{(i)}) - \rho(H^{(i)}) = \frac{1}{2}\big(\Delta(H^{(i)}) + M(H^{(i)})\big)$.
Let $\rho_+(G_i) = \frac{1}{n_i} \sum_{p_i=1}^{n_i} \rho_+(H^{(i)}_{c^i_{p_i} d^i_{p_i}})$, $i = 1, \cdots, N$, we have

$$
\begin{aligned}
& \rho(G_1 \otimes G_2 \otimes \cdots \otimes G_N) \\
= & \frac{1}{2^{N-1} n_1 n_2 \cdots n_N} [\rho(G_1) \otimes \rho_+(G_2) \otimes \cdots \otimes \rho_+(G_N) + \rho_+(G_1) \otimes \rho(G_2) \otimes \cdots \otimes \rho_+(G_N) \\
& + \cdots + \rho_+(G_1) \otimes \rho_+(G_2) \otimes \cdots \otimes \rho(G_N) + \cdots + \rho(G_1) \otimes \rho(G_2) \otimes \cdots \otimes \rho_+(G_N)]. (11)
\end{aligned}
$$

Therefore $\rho(G)$ is N-separable. □

In order to describe the theorem in detail, we give the following example.

Example 10. The density matrix acting on $H_1 \otimes H_2 \otimes H_3 \otimes H_4$ of the tensor of four graphs is 4-separable.

Let G_i be a graph on m_i vertices and n_i edges, $i = 1, 2, 3, 4$, and denote $H^{(i)} = H^{(i)}_{c^i_{p_i} d^i_{p_i}}$, we have

$$
\begin{aligned}
\rho_+(G_1) = \frac{1}{n_1} \sum_{p_1=1}^{n_1} \rho_+(H^{(1)}), \rho_+(G_2) = \frac{1}{n_2} \sum_{p_2=1}^{n_2} \rho_+(H^{(2)}), \\
\rho_+(G_3) = \frac{1}{n_3} \sum_{p_3=1}^{n_3} \rho_+(H^{(3)}), \rho_+(G_4) = \frac{1}{n_4} \sum_{p_4=1}^{n_4} \rho_+(H^{(4)}), \qquad (12)
\end{aligned}
$$

where $\rho_+(H^{(i)}) \overset{\text{def}}{=} \Delta(H^{(i)}) - \rho(H^{(i)}) = \frac{1}{2}\big(\Delta(H^{(i)}) + M(H^{(i)})\big)$. Since $\rho_+(H^{(i)})$ are all density matrices,

$$
\begin{aligned}
& \rho(G_1 \otimes G_2 \otimes G_3 \otimes G_4) \\
= & \frac{1}{8} [\rho_+(G_1) \otimes \rho_+(G_2) \otimes \rho_+(G_3) \otimes \rho(G_4) + \rho_+(G_1) \otimes \rho_+(G_2) \otimes \rho(G_3) \otimes \rho_+(G_4)
\end{aligned}
$$

$$+\rho_+(G_1)\otimes\rho(G_2)\otimes\rho_+(G_3)\otimes\rho_+(G_4)+\rho_+(G_1)\otimes\rho(G_2)\otimes\rho(G_3)\otimes\rho(G_4)$$
$$+\rho(G_1)\otimes\rho_+(G_2)\otimes\rho_+(G_3)\otimes\rho_+(G_4)+\rho(G_1)\otimes\rho_+(G_2)\otimes\rho(G_3)\otimes\rho(G_4)$$
$$+\rho(G_1)\otimes\rho(G_2)\otimes\rho_+(G_3)\otimes\rho(G_4)+\rho(G_1)\otimes\rho(G_2)\otimes\rho(G_3)\otimes\rho_+(G_4)]. \quad (13)$$

Therefore $\rho(G)$ is 4-separable.

Lemma 11. *For the matrix* $\sigma = \frac{1}{5}P[\frac{1}{\sqrt{2}}(|ijkp\rangle - |rstq\rangle)] + \frac{1}{5}P[\frac{1}{\sqrt{2}}(|ijkq\rangle - |rstp\rangle)] + \frac{1}{5}P[\frac{1}{\sqrt{2}}(|ijtp\rangle - |rskq\rangle)] + \frac{1}{5}P[\frac{1}{\sqrt{2}}(|iskp\rangle - |rjtq\rangle)] + \frac{1}{5}P[\frac{1}{\sqrt{2}}(|rjkp\rangle - |istq\rangle)]$, *the following conlusions hold:*

(i) It is a density matrix and 4*-separable.*

(ii) The density matrix of complete graph $\rho(K_n)$ *is also* 4*-separable.*

Proof. (i) Since the project operator is semi-positive, σ is semi-positive. Moreover, the computation shows that $tr(\sigma) = 1$, so σ is a density matrix. Let

$$|u^{\pm}\rangle = \frac{1}{\sqrt{2}}(|i\rangle \pm |r\rangle),\ \ |v^{\pm}\rangle = \frac{1}{\sqrt{2}}(|j\rangle \pm |s\rangle),\ \ |w^{\pm}\rangle = \frac{1}{\sqrt{2}}(|k\rangle \pm |t\rangle),\ \ |\varphi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|p\rangle \pm |q\rangle).$$

Then σ can be written as the linear combination of tensor products, thus σ is 4-separable.

(ii) Since $M(K_n) = J_n - I_n$, where J_n is the $n\times n$ all-ones matrix and I_n is the $n\times n$ identity matrix, whenever there is an edge $\{u_iv_jw_k\varphi_p,\ u_rv_sw_t\varphi_q\}$, there must be entangled edges $\{u_rv_jw_k\varphi_p,\ u_iv_sw_t\varphi_p\}$, $\{u_iv_sw_k\varphi_p,\ u_rv_jw_t\varphi_p\}$, $\{u_iv_jw_t\varphi_p,\ u_rv_sw_k\varphi_p\}$ and $\{u_iv_jw_k\varphi_q,\ u_rv_sw_t\varphi_q\}$, so the density matrix $\rho(K_n)$ is 4-separable. □

Theorem 12. *Given a graph* $G_1\otimes G_2\otimes G_3\otimes G_4$, *the density matrix* $\rho(G_1\otimes G_2\otimes G_3\otimes G_4)$ *is* 4*-separable. However if a density matrix* $\rho(G)$ *is* 4*-separable, it does not necessarily mean that* $G = G_1\otimes G_2\otimes G_3\otimes G_4$ *for some graphs* $G_1,\ G_2,\ G_3,\ G_4$.

Proof. The first result follows from Example 10. On the other side, from Lemma 11, the density matrix of the complete graph $\rho(K_n)$ is 4-separable. Because of the complete graph is not a tensor product of three graphs [10], it is not a tensor product of four graphs. □

4 Separability of Some Special Graphs

In this section, we will derive the results of some special graphs.

Theorem 13. *The density matrix of star graphs* $\rho(K_{1,n-1})$ *is* N*-partite entangled for* $n = n_1n_2\cdots n_N \geqslant 2^N$.

Proof. For a star graph $G = K_{1,n-1}$ on $n = n_1\cdots n_N$ vertices with orthonormal basis $|\alpha_1\rangle,\ |\alpha_2\rangle,\ \cdots,\ |\alpha_n\rangle$, we have

$$\rho(G) = \frac{1}{n-1}\sum_{k=2}^{n}\rho(H_{1k}) = \frac{1}{n-1}\sum_{k=2}^{n}P[\frac{1}{\sqrt{2}}(|\alpha_1\rangle - |\alpha_k\rangle)]. \quad (14)$$

Consider $\rho(G)$ in $C^{n_1}_{A_1}\otimes C^{n_2}_{A_2}\otimes\cdots\otimes C^{n_N}_{A_N}$, where $C^{n_i}_{A_i}$ are associated to $\mathcal{H}_{A_i}$, $i=1,\cdots,N$ respectively. Let $\{|v^i_1\rangle,\ |v^i_2\rangle,\ \cdots,\ |v^i_{n_i}\rangle\}$ be the orthonormal basis of $C^{n_i}_{A_i}$ respectively. So

$$\rho(G)=\frac{1}{n-1}\sum_{k=2}^{n}P[\frac{1}{\sqrt{2}}(|v^1_1\cdots v^N_1\rangle-|v^1_{r_k}\cdots v^N_{r_k}\rangle)], \tag{15}$$

it follows that

$$\begin{aligned}&\rho(G)\\ =&\frac{1}{n-1}\Big\{\sum_{i_1=2}^{n_1}P[\frac{1}{\sqrt{2}}(|v^1_1\rangle-|v^1_{i_1}\rangle)|v^2_1\cdots v^N_1\rangle]+\cdots+\sum_{i_N=2}^{n_N}P[\frac{1}{\sqrt{2}}|v^1_1\cdots v^{n-1}_1\rangle(|v^N_1\rangle-v^N_{i_N}\rangle)]\\ &+\cdots\cdots+\sum_{i_1,\cdots,i_N=2}^{n_1,\cdots,n_N}P[\frac{1}{\sqrt{2}}(|v^1_1\cdots v^N_1\rangle-|v^1_{i_1}\cdots v^N_{i_N}\rangle)]\Big\}.\end{aligned} \tag{16}$$

Consider the following projectors $P_i=|v^i_1\rangle\langle v^i_1|+|v^i_2\rangle\langle v^i_2|$, $i=1,\cdots,N$, then

$$\begin{aligned}&(P_1\otimes\cdots\otimes P_N)\rho(G)(P_1\otimes\cdots\otimes P_N)\\ =&\frac{1}{n-1}\Big\{\frac{n-2^N}{2}P[|v^1_1\cdots v^N_1\rangle]+P[\frac{1}{\sqrt{2}}(|v^1_1v^2_1\cdots v^N_1\rangle-|v^1_2v^2_1\cdots v^N_1\rangle)]\\ &+\cdots+P[\frac{1}{\sqrt{2}}(|v^1_1\cdots v^{N-1}_1v^N_1\rangle-|v^1_1\cdots v^{N-1}_1v^N_2\rangle)]\\ &+\cdots\cdots+P[\frac{1}{\sqrt{2}}(|v^1_1\cdots v^N_1\rangle-|v^1_2\cdots v^N_2\rangle)]\Big\}.\end{aligned} \tag{17}$$

So in the basis$|v^1_1\cdots v^N_1\rangle,|v^1_2\cdots v^N_1\rangle,\cdots,|v^1_2\cdots v^N_2\rangle$, we get a $2^N\times 2^N$ matrix

$$(P_1\otimes\cdots\otimes P_N)\rho(G)(P_1\otimes\cdots\otimes P_N)=\frac{1}{n-1}\begin{pmatrix}\frac{n-1}{2}&-\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}&\cdots&-\frac{1}{2}\\ -\frac{1}{2}&\frac{1}{2}&0&0&\cdots&0\\ -\frac{1}{2}&0&\frac{1}{2}&0&\cdots&0\\ -\frac{1}{2}&0&0&\frac{1}{2}&\cdots&0\\ \vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\ -\frac{1}{2}&0&0&0&\cdots&\frac{1}{2}\end{pmatrix}. \tag{18}$$

The eigenvalues of the partially transposition of $(P_1\otimes\cdots\otimes P_N)\rho(G)(P_1\otimes\cdots\otimes P_N)$ are $\frac{1}{2(n-1)}$ (with multiplicity 2^N-3), $\frac{1}{(n-1)}$ and the roots of the polynomial $(n-1)^2\lambda^2-\frac{(n-1)^2}{2}\lambda-\frac{1}{2}$. Here the matrix has a negative eigenvalue, therefore $\rho(G)$ is N-partite entangled. □

Next we give an example about 4-partite entangled states.

Example 14. For a star graph $G = K_{1,\ n-1}$ on $n = n_1 n_2 n_3 n_4$ vertices, the density matrix $\rho(G)$ is 4-partite entangled. From theorem 13, we can get that the partially transposition of $(P_1 \otimes P_2 \otimes P_3 \otimes P_4)\rho(G)(P_1 \otimes P_2 \otimes P_3 \otimes P_4)$ has negative eigenvalue, so $\rho(G)$ is 4-partite entangled.

Now, let us consider the nearest point graph.

Theorem 15. *Let G be a nearest point graph on $n = n_1 n_2 n_3 n_4$ vertices, then the density matrix $\rho(G)$ is 4-separable in $C_A^m \otimes C_B^p \otimes C_C^q \otimes C_D^q$ if and only if it satisfies the degree condition $\Delta(G) = \Delta(G^{\Gamma_A}) = \Delta(G^{\Gamma_B}) = \Delta(G^{\Gamma_C}) = \Delta(G^{\Gamma_D})$.*

Proof. We first prove that the condition is sufficient. Let $\rho(G)$ be the density matrix of a graph on $n = n_1 n_2 n_3 n_4$ vertices and $\Delta(G) = \Delta(G^{\Gamma_A}) = \Delta(G^{\Gamma_B}) = \Delta(G^{\Gamma_C}) = \Delta(G^{\Gamma_D})$ as the degree condition. The degree condition is equivalent to PPT criterion [16]. If $\rho(G)$ is separable in $C_A^{n_1} \otimes C_B^{n_2} \otimes C_C^{n_3} \otimes C_D^{n_4}$, then $\Delta(G) = \Delta(G^{\Gamma_A}) = \Delta(G^{\Gamma_B}) = \Delta(G^{\Gamma_C}) = \Delta(G^{\Gamma_D})$.

In the following, we prove that for the nearest graphs, the condition is also necessary. Let G be a nearest point graph on $n = n_1 n_2 n_3 n_4$ vertices and f edges. We associate to G the orthonormal basis $\{|\alpha_l\rangle : l = 1,\ 2,\ \cdots,\ n\} = \{|u_i\rangle \otimes |v_j\rangle \otimes |w_k\rangle \otimes |\varphi_p\rangle : i = 1,\ \cdots,\ n_1;\ j = 1, \cdots,\ n_2;\ k = 1,\ \cdots,\ n_3;\ p = 1,\ \cdots,\ n_4\}$, where $|u_i\rangle$, $|v_j\rangle$, $|w_k\rangle$ and $|\varphi_p\rangle$ are the orthonormal basis of $C_A^{n_1}$, $C_B^{n_2}$, $C_C^{n_3}$, $C_D^{n_4}$ respectively. Let $i,\ r \in \{1,\ \cdots,\ n_1\}$, $j,\ s \in \{1,\ \cdots,\ n_2\}$, $k,\ t \in \{1,\ \cdots,\ n_3\}$, $p,\ q \in \{1,\ \cdots,\ n_4\}$. $\lambda_{ijkp,\ rstq} \in \{0,\ 1\}$ is defined as

$$\lambda_{ijkp,\ rstq} = \begin{cases} 1, & \text{if } (u_i v_j w_k \varphi_p,\ u_r v_s w_t \varphi_q) \in E(G); \\ 0, & \text{if } (u_i v_j w_k \varphi_p,\ u_r v_s w_t \varphi_q) \notin E(G), \end{cases} \tag{19}$$

where $i,\ j,\ k,\ r,\ s,\ t$ satisfy the following 14 conditions:

- $i = r,\ j = s,\ k = t,\ p = q + 1;\quad i = r,\ j = s,\ k = t + 1,\ p = q;$
- $i = r,\ j = s + 1,\ k = t,\ p = q;\quad i = r + 1,\ j = s,\ k = t,\ p = q;$
- $i = r + 1,\ j = s + 1,\ k = t,\ p = q;\quad i = r + 1,\ j = s,\ k = t + 1,\ p = q;$
- $i = r + 1,\ j = s,\ k = t,\ p = q + 1;\quad i = r,\ j = s + 1,\ k = t + 1,\ p = q;$
- $i = r,\ j = s + 1,\ k = t,\ p = q + 1;\quad i = r,\ j = s,\ k = t + 1,\ p = q + 1;$
- $i = r + 1,\ j = s + 1,\ k = t + 1,\ p = q;\quad i = r + 1,\ j = s + 1,\ k = t,\ p = q + 1;$
- $i = r + 1,\ j = s,\ k = t + 1,\ p = q + 1;\quad i = r,\ j = s + 1,\ k = t + 1,\ p = q + 1.$

Let $\rho(G)$, $\rho(G^{\Gamma_A})$, $\rho(G^{\Gamma_B})$, $\rho(G^{\Gamma_C})$ and $\rho(G^{\Gamma_D})$ be the density matrices corresponding to the graph G, G^{Γ_A}, G^{Γ_B}, G^{Γ_C} and G^{Γ_D} respectively. Then

$$\rho(G) = \frac{1}{2f}(\Delta(G) - M(G)), \qquad \rho(G^{\Gamma_A}) = \frac{1}{2f}(\Delta(G^{\Gamma_A}) - M(G^{\Gamma_A})),$$

$$\rho(G^{\Gamma_B}) = \frac{1}{2f}(\Delta(G^{\Gamma_B}) - M(G^{\Gamma_B})), \quad \rho(G^{\Gamma_C}) = \frac{1}{2f}(\Delta(G^{\Gamma_C}) - M(G^{\Gamma_C})),$$
$$\rho(G^{\Gamma_D}) = \frac{1}{2f}(\Delta(G^{\Gamma_D}) - M(G^{\Gamma_D})). \tag{20}$$

Let G_1 be the subgraph of G whose edges are all the entangled edges of G. An edge $\{u_iv_jw_k,\ u_rv_sw_t\}$ is entangled if $i \neq r,\ j \neq s,\ k \neq t$. Let $G_1^A,\ \ G_1^B,\ \ G_1^C,\ \ G_1^D$ be the subgraph of G^{Γ_A}, G^{Γ_B}, G^{Γ_C}, G^{Γ_D} corresponding to all the entangled edges. Obviously, $G_1^A = (G_1)^{\Gamma_A},\ G_1^B = (G_1)^{\Gamma_B},\ G_1^C = (G_1)^{\Gamma_C},\ G_1^D = (G_1)^{\Gamma_D}$. We have

$$\rho(G_1) = \frac{1}{f}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3}\sum_{P=1}^{n_4}\lambda_{ijkp,\ rstq}P[\frac{1}{\sqrt{2}}(|u_iv_jw_k\varphi_p\rangle - |u_rv_sw_t\varphi_q\rangle)], \tag{21}$$

where $i,\ j,\ k,\ p;\ r,\ s,\ t,\ q$ must satisfy the 14 conditions. We can get $\rho(G_1^A),\ \rho(G_1^B),\ \rho(G_1^C)$ and $\rho(G_1^D)$ by commuting the index of $u,\ v,\ w$ in the above equation respectively. Also we have

$$\Delta(G_1) = \frac{1}{2f}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3}\sum_{P=1}^{n_4}\lambda_{ijkp,\ rstq}P[|u_iv_jw_k\varphi_p\rangle], \tag{22}$$

where $i,\ j,\ k,\ p;\ r,\ s,\ t,\ q$ must satisfy either the 14 conditions. We can get $\Delta(G_1^A)$, $\Delta(G_1^B)$, $\Delta(G_1^C)$ and $\Delta(G_1^D)$ by commuting the index of λ . Let $G_2,\ G_2^A,\ G_2^B,\ G_2^C$ and G_2^D be the subgraph of $G,\ G^A,\ G^B$, G^C and G^D containing all the unentangled edges, respectively. It is obvious that $\Delta(G_2) = \Delta(G_2^{\Gamma_A}) = \Delta(G_2^{\Gamma_B}) = \Delta(G_2^{\Gamma_C}) = \Delta(G_2^{\Gamma_D})$. So $\Delta(G) = \Delta(G^{\Gamma_A}) = \Delta(G^{\Gamma_B}) = \Delta(G^{\Gamma_C}) = \Delta(G^{\Gamma_D})$ if and only if $\Delta(G_1) = \Delta(G_1^{\Gamma_A}) = \Delta(G_1^{\Gamma_B}) = \Delta(G_1^{\Gamma_C}) = \Delta(G_1^{\Gamma_D})$. The condition implies that

$$\lambda_{ijkp,\ rstq} = \lambda_{rjkp,\ istq} = \lambda_{iskp,\ rjtq} = \lambda_{ijtp,\ rskq} = \lambda_{ijkq,\ rstp}, \tag{23}$$

for $i,\ r \in \{1,\ 2,\ \cdots,\ n_1\},\ j,\ s \in \{1,\ 2,\ \cdots,\ n_2\},\ \ k,\ t \in \{1,\ 2,\ \cdots,\ n_3\ \ p,\ q \in \{1,\ 2,\ \cdots,\ n_4\}$. The above equations show that whenever there is an entangled edge $\{u_iv_jw_k\varphi_p,\ u_rv_sw_t\varphi_q\}$ in G (here $i \neq r,\ j \neq s,\ k \neq t,\ p \neq q$), there must be the entangled edges $\{u_rv_jw_k\varphi_p,\ u_iv_sw_t\varphi_q\}$, $\{u_iv_sw_k\varphi_p,\ u_rv_jw_t\varphi_q\}$, $\{u_iv_jw_t\varphi_p,\ u_rv_sw_k\varphi_q\}$ and $\{u_iv_jw_k\varphi_q,\ u_rv_sw_t\}\varphi_p$ in G. Let

$$\begin{aligned}
&\rho(i,\ j,\ k,\ p;\ r,\ s,\ t,\ q)\\
=\ &\frac{1}{4}(P[\frac{1}{\sqrt{2}}(|u_iv_jw_k\varphi_p\rangle - |u_rv_sw_t\varphi_q\rangle)] + P[\frac{1}{\sqrt{2}}(|u_rv_jw_k\varphi_p\rangle - |u_iv_sw_t\varphi_q\rangle)]\\
&+P[\frac{1}{\sqrt{2}}(|u_iv_sw_k\varphi_p\rangle - |u_rv_jw_t\varphi_q\rangle)] + P[\frac{1}{\sqrt{2}}(|u_iv_jw_t\varphi_p\rangle - |u_rv_sw_k\varphi_q\rangle)])\\
&+P[\frac{1}{\sqrt{2}}(|u_iv_jw_k\varphi_q\rangle - |u_rv_sw_t\varphi_p\rangle)]).
\end{aligned} \tag{24}$$

By Lemma 11, we know that $\rho(i,\ j,\ k,\ p;\ r,\ s,\ t,\ q)$ is 4-separable. By using Theorem 3 in Ref. [6] we can easily get $\rho(G_2)$ is 4-separable. Therefore the density matrix $\rho(G)$ is 4-separable in $C_A^m \otimes C_B^p \otimes C_C^q \otimes C_D^q$ □

Similarly, the result can be generalized to multipartite systems.

Theorem 16. *Let G be a nearest point graph on $n = n_1 n_2 \cdots n_N$ vertices. The density matrix $\rho(G)$ is N-separable in $C^m_{A_1} \otimes C^p_{A_2} \otimes \cdots \otimes C^q_{A_N}$ if and only if* $\Delta(G) = \Delta(G^{\Gamma_{A_1}}) = \Delta(G^{\Gamma_{A_2}}) = \cdots = \Delta(G^{\Gamma_{A_N}})$.

5 Comments and Conclusions

We have studied the separability of Laplacian matrices of graphs. Using the matrix realignment, a entanglement criterion for graphs has been presented. We have proved that if the density matrix ρ is the tensor product of N graphs, then it is N-separable. But if a density matrix is 4-separable, it can not necessarily be written as tensor product of four graphs. Furthermore, we studied the entanglement of star graphs and nearest point graphs. We proved that the star graph $\rho(K_{1,\ n-1})$ is N-partite entangled for $n = n_1 n_2 \cdots n_N \geqslant 2^N$. And the density matrix $\rho(G)$ of nearest point graph is 4-separable if and only if it satisfies the degree condition. These results are useful to better understand the physical characteristics and mathematical structures of entangled states.

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