A basis for the diagonally signed-symmetric polynomials

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Abstract

Let $n \geqslant 1$ be an integer and let B_n denote the hyperoctahedral group of rank n. The group B_n acts on the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by signed permutations simultaneously on both of the sets of variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$. The invariant ring $M^{B_n} := \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]^{B_n}$ is the ring of diagonally signed-symmetric polynomials. In this article, we provide an explicit free basis of M^{B_n} as a module over the ring of symmetric polynomials on both of the sets of variables $x_1^2, \ldots, x_n^2$ and $y_1^2, \ldots, y_n^2$ using signed descent monomials.

Keywords: hyperoctahedral group, symmetric polynomials.

1 Introduction

Let V be an n-dimensional vector space over a field k of characteristic zero. Suppose that W is a finite reflection group in V; that is, W is finite subgroup of $GL(V)$ generated by elements of order 2 that fix a hyperplane pointwise. Then W acts by ring automorphisms on the symmetric algebra $S(V^*)$, where V^* is the dual of V. If we give V a basis, then $S(V^*)$ can be identified with a polynomial ring $k[\mathbf{x}] := k[x_1, \ldots, x_n]$. The action of the group W on the polynomial ring $k[\mathbf{x}]$, under the above identification, has been classically studied. For example, by [7, Theorem A] the ring $k[\mathbf{x}]^W$ consisting of all W-invariant polynomials is itself a polynomial ring on n homogeneous generators. Consider now the diagonal action of W on the symmetric algebra $S(V^* \oplus V^*)$. If we give V a basis as before, then $S(V^* \oplus V^*)$ can be identified with a polynomial algebra $k[\mathbf{x}, \mathbf{y}] := k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and W acts diagonally on it. In this case, the ring $M^W := k[\mathbf{x}, \mathbf{y}]^W$ consisting of all diagonally W-invariant polynomials is no longer a polynomial algebra.

The ring $R^W := k[\mathbf{x}]^W \otimes k[\mathbf{y}]^W$ of all polynomials that are W-invariant in both of the sets of variables $\mathbf{x}$ and $\mathbf{y}$ is naturally a subring of M^W. Therefore we can see M^W as a module over R^W. It can be seen that in fact M^W is a free module over R^W of rank $|W|$. This relies on the fact that M^W is a Cohen-Macaulay ring which is true by [9, Proposition 13]. This article is concerned with the determination of explicit free bases of M^W as a module over R^W for a particular class of groups using elementary methods. For simplicity, we work with rational coefficients although all the constructions provided here work for any field of characteristic zero.

In [5], Allen provided an explicit basis for the case of the symmetric group. More precisely, suppose that $W = \Sigma_n$ acts on the polynomial algebra $\mathbb{Q}[\mathbf{x}] = \mathbb{Q}[x_1, \ldots, x_n]$ by permutations of the variables $x_1, \ldots, x_n$. In this case the invariant ring $\mathbb{Q}[\mathbf{x}]^{\Sigma_n}$ is the ring of symmetric polynomials. This ring is a polynomial algebra on the elementary symmetric polynomials. Let Σ_n act diagonally on $\mathbb{Q}[\mathbf{x}, \mathbf{y}] := \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Then $M^{\Sigma_n} = \mathbb{Q}[\mathbf{x}, \mathbf{y}]^{\Sigma_n}$ is the ring of diagonally symmetric or multisymmetric polynomials. Given $\pi \in \Sigma_n$ define the diagonal descent monomial

$$e_\pi := \prod_{i \in Des(\pi^{-1})} (x_1 \cdots x_i) \prod_{j \in Des(\pi)} (y_{\pi(1)} \cdots y_{\pi(j)}) = \prod_{i=1}^{n} x_i^{d_i(\pi^{-1})} y_i^{d_{\pi^{-1}(i)}(\pi)},$$

where $Des(\pi)$ denotes the descent set of π, and $d_i(\pi^{-1})$ and $d_{\pi^1(i)}(\pi)$ are integers (see Section 2 for the definitions). By [5, Theorem 1.3] the collection $\{\rho_{\Sigma_n}(e_\pi)\}_{\pi \in \Sigma_n}$ forms a free basis of M^{Σ_n} as a module over $R^{\Sigma_n} = \mathbb{Q}[\mathbf{x}]^{\Sigma_n} \otimes \mathbb{Q}[\mathbf{y}]^{\Sigma_n}$, where ρ_{Σ_n} is the averaging operator defined below.

The goal of this article is to show that an analogous construction works for the hyperoctahedral group B_n acting on the polynomial algebra $\mathbb{Q}[\mathbf{x}] = \mathbb{Q}[x_1, \ldots, x_n]$ by signed permutations. In this case, the invariant ring $\mathbb{Q}[\mathbf{x}]^{B_n}$ consists of all symmetric polynomials on the variables $x_1^2, \ldots, x_n^2$. Suppose that B_n acts diagonally on the polynomial ring $\mathbb{Q}[\mathbf{x}, \mathbf{y}] = \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by signed permutations. Then the invariant ring $M^{B_n} = \mathbb{Q}[\mathbf{x}, \mathbf{y}]^{B_n}$ is the ring of diagonally signed-symmetric polynomials. A free basis of it as a module over $R^{B_n} = \mathbb{Q}[\mathbf{x}]^{B_n} \otimes \mathbb{Q}[\mathbf{y}]^{B_n}$ can be constructed in the same spirit as in the case of permutations. Given $\sigma \in B_n$, define the diagonal signed descent monomial

$$c_\sigma := \left(\prod_{i=1}^{n} x_i^{f_i(\sigma^{-1})} \right) \left(\prod_{i=1}^{n} y_{|\sigma(i)|}^{f_i(\sigma)}, \right).$$

See Section 3 for the definition of the numbers $f_i(\sigma)$. The goal of this article is to prove the following theorem.

Theorem 1. *Suppose that $n \geqslant 1$. Then the collection $\{\rho(c_\sigma)\}_{\sigma \in B_n}$ forms a free basis of M^{B_n} as a module over R^{B_n}, where ρ is the averaging operator.*

A similar basis to the one given in the previous theorem was constructed in [6]. Moreover, in there a nice combinatorial interpretation of the basis monomials was provided in terms of certain diagrams of the square lattice. The author would like to thank F. Bergeron and R. Biagioli for pointing out their work to him.

2 The symmetric group

In this section we provide a brief review of an explicit basis for the coinvariant ring for groups of type A. This basis was constructed by Garsia and Stanton in [8] using descent monomials. A construction of a free basis for the ring of diagonally symmetric polynomials as a module over the symmetric polynomials, constructed by Allen in [5], is also reviewed.

2.1 Major index

For every integer $n \geqslant 1$, let Σ_n denote the symmetric group of self bijections of the set $\{1, 2, \ldots, n\}$. We use the notation $\pi = [\pi_1, \ldots, \pi_n]$ for an element $\pi \in \Sigma_n$ with $\pi_i = \pi(i)$ for $1 \leqslant i \leqslant n$. Given $\pi \in \Sigma_n$ define its descent to be the set

$$Des(\pi) := \{1 \leqslant i \leqslant n-1 \mid \pi(i) > \pi(i+1)\}.$$

Also, for any $1 \leqslant i \leqslant n$ let

$$d_i(\pi) := |\{j \in Des(\pi) \mid j \geqslant i\}|.$$

The numbers $d_i(\pi)$ satisfy the following properties:

$$d_1(\pi) \geqslant d_2(\pi) \geqslant \cdots \geqslant d_{n-1}(\pi) \geqslant d_n(\pi) = 0, \text{ and} \tag{1}$$

$$\text{if } i < j \text{ and } d_i(\pi) = d_j(\pi), \text{ then } \pi(i) < \pi(i+1) < \cdots < \pi(j). \tag{2}$$

The major index of $\pi \in \Sigma_n$, denoted by $\mathrm{maj}(\pi)$, is defined to be

$$\mathrm{maj}(\pi) := \sum_{i=1}^{n} d_i(\pi) = \sum_{i \in Des(\pi)} i.$$

Example 2. Suppose that π is the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 1 & 4 & 3 & 5 \end{pmatrix}.$$

In our notation this is the permutation $\pi = [6, 2, 1, 4, 3, 5]$. The descent set of this permutation is $Des(\pi) = \{1, 2, 4\}$ and

$$d_1(\pi) = 3, d_2(\pi) = 2, d_3(\pi) = d_4(\pi) = 1, \text{ and } d_5(\pi) = d_6(\pi) = 0.$$

Also, the major index of π is $\mathrm{maj}(\pi) = 3 + 2 + 1 + 1 + 0 + 0 = 7$.

In [10], MacMahon showed that the major index is equidistributed with respect to the length function. This means that the number of permutations of length n with k inversions is the same as the number of permutations of length n with major index equal to k. The numbers $d_1(\pi) \geqslant d_2(\pi) \geqslant \cdots \geqslant d_{n-1}(\pi)$ are defined exactly to provide a partition of the integer $\mathrm{maj}(\pi)$.

2.2 Descent monomials

Suppose that $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a set of algebraically independent commuting variables. Consider the polynomial algebra $\mathbb{Q}[\mathbf{x}] := \mathbb{Q}[x_1, \ldots, x_n]$ seen as a graded ring with $\deg(x_i) = 1$ for $1 \leqslant i \leqslant n$. The group Σ_n acts naturally on $\mathbb{Q}[\mathbf{x}]$ by permuting the variables $x_1, \ldots, x_n$. The invariant ring $\mathbb{Q}[\mathbf{x}]^{\Sigma_n}$ consists of all symmetric polynomials on the variables $x_1, \ldots, x_n$. This ring is a polynomial algebra on the generators $e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n)$, where $e_k(x_1, \ldots, x_n)$ is the k-th elementary symmetric polynomial

$$e_k(x_1, \ldots, x_n) := \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Suppose that $\pi \in \Sigma_n$. Define the descent monomial associated to π to be

$$a_\pi := \prod_{i \in Des(\pi)} x_{\pi(1)} \cdots x_{\pi(i)} = \prod_{i=1}^{n} x_{\pi(i)}^{d_i(\pi)} = \prod_{i=1}^{n} x_i^{d_{\pi^{-1}(i)}(\pi)}. \tag{3}$$

It follows that for any $\pi \in \Sigma_n$ we have

$$\deg(a_\pi) = d_1(\pi) + d_2(\pi) + \cdots + d_n(\pi) = \mathrm{maj}(\pi).$$

Example 3. Suppose that $\pi = [6, 2, 1, 4, 3, 5]$. Then, as in Example 2, the descent of π is the set $Des(\pi) = \{1, 2, 4\}$ and the corresponding descent monomial is the monomial

$$a_\pi = (x_{\pi(1)})(x_{\pi(1)} x_{\pi(2)})(x_{\pi(1)} x_{\pi(2)} x_{\pi(3)} x_{\pi(4)}) = x_1 x_2^2 x_4 x_6^3.$$

In [8], Garsia and Stanton used Stanley–Reisner rings to show that descent monomials provide a basis for the coinvariant algebra of type A. More precisely, let I_n^A denote the ideal in $\mathbb{Q}[\mathbf{x}]$ generated by the symmetric polynomials $e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n)$. Then $\mathbb{Q}[\mathbf{x}]/I_n^A$ is the coinvariant algebra of type A. Let $\bar{a}_\pi$ denote the image of a_π in the coinvariant algebra under the natural map. In [8], it was proved that the collection $\{\bar{a}_\pi\}_{\pi \in \Sigma_n}$ forms a basis of $\mathbb{Q}[\mathbf{x}]/I_n^A$ as a $\mathbb{Q}$-vector space. Moreover, the collection $\{a_\pi\}_{\pi \in \Sigma_n}$ provides a free basis for $\mathbb{Q}[\mathbf{x}]$ as a module over the symmetric polynomials $\mathbb{Q}[\mathbf{x}]^{\Sigma_n}$. This result has an interesting geometric application. Consider the flag manifold $U(n)/T$, where $T \subset U(n)$ is a maximal torus. Then $H^*(U(n)/T; \mathbb{Q})$ can be identified with the invariant algebra $\mathbb{Q}[\mathbf{x}]/I_n^A$, but under this identification the variables $x_1, \ldots, x_n$ are graded with degree 2. This shows that descent monomials provide an explicit basis for the cohomology of the flag manifold $U(n)/T$.

2.3 Diagonal descent monomials

Let $\mathbf{y} = \{y_1, \ldots, y_n\}$ be another set of algebraically independent commuting variables of degree 1 and consider the polynomial algebra $\mathbb{Q}[\mathbf{x}, \mathbf{y}]$. The symmetric group Σ_n acts diagonally on this polynomial ring and the ring of Σ_n-invariants, $M^{\Sigma_n} := \mathbb{Q}[\mathbf{x}, \mathbf{y}]^{\Sigma_n}$, is known as the ring of diagonally symmetric or multisymmetric polynomials. The ring of polynomials that are symmetric in both the variables $\mathbf{x}$ and $\mathbf{y}$, $R^{\Sigma_n} := \mathbb{Q}[\mathbf{x}]^{\Sigma_n} \otimes \mathbb{Q}[\mathbf{y}]^{\Sigma_n}$, is a subring of M^{Σ_n}. Therefore M^{Σ_n} can be seen as a module over R^{Σ_n}. In [5], Allen constructed a free basis for the module M^{Σ_n} using a variation of the descent monomials which is described next. For any $\pi \in \Sigma_n$ define the diagonal descent monomial

$$e_\pi := \left(\prod_{i=1}^{n} x_i^{d_i(\pi^{-1})}\right)\left(\prod_{i=1}^{n} y_{\pi(i)}^{d_i(\pi)}\right) = \prod_{i=1}^{n} x_i^{d_i(\pi^{-1})} y_i^{d_{\pi^{-1}(i)}(\pi)}. \tag{4}$$

Example 4. Suppose that $\pi = [6, 2, 1, 4, 3, 5]$. Then $\pi^{-1} = [3, 2, 5, 4, 6, 1]$, $Des(\pi) = \{1, 2, 4\}$, $Des(\pi^{-1}) = \{1, 3, 5\}$ and we have

$$\begin{aligned} &d_{\pi^{-1}(1)}(\pi) = d_3(\pi) = 1, d_{\pi^{-1}(2)}(\pi) = d_2(\pi) = 2, d_{\pi^{-1}(3)}(\pi) = d_5(\pi) = 0, \\ &d_{\pi^{-1}(4)}(\pi) = d_4(\pi) = 1, d_{\pi^{-1}(5)}(\pi) = d_6(\pi) = 0 \text{ and } d_{\pi^{-1}(6)}(\pi) = d_1(\pi) = 3. \end{aligned}$$

Also

$$d_1(\pi^{-1}) = 3, d_2(\pi^{-1}) = d_3(\pi^{-1}) = 2, d_4(\pi^{-1}) = d_5(\pi^{-1}) = 1 \text{ and } d_6(\pi) = 0.$$

Therefore the diagonal descent monomial associated to π is the monomial

$$e_\pi = x_1^3 x_2^2 x_3^2 x_4 x_5 y_1 y_2^2 y_4 y_6^3.$$

In an analogous way as above, for any $\pi \in \Sigma_n$ the total degree of e_π is given by $\deg(e_\pi) = \mathrm{maj}(\pi) + \mathrm{maj}(\pi^{-1})$. On the other hand, consider the averaging operator

$$\begin{aligned} \rho_{\Sigma_n} : \mathbb{Q}[\mathbf{x}, \mathbf{y}] &\to \mathbb{Q}[\mathbf{x}, \mathbf{y}]^{\Sigma_n} = M^{\Sigma_n} \\ f &\mapsto \frac{1}{n!} \sum_{\pi \in \Sigma_n} \pi \cdot f. \end{aligned}$$

Thus, by definition, $\rho_{\Sigma_n}(e_\pi)$ is a diagonally symmetric polynomial. By [5, Theorem 1.3], the collection $\{\rho_{\Sigma_n}(e_\pi)\}_{\pi \in \Sigma_n}$ forms a free basis of M^{Σ_n} as a module over R^{Σ_n}. It turns out that this result also has an interesting geometric application. Let $B_{com}U(n)$ be the geometric realization of the simplicial space obtained by defining $[B_{com}U(n)]_k := \mathrm{Hom}(\mathbb{Z}^k, U(n))$, where $\mathrm{Hom}(\mathbb{Z}^k, U(n))$ is the space of ordered commuting k-tuples in $U(n)$. The space $B_{com}U(n)$ is the classifying space for commutativity in the group $U(n)$. In [1, Section 8], it is proved that the diagonal descent monomials can be used to obtain an explicit basis of $H^*(B_{com}U(n); \mathbb{Q})$ seen as a module over $H^*(BU(n); \mathbb{Q})$, where $BU(n)$ is the classifying space of $U(n)$.

3 The hyperoctahedral group

In this section we provide analogue constructions to the ones presented in the previous section where the symmetric group is replaced by the group of signed permutations.

3.1 Flag major index

Suppose that $n \geqslant 1$ is an integer. Denote by $\mathbb{I}_n$ the set of integers between $-n$ and n not including 0; that is,

$$\mathbb{I}_n := \{-n, -n+1, \ldots, -1, 1, \ldots, n-1, , n\}.$$

Let B_n denote the group of bijections $\sigma : \mathbb{I}_n \to \mathbb{I}_n$ such that $\sigma(-k) = -\sigma(k)$ for all $k \in \mathbb{I}_n$, with the composition of functions as the group operation. Thus, the group B_n is the group of signed permutations, also known as the hyperoctahedral group of rank n. It is easy to see that B_n is isomorphic to the semidirect product $\Sigma_n \ltimes (\mathbb{Z}/2)^n$. We use the following notation for elements $\sigma \in B_n$. Let $\sigma_i = \sigma(i)$ for $1 \leqslant i \leqslant n$, then we write $\sigma = [\sigma_1, \ldots, \sigma_n]$. The condition $\sigma(-k) = -\sigma(k)$ for all $k \in \mathbb{I}_n$ implies that the element $\sigma \in B_n$ is uniquely determined by the numbers $\sigma_1, \ldots, \sigma_n$. The group B_n is the Weyl group associated to Lie groups of type B_n and C_n and the symmetric group Σ_n is naturally a subgroup of B_n. As in the case of the symmetric group, given $\sigma \in B_n$ define its descent to be the set

$$Des(\sigma) := \{1 \leqslant i \leqslant n-1 \mid \sigma(i) > \sigma(i+1)\}$$

and for $1 \leqslant i \leqslant n$ let

$$d_i(\sigma) := |\{j \in Des(\sigma) \mid j \geqslant i\}|.$$

As before the numbers $d_i(\sigma)$ satisfy the following important properties:

$$d_1(\sigma) \geqslant d_2(\sigma) \geqslant \cdots \geqslant d_{n-1}(\sigma) \geqslant d_n(\sigma) = 0, \text{ and} \tag{5}$$

$$\text{if } i < j \text{ and } d_i(\sigma) = d_j(\sigma), \text{ then } \sigma(i) < \sigma(i+1) < \cdots < \sigma(j). \tag{6}$$

On the other hand, define

$$\varepsilon_i(\sigma) := \begin{cases} 0 & \text{if} \quad \sigma(i) > 0, \\ 1 & \text{if} \quad \sigma(i) < 0, \end{cases}$$

and

$$f_i(\sigma) := 2d_i(\sigma) + \varepsilon_i(\sigma).$$

The numbers $f_i(\sigma)$ also satisfy the properties:

$$f_1(\sigma) \geqslant f_2(\sigma) \geqslant \cdots \geqslant f_n(\sigma), \text{ and} \tag{7}$$

if $i < j$ and $f_i(\sigma) = f_j(\sigma)$, then $\sigma(i) < \sigma(i+1) < \cdots < \sigma(j)$ and all of these numbers have the same sign. (8)

The flag major index of $\sigma \in B_n$, denoted by $\operatorname{fmaj}(\sigma)$, is defined to be

$$\operatorname{fmaj}(\sigma) := \sum_{i=1}^{n} f_i(\sigma) = 2\operatorname{maj}(\sigma) + \operatorname{neg}(\sigma),$$

where $\operatorname{maj}(\sigma) = \sum_{i \in Des(\sigma)} i$ is the major index of σ and $\operatorname{neg}(\sigma) = |\{1 \leqslant i \leqslant n \mid \sigma(i) < 0\}|$.

Example 5. Consider the signed permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ -6 & 2 & -1 & -4 & 3 & 5 \end{pmatrix}.$$

In our notation this is $\sigma = [-6, 2, -1, -4, 3, 5]$. The Descent of σ is the set $Des(\sigma) = \{2, 3\}$. Therefore

$$d_1(\sigma) = d_2(\sigma) = 2, d_3(\sigma) = 1 \text{ and } d_4(\sigma) = d_5(\sigma) = d_6(\sigma) = 0,$$

and

$$\varepsilon_1(\sigma) = \varepsilon_3(\sigma) = \varepsilon_4(\sigma) = 1 \text{ and } \varepsilon_2(\sigma) = \varepsilon_5(\sigma) = \varepsilon_6(\sigma) = 0,$$

We conclude that

$$f_1(\sigma) = 5, f_2(\sigma) = 4, f_3(\sigma) = 3, f_4(\sigma) = 1 \text{ and } f_5(\sigma) = f_6(\sigma) = 0$$

and $\operatorname{fmaj}(\sigma) = 5 + 4 + 3 + 1 + 0 + 0 = 13$.

The flag major index was introduced in [4] and further studied in [2] and [3]. This statistic was introduced as a generalization of the major index for the hyperoctahedral group. This tool has successfully been used to study representation theoretical properties of the group B_n (see for example [3]). The numbers $f_1(\sigma) \geqslant \cdots \geqslant f_n(\sigma)$ are defined so that they provide a partition of the flag major index of σ. Moreover, if $\sigma \in \Sigma_n$ then $\operatorname{fmaj}(\sigma) = 2\operatorname{maj}(\sigma)$ so the flag major index is indeed a natural generalization of the major index.

3.2 Signed descent monomials

Suppose that $\mathbf{x} = \{x_1, \dots, x_n\}$ is a set of algebraically independent commuting variables. Consider the polynomial algebra $\mathbb{Q}[\mathbf{x}]$ seen as a graded ring with $\deg(x_i) = 1$ for $1 \leqslant i \leqslant n$. The group B_n acts naturally on the polynomial algebra $\mathbb{Q}[\mathbf{x}]$ by degree preserving ring homomorphisms in the following way. If $\sigma \in B_n$, then

$$\sigma \cdot (x_1^{p_1} \cdots x_n^{p_n}) := \left(\frac{\sigma(1)}{|\sigma(1)|}\right)^{p_1} \cdots \left(\frac{\sigma(n)}{|\sigma(n)|}\right)^{p_n} x_{|\sigma(1)|}^{p_1} \cdots x_{|\sigma(n)|}^{p_n}.$$

In other words, each σ permutes the variables $x_1, \dots, x_n$ with a suitable sign change. The ring of B_n-invariants, $\mathbb{Q}[\mathbf{x}]^{B_n}$, consists of the symmetric polynomials in the variables $x_1^2, \dots, x_n^2$. It follows that $\mathbb{Q}[\mathbf{x}]^{B_n}$ is a polynomial algebra on the symmetric polynomials $e_1(x_1^2, \dots, x_n^2), \dots, e_n(x_1^2, \dots, x_n^2)$. Suppose that $\sigma \in B_n$. Define the signed descent monomial to be

$$b_\sigma := \prod_{i=1}^{n} x_{|\sigma(i)|}^{f_i(\sigma)} = \prod_{i=1}^{n} x_i^{f_{|\sigma^{-1}(i)|}(\sigma)}. \tag{9}$$

Example 6. Let $\sigma = [-6, 2, -1, -4, 3, 5]$. In this case $\sigma^{-1} = [-3, 2, 5, -4, 6, -1]$ and from Example 5 we conclude that

$$f_{|\sigma^{-1}(1)|}(\sigma) = f_3(\sigma) = 3, f_{|\sigma^{-1}(2)|}(\sigma) = f_2(\sigma) = 4, f_{|\sigma^{-1}(3)|}(\sigma) = f_5(\sigma) = 0,$$
$$f_{|\sigma^{-1}(4)|}(\sigma) = f_4(\sigma) = 1, f_{|\sigma^{-1}(5)|}(\sigma) = f_6(\sigma) = 0 \text{ and } f_{|\sigma^{-1}(6)|}(\sigma) = f_1(\sigma) = 5.$$

Therefore the corresponding signed descent monomial is

$$b_\sigma = x_1^3 x_2^4 x_4 x_6^5.$$

We observe that $\deg(b_\sigma) = \text{fmaj}(\sigma)$ for every $\sigma \in B_n$. Signed descent monomials can be used to obtain a basis for the coinvariant algebra for groups of type B, C as follows. Let I_n^B denote the ideal in $\mathbb{Q}[\mathbf{x}]$ generated by the elements $e_1(x_1^2, \ldots, x_n^2), \ldots, e_n(x_1^2, \ldots, x_n^2)$. Then the quotient $\mathbb{Q}[\mathbf{x}]/I_n^B(\mathbf{x})$ is the coinvariant algebra in this case. Let $\bar{b}_\sigma$ denote the image of b_σ in the coinvariant algebra under the natural map. By [3, Corollary 5.3], the collection $\{\bar{b}_\sigma\}_{\sigma \in B_n}$ forms a basis of $\mathbb{Q}[\mathbf{x}]/I_n^B(\mathbf{x})$ as a $\mathbb{Q}$-vector space. We can also see $\mathbb{Q}[x]$ as a module over the invariant ring $\mathbb{Q}[\mathbf{x}]^{B_n}$. As $\{\bar{b}_\sigma\}_{\sigma \in B_n}$ forms a basis of $\mathbb{Q}[\mathbf{x}]/I_n^B(\mathbf{x})$ as a $\mathbb{Q}$-vector space, then using [5, Theorem 1.2] it can be seen that $\{b_\sigma\}_{\sigma \in B_n}$ forms a free basis of $\mathbb{Q}[\mathbf{x}]$ as a module over $\mathbb{Q}[\mathbf{x}]^{B_n}$. This result has a geometric application as in the case of the symmetric group, namely, the signed descent monomials provide an explicit basis for the rational cohomology of the flag manifold G/T, for a compact connected Lie group G of type B_n, C_n and a maximal torus $T \subset G$.

3.3 Diagonal signed descent monomials

Consider now $\mathbf{y} = \{y_1, \ldots, y_n\}$ another set of algebraically independent commuting variables of degree 1 and consider the polynomial algebra $\mathbb{Q}[\mathbf{x}, \mathbf{y}] := \mathbb{Q}[\mathbf{x}] \otimes \mathbb{Q}[\mathbf{y}] = \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. The group B_n acts diagonally on this polynomial ring; that is, B_n acts as signed permutations simultaneously on the variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$. Define $M^{B_n} := \mathbb{Q}[\mathbf{x}, \mathbf{y}]^{B_n}$. In other words, M^{B_n} is the ring of diagonally signed-symmetric polynomials. The ring of polynomials that are signed-symmetric on both the variables $\mathbf{x}$ and $\mathbf{y}$, $R^{B_n} := \mathbb{Q}[\mathbf{x}]^{B_n} \otimes \mathbb{Q}[\mathbf{y}]^{B_n}$, is a subring of M^{B_n} and thus we can see M^{B_n} as a module over R^{B_n}. As it was pointed out before, M^{B_n} is a free module over R^{B_n} and the goal of this article is to construct an explicit basis for M^{B_n} as a module over R^{B_n}. For this we will consider the following monomials.

Definition 7. Suppose that $\sigma \in B_n$. The diagonal signed descent monomial associated to σ is defined to be

$$c_\sigma := \left(\prod_{i=1}^n x_i^{f_i(\sigma^{-1})}\right)\left(\prod_{i=1}^n y_{|\sigma(i)|}^{f_i(\sigma)}\right) = \prod_{i=1}^n x_i^{f_i(\sigma^{-1})} y_i^{f_{|\sigma^{-1}(i)|}(\sigma)}. \tag{10}$$

We observe that for any $\sigma \in B_n$ we have $\deg(c_\sigma) = \text{fmaj}(\sigma) + \text{fmaj}(\sigma^{-1})$.

Example 8. Suppose that $\sigma = [-6, 2, -1, -4, 3, 5]$. Then $\sigma^{-1} = [-3, 2, 5, -4, 6, -1]$ and thus $Des(\sigma^{-1}) = \{3, 5\}$. In this case, we obtain

$$f_1(\sigma^{-1}) = 5, f_2(\sigma^{-1}) = f_3(\sigma^{-1}) = 4, f_4(\sigma^{-1}) = 3, f_5(\sigma^{-1}) = 2 \text{ and } f_6(\sigma^{-1}) = 1.$$

Also, from Example 6 we have

$$f_{|\sigma^{-1}(1)|}(\sigma) = 3, f_{|\sigma^{-1}(2)|}(\sigma) = 4, f_{|\sigma^{-1}(3)|}(\sigma) = 0,$$
$$f_{|\sigma^{-1}(4)|}(\sigma) = 1, f_{|\sigma^{-1}(5)|}(\sigma) = 0 \text{ and } f_{|\sigma^{-1}(6)|}(\sigma) = 5.$$

We conclude that the corresponding diagonal signed descent monomial is

$$c_\sigma = x_1^5 x_2^4 x_3^4 x_4^3 x_5^2 x_6 y_1^3 y_2^4 y_4 y_6^5.$$

3.4 Averaging polynomials

Consider the averaging operator

$$\rho : \mathbb{Q}[\mathbf{x}, \mathbf{y}] \to \mathbb{Q}[\mathbf{x}, \mathbf{y}]^{B_n} = M^{B_n}$$
$$f \mapsto \frac{1}{|B_n|} \sum_{\sigma \in B_n} \sigma \cdot f.$$

The map ρ is a ring homomorphism that is surjective. Moreover, as a $\mathbb{Q}$-vector space M^{B_n} is generated by elements of the form $\rho(m(\mathbf{x}, \mathbf{y}))$, where $m(\mathbf{x}, \mathbf{y})$ is a monomial in $\mathbb{Q}[\mathbf{x}, \mathbf{y}]$. We will use the following notation. A sequence of non-negative integers will be denoted in the form $\mathbf{p} = (p_1, \ldots, p_n)$. Also, for such a sequence of integers we write $x^{\mathbf{p}}$ to denote the monomial $x_1^{p_1} \cdots x_n^{p_n}$.

Lemma 9. *Suppose that $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ are sequences of non-negative integers and let $m(\mathbf{x}, \mathbf{y}) = x^{\mathbf{p}} y^{\mathbf{q}}$. If $p_k + q_k$ is odd for some $1 \leqslant k \leqslant n$, then $\rho(m(\mathbf{x}, \mathbf{y})) = 0$.*

Proof. Suppose that $1 \leqslant k \leqslant n$ is such that $p_k + q_k$ is odd. Define

$$B_n^+ = \{\sigma \in B_n \mid \sigma(k) > 0\} \text{ and } B_n^- = \{\sigma \in B_n \mid \sigma(k) < 0\}.$$

Note that $B_n = B_n^+ \sqcup B_n^-$. Moreover, there is a bijection $\tau : B_n^+ \to B_n^-$ defined by

$$\tau(\sigma)(i) := \begin{cases} \sigma(i) & \text{if} \quad i \neq k, \\ -\sigma(i) & \text{if} \quad i = k. \end{cases}$$

By definition,

$$\rho(m(\mathbf{x}, \mathbf{y})) = \frac{1}{|B_n|} \sum_{\sigma \in B_n} c_\sigma x_{|\sigma(1)|}^{p_1} \cdots x_{|\sigma(n)|}^{p_n} y_{|\sigma(1)|}^{q_1} \cdots y_{|\sigma(n)|}^{q_n},$$

where

$$c_\sigma = \left(\frac{\sigma(1)}{|\sigma(1)|}\right)^{p_1+q_1} \cdots \left(\frac{\sigma(n)}{|\sigma(n)|}\right)^{p_n+q_n}.$$

For any $\sigma \in B_n^+$, we have $c_{\tau(\sigma)} = -c_\sigma$ since $p_k + q_k$ is odd. Therefore,

$$\begin{aligned}\rho(m(\mathbf{x},\mathbf{y})) &= \frac{1}{|B_n|}\sum_{\sigma\in B_n} c_\sigma x^{p_1}_{|\sigma(1)|}\cdots x^{p_n}_{|\sigma(n)|} y^{q_1}_{|\sigma(1)|}\cdots y^{q_n}_{|\sigma(n)|} \\ &= \frac{1}{|B_n|}\sum_{\sigma\in B_n^+} (c_\sigma + c_{\tau(\sigma)}) x^{p_1}_{|\sigma(1)|}\cdots x^{p_n}_{|\sigma(n)|} y^{q_1}_{|\sigma(1)|}\cdots y^{q_n}_{|\sigma(n)|} = 0.\end{aligned}$$

□

By the previous lemma, it follows that M^{B_n} is generated as a vector space over $\mathbb{Q}$ by the elements of the form $\rho(m(\mathbf{x},\mathbf{y}))$, where $m(\mathbf{x},\mathbf{y}) = x^{\mathbf{p}}y^{\mathbf{q}}$, and $\mathbf{p} = (p_1,\ldots,p_n)$ and $\mathbf{q} = (q_1,\ldots,q_n)$ are sequences of integers such that $p_k + q_k$ is even for all $1 \leqslant k \leqslant n$. Note that for any such monomial we have

$$\rho(m(\mathbf{x},y)) = \frac{1}{n!}\sum_{\alpha\in\Sigma_n} x^{p_1}_{\alpha(1)}\cdots x^{p_n}_{\alpha(n)} y^{q_1}_{\alpha(1)}\cdots y^{q_n}_{\alpha(n)}.$$

Thus, if $p_k + q_k$ is even for all $1 \leqslant k \leqslant n$, then $\rho(x^{\mathbf{p}}y^{\mathbf{q}}) \neq 0$.

Suppose now that $\sigma \in B_n$ and consider the monomial $c_\sigma = \prod_{i=1}^n x_i^{f_i(\sigma^{-1})} y_i^{f_{|\sigma^{-1}(i)|}(\sigma)}$ defined in equation (10). We claim that for every $1 \leqslant i \leqslant n$ the numbers $f_i(\sigma^{-1})$ and $f_{|\sigma^{-1}(i)|}(\sigma)$ have the same parity. To see this recall that

$$\begin{aligned} f_i(\sigma^{-1}) &= 2d_i(\sigma^{-1}) + \varepsilon_i(\sigma^{-1}), \text{ and} \\ f_{|\sigma^{-1}(i)|}(\sigma) &= 2d_{|\sigma^{-1}(i)|}(\sigma) + \varepsilon_{|\sigma^{-1}(i)|}(\sigma).\end{aligned}$$

Because of this, to show that $f_i(\sigma^{-1})$ and $f_{|\sigma^{-1}(i)|}(\sigma)$ have the same parity, it suffices to show that $\varepsilon_i(\sigma^{-1}) = \varepsilon_{|\sigma^{-1}(i)|}(\sigma)$ for any $1 \leqslant i \leqslant n$. Let $k = \sigma^{-1}(i)$ so that $\sigma(k) = i$. We consider the following two cases.

- Case 1. Suppose that $k = \sigma^{-1}(i) > 0$. Then in this case $\varepsilon_i(\sigma^{-1}) = 0$. Also, $|\sigma^{-1}(i)| = k$ and thus $\varepsilon_{|\sigma^{-1}(i)|}(\sigma) = \varepsilon_k(\sigma) = 0$ since $\sigma(k) = i > 0$.

- Case 2. Suppose that $k = \sigma^{-1}(i) < 0$. Then $\varepsilon_i(\sigma^{-1}) = 1$. On the other hand, $|\sigma^{-1}(i)| = -k$ and $\sigma(-k) = -i < 0$. Therefore $\varepsilon_{|\sigma^{-1}(i)|}(\sigma) = \varepsilon_{-k}(\sigma) = 1$.

In either case we conclude that $\varepsilon_i(\sigma^{-1}) = \varepsilon_{|\sigma^{-1}(i)|}(\sigma)$ for any $1 \leqslant i \leqslant n$. This shows that $f_i(\sigma^{-1})$ and $f_{|\sigma^{-1}(i)|}(\sigma)$ have the same parity. As a consequence, we conclude that $\rho(c_\sigma) \neq 0$ for all $\sigma \in B_n$. By definition $\rho(c_\sigma) \in M^{B_n}$. We will show below that the collection $\{\rho(c_\sigma)\}_{\sigma\in B_n}$ forms a free basis of M^{B_n} as a module over R^{B_n}.

3.5 Ordering of monomials

Our next goal is to define a total order on a subset of the set of monomials of the form $m(\mathbf{x},\mathbf{y}) = x^{\mathbf{p}}y^{\mathbf{q}}$. We will only consider monomials that are ordered in a suitable way. To motivate the ordering that we use suppose that $\sigma \in B_n$ and consider the corresponding diagonal signed descent monomial monomial c_σ. This monomial can be written in the form $c_\sigma = x^{\boldsymbol{\delta}}y^{\boldsymbol{\gamma}}$, where

$$\boldsymbol{\delta} = (f_1(\sigma^{-1}), \dots, f_n(\sigma^{-1})), \text{ and}$$
$$\boldsymbol{\gamma} = (f_{|\sigma^{-1}(1)|}(\sigma), \dots, f_{|\sigma^{-1}(n)|}(\sigma)).$$

We observe that the sequences $\boldsymbol{\delta}$ and $\boldsymbol{\gamma}$ satisfy the next key properties. From equation (7) we obtain right away

$$f_1(\sigma^{-1}) \geqslant f_2(\sigma^{-1}) \geqslant \dots \geqslant f_n(\sigma^{-1}). \tag{11}$$

Furthermore, assume $1 \leqslant i < n$ is such that $f_i(\sigma^{-1}) = f_{i+1}(\sigma^{-1})$ and these are both even numbers. This implies that $\varepsilon_i(\sigma^{-1}) = \varepsilon_{i+1}(\sigma^{-1}) = 0$ which means that $\sigma^{-1}(i) > 0$ and $\sigma^{-1}(i+1) > 0$. On the other hand, by equation (8), we see that $0 < \sigma^{-1}(i) < \sigma^{-1}(i+1)$. Using equation (7), we conclude $f_{|\sigma^{-1}(i)|}(\sigma) = f_{\sigma^{-1}(i)}(\sigma) \geqslant f_{\sigma^{-1}(i+1)}(\sigma) = f_{|\sigma^{-1}(i+1)|}(\sigma)$. Thus we obtain:

$$\text{if } f_i(\sigma^{-1}) = f_{i+1}(\sigma^{-1}) \text{ are even, then } f_{|\sigma^{-1}(i)|}(\sigma) \geqslant f_{|\sigma^{-1}(i+1)|}(\sigma). \tag{12}$$

Finally, assume that $1 \leqslant i < n$ is such that $f_i(\sigma^{-1}) = f_{i+1}(\sigma^{-1})$ and these are both odd. Then, in this case, we conclude that $\varepsilon_i(\sigma^{-1}) = \varepsilon_{i+1}(\sigma^{-1}) = 1$. This implies $\sigma^{-1}(i) < 0$ and $\sigma^{-1}(i+1) < 0$. Using equation (8), we see that $\sigma^{-1}(i) < \sigma^{-1}(i+1) < 0$ and thus $-\sigma^{-1}(i) > -\sigma^{-1}(i+1) > 0$. Equation (7) implies then that $f_{|\sigma^{-1}(i)|}(\sigma) = f_{-\sigma^{-1}(i)}(\sigma) \leqslant f_{-\sigma^{-1}(i+1)}(\sigma) = f_{|\sigma^{-1}(i+1)|}(\sigma)$. In this case, we conclude that

$$\text{if } f_i(\sigma^{-1}) = f_{i+1}(\sigma^{-1}) \text{ are odd, then } f_{|\sigma^{-1}(i)|}(\sigma) \leqslant f_{|\sigma^{-1}(i+1)|}(\sigma). \tag{13}$$

Equations (11), (12) and (13) motivate us to work with monomials $m(\mathbf{x},\mathbf{y}) = x^{\mathbf{p}}y^{\mathbf{q}}$ whose exponents are ordered in a similar way.

Definition 10. Suppose that $\mathbf{p} = (p_1, \dots, p_n)$ and $\mathbf{q} = (q_1, \dots, q_n)$ are two sequences of non-negative integers with $p_k + q_k$ even for all $1 \leqslant k \leqslant n$. We say that the monomial $m(\mathbf{x},\mathbf{y}) = x^{\mathbf{p}}y^{\mathbf{q}}$ is ordered and write $m(\mathbf{x},\mathbf{y}) \in \mathcal{O}_n$ if the exponents of $m(\mathbf{x},\mathbf{y})$ satisfy the following conditions:

1. $p_1 \geqslant p_2 \geqslant \dots \geqslant p_n$,
2. if $p_i = p_{i+1}$ are even, then $q_i \geqslant q_{i+1}$, and
3. if $p_i = p_{i+1}$ are odd, then $q_i \leqslant q_{i+1}$.

The previous ordering can be described in the following way. For each integer q, define

$$\mathfrak{s}(q) := \begin{cases} q & \text{if} \quad q \text{ is even,} \\ -q & \text{if} \quad q \text{ is odd.} \end{cases}$$

Suppose that $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ are sequences of non-negative integers. Then $m(\mathbf{x}, \mathbf{y}) = x^{\mathbf{p}} y^{\mathbf{q}} \in \mathcal{O}_n$ if and only if $(p_1, \mathfrak{s}(q_1)) \geqslant_\ell \ldots \geqslant_\ell (p_n, \mathfrak{s}(q_n))$, where $\geqslant_\ell$ denotes the lexicographic order. Equations (11), (12) and (13) imply that for every $\sigma \in B_n$ the diagonal signed descent monomial c_σ is ordered in this way; that is, $c_\sigma \in \mathcal{O}_n$ for all $\sigma \in B_n$. Suppose now that $m(\mathbf{x}, \mathbf{y}) = x^{\mathbf{p}} y^{\mathbf{q}}$ is a monomial with $p_k + q_k$ even for all $1 \leqslant k \leqslant n$ but whose exponents are not necessarily ordered as above. Consider the signed-symmetric polynomial $\rho(m(\mathbf{x}, \mathbf{y}))$. In this polynomial there exists a unique monomial $n(\mathbf{x}, \mathbf{y})$ whose exponents are ordered as above; that is, $\rho(m(\mathbf{x}, \mathbf{y}))$ contains a unique monomial $n(\mathbf{x}, \mathbf{y}) \in \mathcal{O}_n$ and $\rho(m(\mathbf{x}, \mathbf{y})) = \rho(n(\mathbf{x}, \mathbf{y}))$. Because of this, it suffices to work with monomials $m(\mathbf{x}, \mathbf{y})$ that are ordered as above.

Next we define a total order on $\mathcal{O}_n$ so that we can compare monomials. For this suppose that $\mathbf{q} = (q_1, \ldots, q_n)$ is a sequence of integers. Define the ordering of $\mathbf{q}$ to be the sequence

$$\mathfrak{o}(\mathbf{q}) := (q_{\alpha(1)}, \ldots, q_{\alpha(n)}),$$

where $(q_{\alpha(1)}, \ldots, q_{\alpha(n)})$ is a rearrangement of the sequence $\mathbf{q}$ in decreasing order; that is, $\alpha \in \Sigma_n$ is such that $q_{\alpha(1)} \geqslant \ldots \geqslant q_{\alpha(n)}$. For example, if $\mathbf{q} = (2, 3, 4, 1, 1)$ then $\mathfrak{o}(\mathbf{q}) = (4, 3, 2, 1, 1)$. Suppose now that $m(\mathbf{x}, \mathbf{y}) = x^{\mathbf{p}} y^{\mathbf{q}}$ is a monomial with $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$. Define

$$\mathfrak{o}(m(\mathbf{x}, \mathbf{y})) := (\mathfrak{o}(\mathbf{p}), \mathfrak{o}(\mathbf{q})).$$

In other words, $\mathfrak{o}(m(\mathbf{x}, \mathbf{y}))$ recovers the exponents $\mathbf{x}$ and $\mathbf{y}$ in the monomial $m(\mathbf{x}, \mathbf{y})$ ordered in a decreasing fashion. For example, if $n = 4$ and $m(\mathbf{x}, \mathbf{y}) = x_1^2 x_3 x_4^5 y_1 y_2^4 y_3 y_4^2$, then $\mathfrak{o}(m(\mathbf{x}, \mathbf{y})) = ((5, 2, 1, 0), (4, 2, 1, 1))$. Using this ordering of exponents, we can define the following total order on $\mathcal{O}_n$.

Definition 11. Suppose that $m(\mathbf{x}, \mathbf{y}) = x^{\mathbf{p}} y^{\mathbf{q}}$ and $n(\mathbf{x}, \mathbf{y}) = x^{\mathbf{i}} y^{\mathbf{j}}$ are two monomials in $\mathcal{O}_n$. We write $m(\mathbf{x}, \mathbf{y}) \succcurlyeq n(\mathbf{x}, \mathbf{y})$ if and only if

1. $\mathfrak{o}(m(\mathbf{x}, \mathbf{y})) \geqslant_\ell \mathfrak{o}(n(\mathbf{x}, \mathbf{y}))$, and

2. if $\mathfrak{o}(m(\mathbf{x}, \mathbf{y})) = \mathfrak{o}(n(\mathbf{x}, \mathbf{y}))$, then $(\mathbf{p}, \mathfrak{s}(\mathbf{q})) \geqslant_\ell (\mathbf{i}, \mathfrak{s}(\mathbf{j}))$.

In the above equation we used the notation $\mathfrak{s}(\mathbf{q}) := (\mathfrak{s}(q_1), \ldots, \mathfrak{s}(q_n))$ for a sequence of integers $\mathbf{q} = (q_1, \ldots, q_n)$. Also, $\geqslant_\ell$ denotes the lexicographic order.

Example 12. Suppose that $n = 4$. Consider the monomials $m(\mathbf{x}, \mathbf{y}) = x_1^7 x_2^6 x_3^6 x_4^5 y_1^3 y_2^8 y_3^6 y_4^5$ and $n(\mathbf{x}, \mathbf{y}) = x_1^7 x_2^6 x_3^6 x_4^5 y_1^5 y_2^8 y_3^6 y_4^3$. Then $m(\mathbf{x}, \mathbf{y}), n(\mathbf{x}, \mathbf{y}) \in \mathcal{O}_4$ are such that

$$\mathfrak{o}(m(\mathbf{x}, \mathbf{y})) = \mathfrak{o}(n(\mathbf{x}, \mathbf{y})) = ((7, 6, 6, 5), (8, 6, 5, 3))$$

and $m(\mathbf{x}, y) \succcurlyeq n(\mathbf{x}, \mathbf{y})$.

3.6 Signed index permutation

Suppose that $m(\mathbf{x}, \mathbf{y}) = x^{\mathbf{p}} y^{\mathbf{q}} \in \mathcal{O}_n$. We observe that by construction the sequence $\mathbf{p} = (p_1, \ldots, p_n)$ is in decreasing order. This is not necessarily true for the sequence $\mathbf{q} = (q_1, \ldots, q_n)$. With this in mind, we can associate to the monomial $m(\mathbf{x}, \mathbf{y})$ the unique element $\sigma \in B_n$, which we call its signed index permutation, that satisfies the following properties:

$$q_{|\sigma(1)|} \geqslant q_{|\sigma(2)|} \geqslant \cdots \geqslant q_{|\sigma(n)|}, \tag{14}$$

$$\text{if } 0 < i < j \text{ and } q_{|\sigma(i)|} = q_{|\sigma(j)|}, \text{ then } \sigma(i) < \sigma(i+1) < \cdots < \sigma(j), \text{ and} \tag{15}$$

$$q_{|\sigma(i)|} \text{ is even if and only if } \sigma(i) > 0. \tag{16}$$

In other words, the signed permutation σ is the unique element in B_n whose signs are determined by the parity of the q_i's and that orders the elements in the sequence $\mathbf{q} = (q_1, \ldots, q_n)$ in decreasing way breaking ties from left to right for even values of q_i and from right to left for odd values of q_i.

Example 13. Suppose that $n = 6$ and that $m(\mathbf{x}, \mathbf{y}) = x_1^7 x_2^6 x_3^6 x_4^5 x_5^5 x_6^3 y_1^3 y_2^8 y_3^6 y_4^3 y_5^5 y_6^5 \in \mathcal{O}_6$. In this case $\mathbf{q} = (3, 8, 6, 3, 5, 5)$ and the signed index permutation associated to $m(\mathbf{x}, \mathbf{y})$ is $\sigma = [2, 3, -6, -5, -4, -1]$.

3.7 Exponent decomposition

Suppose that $m(\mathbf{x}, \mathbf{y}) = x^{\mathbf{p}} y^{\mathbf{q}}$ is a monomial in $\mathcal{O}_n$ and let $\sigma \in B_n$ be the signed index permutation associated to $m(\mathbf{x}, \mathbf{y})$ as explained above. We can use the signed permutation σ to obtain a decomposition of the sequences $\mathbf{p}$ and $\mathbf{q}$ as is explained next. We start by decomposing $\mathbf{q}$. For this we need the following proposition.

Proposition 14. *Suppose that $m(\mathbf{x}, \mathbf{y}) = x^{\mathbf{p}} y^{\mathbf{q}} \in \mathcal{O}_n$. Then, the sequence of integers $\{q_{|\sigma(i)|} - f_i(\sigma)\}_{i=1}^n$ is a decreasing sequence of non-negative even integers.*

Proof. By the definition and equation (16) we have that for any $1 \leqslant i \leqslant n$

$$\begin{aligned} q_{|\sigma(i)|} - f_i(\sigma) &= q_{|\sigma(i)|} - 2d_i(\sigma) - \varepsilon_i(\sigma) \\ &\equiv q_{|\sigma(i)|} - \varepsilon_i(\sigma) \pmod 2 \\ &\equiv 0 \pmod 2. \end{aligned}$$

This proves that $q_{|\sigma(i)|} - f_i(\sigma)$ is even for all $1 \leqslant i \leqslant n$. Observe that $q_{|\sigma(n)|} - f_n(\sigma) = q_{|\sigma(n)|} - \varepsilon_n(\sigma)$. We have $q_{|\sigma(n)|} \geqslant 0$, and $q_{|\sigma(n)|}$ and $\varepsilon_n(\sigma)$ have the same parity with $\varepsilon_n(\sigma) \in \{0, 1\}$. It follows then that in either case $q_{|\sigma(n)|} - f_n(\sigma) \geqslant 0$. It remains to prove $q_{|\sigma(i)|} - f_i(\sigma) \geqslant q_{|\sigma(i+1)|} - f_{i+1}(\sigma)$ for all $1 \leqslant i \leqslant n-1$. For this we consider the following cases.

• Case 1. Suppose that $\sigma(i) < \sigma(i+1)$. This implies $d_i(\sigma) = d_{i+1}(\sigma)$. Thus, in this case, we need to prove that $q_{|\sigma(i)|} - \varepsilon_i(\sigma) \geqslant q_{|\sigma(i+1)|} - \varepsilon_{i+1}(\sigma)$. Since $q_{|\sigma(i)|} \geqslant q_{|\sigma(i+1)|}$ the only case we need to inspect is the case $\varepsilon_i(\sigma) = 1$ and $\varepsilon_{i+1}(\sigma) = 0$. However, under this assumption $q_{|\sigma(i)|}$ is odd and $q_{|\sigma(i+1)|}$ is even and thus $q_{|\sigma(i)|} - 1 \geqslant q_{|\sigma(i+1)|}$.

• Case 2. Suppose that $\sigma(i) > \sigma(i+1)$ and $\varepsilon_i(\sigma) \neq \varepsilon_{i+1}(\sigma)$. This implies $\varepsilon_i(\sigma) = 0$ and $\varepsilon_{i+1}(\sigma) = 1$. We have $d_i(\sigma) = d_{i+1}(\sigma) + 1$. In this case, we need to show that $q_{|\sigma(i)|} - 1 \geqslant q_{|\sigma(i+1)|}$. Note that $q_{|\sigma(i)|}$ must be even and $q_{|\sigma(i+1)|}$ must be odd and by equation (14) we have $q_{|\sigma(i)|} \geqslant q_{|\sigma(i+1)|}$. Therefore $q_{|\sigma(i)|} > q_{|\sigma(i+1)|}$ which means $q_{|\sigma(i)|} - 1 \geqslant q_{|\sigma(i+1)|}$.

• Case 3. Suppose that $\sigma(i) > \sigma(i+1)$ and $\varepsilon_i(\sigma) = \varepsilon_{i+1}(\sigma)$. Then $d_i(\sigma) = d_{i+1}(\sigma) + 1$. In this case we need to show that $q_{|\sigma(i)|} - 2 \geqslant q_{|\sigma(i+1)|}$. Since $\varepsilon_i(\sigma) = \varepsilon_{i+1}(\sigma)$, then $q_{|\sigma(i)|}$ and $q_{|\sigma(i+1)|}$ must have the same parity. By equation (14) we have $q_{|\sigma(i)|} \geqslant q_{|\sigma(i+1)|}$. Thus we only need to prove that $q_{|\sigma(i)|} > q_{|\sigma(i+1)|}$. Assume by contradiction that $q_{|\sigma(i)|} = q_{|\sigma(i+1)|}$. Using equation (15) we conclude $\sigma(i) < \sigma(i+1)$ which contradicts our original assumption. □

By the previous proposition, for every $1 \leqslant i \leqslant n$ we can find a non-negative number $\mu_{|\sigma(i)|}$ such that $q_{|\sigma(i)|} = 2\mu_{|\sigma(i)|} + f_i(\sigma)$. Define $\gamma_{|\sigma(i)|} := f_i(\sigma)$ so that $\gamma_i = f_{|\sigma^{-1}(i)|}(\sigma)$.

Proposition 15. *The sequences* $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$ *and* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ *are sequences of non-negative integers that satisfy the following properties:*

1. $\mathbf{q} = 2\boldsymbol{\mu} + \boldsymbol{\gamma}$,
2. $\mu_{|\sigma(1)|} \geqslant \mu_{|\sigma(2)|} \geqslant \cdots \geqslant \mu_{|\sigma(n)|}$,
3. $\gamma_{|\sigma(1)|} \geqslant \gamma_{|\sigma(2)|} \geqslant \cdots \geqslant \gamma_{|\sigma(n)|}$,
4. *if* $0 < i < j \leqslant n$ *and* $\gamma_i = \gamma_j$*, then* $\mathfrak{s}(q_i) \geqslant \mathfrak{s}(q_j)$.

Proof. Property (1) follows directly from the definition of $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$. Property (2) was proved in Proposition 14 and property (3) follows from equation (7). Suppose now that $0 < i < j \leqslant n$ and $\gamma_i = \gamma_j$. This means $f_{|\sigma^{-1}(i)|}(\sigma) = f_{|\sigma^{-1}(j)|}(\sigma)$; that is, $2d_{|\sigma^{-1}(i)|}(\sigma) + \varepsilon_{|\sigma^{-1}(i)|}(\sigma) = 2d_{|\sigma^{-1}(j)|}(\sigma) + \varepsilon_{|\sigma^{-1}(j)|}(\sigma)$. From here we obtain $\varepsilon_{|\sigma^{-1}(i)|}(\sigma) = \varepsilon_{|\sigma^{-1}(j)|}(\sigma)$ and $d_{|\sigma^{-1}(i)|}(\sigma) = d_{|\sigma^{-1}(j)|}(\sigma)$. By definition

$$q_i = \gamma_i + 2\mu_i = 2(\mu_i + 2d_{|\sigma^{-1}(i)|}(\sigma)) + \varepsilon_{|\sigma^{-1}(i)|}(\sigma),$$
$$q_j = \gamma_j + 2\mu_j = 2(\mu_j + 2d_{|\sigma^{-1}(j)|}(\sigma)) + \varepsilon_{|\sigma^{-1}(j)|}(\sigma).$$

In particular, we conclude that q_i and q_j have the same parity. We need to consider two cases according to the parity of these numbers. Suppose first that q_i and q_j are even. Let $k = |\sigma^{-1}(i)|$ and $l = |\sigma^{-1}(j)|$. Since q_i and q_j are even, then $\varepsilon_k(\sigma) = \varepsilon_l(\sigma) = 0$ and this implies that $\sigma^{-1}(i), \sigma^{-1}(j) > 0$; that is, $k = \sigma^{-1}(i) > 0$ and $l = \sigma^{-1}(j) > 0$. Let's show that $k < l$. Assume, by contradiction, that $l < k$. Since $d_l(\sigma) = d_k(\sigma)$ and $l < k$, then by equation (6) it follows that

$$j = \sigma(l) < \sigma(l+1) < \cdots < \sigma(k) = i$$

which contradicts the assumption $i < j$. Therefore $0 < \sigma^{-1}(i) < \sigma^{-1}(j)$ and by equation (14) we conclude $q_i = q_{|\sigma(\sigma^{-1}(i))|} \geqslant q_{|\sigma(\sigma^{-1}(j))|} = q_j$. The case where q_i and q_j are odd is handled in a similar way. □

Next we obtain a similar decomposition for the sequence $\mathbf{p} = (p_1, \dots, p_n)$. To start assume $m(\mathbf{x}, \mathbf{y}) = x^{\mathbf{p}} y^{\mathbf{q}} \in \mathcal{O}_n$. This implies that $p_i + q_i$ is even for all $1 \leqslant i \leqslant n$. On the other hand, $\varepsilon_i(\sigma^{-1}) = 0$ if and only if $k := \sigma^{-1}(i) > 0$ and this is the case if and only if $q_i = q_{\sigma(k)}$ is even. We conclude that if $m(\mathbf{x}, \mathbf{y}) \in \mathcal{O}_n$ then for every $1 \leqslant i \leqslant n$

$$p_i \equiv q_i \equiv \varepsilon_i(\sigma^{-1}) \pmod 2.$$

Proposition 16. *Suppose that* $m(\mathbf{x}, \mathbf{y}) = x^{\mathbf{p}} y^{\mathbf{q}} \in \mathcal{O}_n$. *Then* $\{p_i - f_i(\sigma^{-1})\}_{i=1}^n$ *is a decreasing sequence of non-negative even integers.*

Proof. By the above comment

$$\begin{aligned} p_i - f_i(\sigma^{-1}) &= p_i - 2d_i(\sigma^{-1}) - \varepsilon_i(\sigma^{-1}) \\ &\equiv q_i - \varepsilon_i(\sigma^{-1}) \pmod 2 \\ &\equiv 0 \pmod 2. \end{aligned}$$

This proves that $p_i - f_i(\sigma^{-1})$ is even for all $1 \leqslant i \leqslant n$. On the other hand, we have $p_n - f_n(\sigma^{-1}) = p_n - \varepsilon_n(\sigma^{-1}) \geqslant 0$ because p_n is odd if and only if $\varepsilon_n(\sigma^{-1}) = 1$. We are left to prove that $p_i - f_i(\sigma^{-1}) \geqslant p_{i+1} - f_{i+1}(\sigma^{-1})$ for all $1 \leqslant i \leqslant n-1$. For this the following cases are considered.

• Case 1. Suppose that $\sigma^{-1}(i) < \sigma^{-1}(i+1)$. This implies $d_i(\sigma^{-1}) = d_{i+1}(\sigma^{-1})$. Thus in this case we need to prove that $p_i - \varepsilon_i(\sigma^{-1}) \geqslant p_{i+1} - \varepsilon_{i+1}(\sigma^{-1})$. Since $p_i \geqslant p_{i+1}$, the only case we need to inspect is the case $\varepsilon_i(\sigma^{-1}) = 1$ and $\varepsilon_{i+1}(\sigma^{-1}) = 0$. However, under this assumption p_i is odd and p_{i+1} is even and thus $p_i - 1 \geqslant p_{i+1}$ as $p_i \geqslant p_{i+1}$.

• Case 2. Suppose that $\sigma^{-1}(i) > \sigma^{-1}(i+1)$ and $\varepsilon_i(\sigma^{-1}) \neq \varepsilon_{i+1}(\sigma^{-1})$. This is only possible if $\varepsilon_i(\sigma^{-1}) = 0$ and $\varepsilon_{i+1}(\sigma^{-1}) = 1$. Then $i \in Des(\sigma^{-1})$ and $d_i(\sigma^{-1}) = d_{i+1}(\sigma^{-1}) + 1$. In this case we need to show that $p_i - 1 \geqslant p_{i+1}$. Note that p_i must be even and p_{i+1} must be odd, and by assumption $p_i \geqslant p_{i+1}$. Therefore $p_i > p_{i+1}$ and thus $p_i - 1 \geqslant p_{i+1}$ as desired.

• Case 3. Suppose that $\sigma^{-1}(i) > \sigma^{-1}(i+1)$ and $\varepsilon_i(\sigma^{-1}) = \varepsilon_{i+1}(\sigma^{-1})$. Then $i \in Des(\sigma^{-1})$ and therefore $d_i(\sigma^{-1}) = d_{i+1}(\sigma^{-1}) + 1$. In this case, we need to show that $p_i - 2 \geqslant p_{i+1}$. Note that $\varepsilon_i(\sigma^{-1}) = \varepsilon_{i+1}(\sigma^{-1})$ implies that p_i and p_{i+1} have the same parity. We know that $p_i \geqslant p_{i+1}$. Therefore we only need to prove that $p_i > p_{i+1}$. Assume by contradiction that $p_i = p_{i+1}$. By assumption $m(\mathbf{x}, \mathbf{y}) = x^{\mathbf{p}} y^{\mathbf{q}} \in \mathcal{O}_n$ and $p_i = p_{i+1}$. This implies $\mathfrak{s}(q_i) \geqslant \mathfrak{s}(q_{i+1})$. Let $k = |\sigma^{-1}(i)|$ and $l = |\sigma^{-1}(i+1)|$. If $\varepsilon_i(\sigma^{-1}) = \varepsilon_{i+1}(\sigma^{-1}) = 0$ then we obtain $0 < l < k$. We conclude $q_{i+1} = q_{|\sigma(l)|} \geqslant q_{|\sigma(k)|} = q_i$ by equation (14). Observe that q_i and q_{i+1} must be even since they have the same parity as $\varepsilon_i(\sigma^{-1}) = \varepsilon_{i+1}(\sigma^{-1}) = 0$. Thus $q_i = \mathfrak{s}(q_i) \geqslant \mathfrak{s}(q_{i+1}) = q_{i+1}$ and in turn $q_{i+1} = q_{|\sigma(l)|} = q_{|\sigma(k)|} = q_i$. This together with equation (15) imply $i + 1 = \sigma(l) < \sigma(k) = i$ which is a contradiction. Suppose now that $\varepsilon_i(\sigma^{-1}) = \varepsilon_{i+1}(\sigma^{-1}) = 1$. Then $k = -\sigma^{-1}(i)$ and $l = -\sigma^{-1}(i+1)$ and by assumption $\sigma^{-1}(i) > \sigma^{-1}(i+1)$. Thus $0 > -k > -l$; that is, $0 < k < l$. Using equation (14) we conclude $q_i = q_{|\sigma(k)|} \geqslant q_{|\sigma(l)|} = q_{i+1}$. Under the given assumptions q_i and q_{i+1} must be odd and $\mathfrak{s}(q_i) \geqslant \mathfrak{s}(q_{i+1})$; that is, $q_i \leqslant q_{i+1}$. Again we conclude that $q_i = q_{|\sigma(k)|} = q_{|\sigma(l)|} = q_{i+1}$. Since $0 < k < l$ using equation (15) as before, we conclude that $-i = \sigma(k) < \sigma(l) = -i - 1$ deriving a contradiction in either case. □

By the previous proposition, for every $1 \leqslant i \leqslant n$ we can find a non-negative integer ν_i such that $2\nu_i = p_i - f_i(\sigma^{-1})$; that is, $p_i = 2\nu_i + f_i(\sigma^{-1})$. Define $\delta_i := f_i(\sigma^{-1})$.

Proposition 17. *The sequences $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n)$ and $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$ are sequences of non-negative integers that satisfy the following properties:*

1. $\mathbf{p} = 2\boldsymbol{\nu} + \boldsymbol{\delta}$,
2. $\nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_n$,
3. $\delta_1 \geqslant \delta_2 \geqslant \cdots \geqslant \delta_n$,
4. *if $0 < i < j \leqslant n$ and $\delta_i = \delta_j$, then $\mathfrak{s}(q_i) \geqslant \mathfrak{s}(q_j)$.*

Proof. Property (1) is given directly by the definition of the sequences $\boldsymbol{\nu}$ and $\boldsymbol{\delta}$. Also, property (2) is proved in Proposition 16 and property (3) follows from equation (7) applied to the signed permutation σ^{-1}. To prove property (4) suppose that $0 < i < j \leqslant n$ and $\delta_i = \delta_j$. This means $f_i(\sigma^{-1}) = f_j(\sigma^{-1})$ and in turn $d_i(\sigma^{-1}) = d_j(\sigma^{-1})$ and $\varepsilon_i(\sigma^{-1}) = \varepsilon_j(\sigma^{-1})$. Since $i < j$, applying equation (6) to the numbers $d_i(\sigma^{-1})$, it follows that $\sigma^{-1}(i) < \sigma^{-1}(j)$. Also, as $\varepsilon_i(\sigma^{-1}) = \varepsilon_j(\sigma^{-1})$ then q_i and q_j have the same parity. Assume q_i and q_j are both even, then $0 < \sigma^{-1}(i) < \sigma^{-1}(j)$ and by equation (14) we have

$$q_i = q_{|\sigma(\sigma^{-1}(i))|} \geqslant q_{|\sigma(\sigma^{-1}(j))|} = q_j.$$

Similarly, if q_i and q_j are odd, then $\sigma^{-1}(i) < \sigma^{-1}(j) < 0$ and thus $-\sigma^{-1}(i) > -\sigma^{-1}(j) > 0$. Again by property (14) we have

$$q_i = q_{|\sigma(\sigma^{-1}(i))|} \leqslant q_{|\sigma(\sigma^{-1}(j))|} = q_j.$$

In either case we obtain $\mathfrak{s}(q_i) \geqslant \mathfrak{s}(q_j)$. □

3.8 Monomial decomposition

Next we derive a decomposition for monomials that are ordered as in Definition 10. For that suppose that $m(\mathbf{x}, \mathbf{y}) = x^{\mathbf{p}} y^{\mathbf{q}} \in \mathcal{O}_n$. We can decompose $\mathbf{p} = 2\boldsymbol{\nu} + \boldsymbol{\delta}$, where $\boldsymbol{\nu}$ and $\boldsymbol{\delta}$ are the sequences of integers provided in Propositions 17. Similarly, we can write $\mathbf{q} = 2\boldsymbol{\mu} + \boldsymbol{\gamma}$, where $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ are sequences of integers given by Proposition 15. Let $\sigma \in B_n$ be the index permutation associated to the monomial $m(\mathbf{x}, \mathbf{y})$. Recall that by definition $\delta_i = f_i(\sigma^{-1})$ and $\gamma_i = f_{|\sigma^{-1}(i)|}(\sigma)$. Therefore the diagonal signed descent monomial c_σ is given by

$$c_\sigma = \prod_{i=1}^{n} x_i^{f_i(\sigma^{-1})} y_i^{f_{|\sigma^{-1}(i)|}(\sigma)} = x^{\boldsymbol{\delta}} y^{\boldsymbol{\gamma}}.$$

This means that we have a decomposition

$$m(\mathbf{x}, \mathbf{y}) = x^{\mathbf{p}} y^{\mathbf{q}} = x^{2\boldsymbol{\nu}+\boldsymbol{\delta}} y^{2\boldsymbol{\mu}+\boldsymbol{\gamma}} = x^{2\boldsymbol{\nu}} y^{2\boldsymbol{\mu}} c_\sigma.$$

Given a sequence of integers $\mathbf{i} = (i_1, \ldots, i_n)$, let $\Sigma_n(\mathbf{i})$ denote the stabilizer of $\mathbf{i}$ under the permutation action of Σ_n; that is, $\Sigma_n(\mathbf{i})$ is the subgroup of elements in Σ_n that fix $\mathbf{i}$. Define

$$s_{2\boldsymbol{\nu}}(\mathbf{x}) = \sum_{[\alpha]\in\Sigma_n/\Sigma_n(\boldsymbol{\nu})} x^{2\alpha(\boldsymbol{\nu})}, \quad \text{and} \quad s_{2\mu}(\mathbf{y}) = \sum_{[\beta]\in\Sigma_n/\Sigma_n(\boldsymbol{\mu})} y^{2\beta(\boldsymbol{\mu})}.$$

By definition, the functions $s_{2\boldsymbol{\nu}}(\mathbf{x})$ and $s_{2\boldsymbol{\mu}}(\mathbf{y})$ are symmetric polynomials on the variables $x_1^2, \ldots, x_n^2$ and $y_1^2, \ldots, y_n^2$ induced by the monomials $x^{2\boldsymbol{\nu}}$ and $y^{2\boldsymbol{\mu}}$ respectively.[1] In particular, $s_{2\boldsymbol{\nu}}(\mathbf{x})s_{2\boldsymbol{\mu}}(\mathbf{y}) \in R^{B_n}$. In the same way as in [5, Proposition 3.2], the polynomial $s_{2\boldsymbol{\nu}}(\mathbf{x})s_{2\boldsymbol{\mu}}(\mathbf{y})\rho(c_\sigma)$ can be decomposed as we prove next.

Theorem 18. *Suppose that $m(\mathbf{x},\mathbf{y}) = x^{\mathbf{p}}y^{\mathbf{q}} \in \mathcal{O}_n$ and let $\sigma \in B_n$ be the corresponding signed index permutation. Let $\boldsymbol{\nu}$, $\boldsymbol{\mu}$, $\boldsymbol{\delta}$ and $\boldsymbol{\gamma}$ be as defined above. Then*

$$s_{2\boldsymbol{\nu}}(\mathbf{x})s_{2\boldsymbol{\mu}}(\mathbf{y})\rho(c_\sigma) = k_{m(\mathbf{x},y)}\rho(m(\mathbf{x},\mathbf{y})) + \sum_{n(\mathbf{x},\mathbf{y})\precneqq m(\mathbf{x},\mathbf{y})} k_{n(\mathbf{x},\mathbf{y})}\rho(n(\mathbf{x},\mathbf{y})).$$

In the above equation $n(\mathbf{x},y) \in \mathcal{O}_n$ runs through the collection of ordered monomials with same total degree as $m(\mathbf{x},y)$ with $n(\mathbf{x},\mathbf{y}) \precneqq m(\mathbf{x},\mathbf{y})$, $k_{m(\mathbf{x},y)} > 0$ and $k_{n(\mathbf{x},\mathbf{y})}$ are constants.

Proof. Using the definition we have

$$\begin{aligned} s_{2\boldsymbol{\nu}}(\mathbf{x})s_{2\boldsymbol{\mu}}(\mathbf{y})\rho(c_\sigma) = \rho(s_{2\boldsymbol{\nu}}(\mathbf{x})s_{2\boldsymbol{\mu}}(\mathbf{y})c_\sigma) &= \sum_{[\alpha]\in\Sigma_n/\Sigma_n(\boldsymbol{\nu})} \sum_{[\beta]\in\Sigma_n/\Sigma_n(\boldsymbol{\mu})} \rho(x^{2\alpha\boldsymbol{\nu}}y^{2\beta\boldsymbol{\mu}}c_\sigma) \\ &= \sum_{[\alpha]\in\Sigma_n/\Sigma_n(\boldsymbol{\nu})} \sum_{[\beta]\in\Sigma_n/\Sigma_n(\boldsymbol{\mu})} \rho(x^{2\alpha\boldsymbol{\nu}+\boldsymbol{\delta}}y^{2\beta\boldsymbol{\mu}+\boldsymbol{\gamma}}). \end{aligned}$$

Fix $\alpha, \beta \in \Sigma$ and let $n(\mathbf{x},\mathbf{y}) \in \mathcal{O}_n$ be the unique ordered monomial such that $\rho(n(\mathbf{x},\mathbf{y})) = \rho(x^{2\alpha\boldsymbol{\nu}+\boldsymbol{\delta}}y^{2\beta\boldsymbol{\mu}+\boldsymbol{\gamma}})$. To prove the theorem we need to prove that $n(\mathbf{x},\mathbf{y}) \preccurlyeq m(\mathbf{x},\mathbf{y})$. Let $[\Sigma_n(\boldsymbol{\delta})]$ denote the image of $\Sigma_n(\boldsymbol{\delta})$ in $\Sigma_n/\Sigma_n(\boldsymbol{\nu})$ under the natural map. If $[\alpha] \in [\Sigma_n(\boldsymbol{\delta})]$ then $\mathfrak{o}(2\alpha\boldsymbol{\nu}+\boldsymbol{\delta}) = \mathfrak{o}(\alpha(2\boldsymbol{\nu}+\boldsymbol{\delta})) = \mathfrak{o}(\mathbf{p})$. Also, if $[\alpha] \notin [\Sigma_n(\boldsymbol{\delta})]$, then by parts (2) and (3) of Proposition 17 it follows that if $\mathfrak{o}(2\alpha\boldsymbol{\nu}+\boldsymbol{\delta}) <_\ell \mathfrak{o}(\mathbf{p})$. Similarly, let $[\Sigma_n(\boldsymbol{\gamma})]$ denote the image of $\Sigma_n(\boldsymbol{\gamma})$ in $\Sigma_n/\Sigma_n(\boldsymbol{\mu})$. Then if $[\beta] \in [\Sigma_n(\boldsymbol{\gamma})]$ then $\mathfrak{o}(2\beta\boldsymbol{\mu}+\boldsymbol{\gamma}) = \mathfrak{o}(\mathbf{q})$. Also, if $[\beta] \notin [\Sigma_n(\boldsymbol{\gamma})]$ then by parts (2) and (3) of Proposition 15 we have $\mathfrak{o}(2\beta\boldsymbol{\mu}+\boldsymbol{\gamma}) <_\ell \mathfrak{o}(\mathbf{q})$. With this in mind we have the following cases.

- Case 1. Suppose that $[\alpha] \notin [\Sigma_n(\boldsymbol{\delta})]$ or $[\beta] \notin [\Sigma_n(\boldsymbol{\gamma})]$. If $[\alpha] \notin [\Sigma_n(\boldsymbol{\delta})]$ then by the previous comment $\mathfrak{o}(2\alpha\boldsymbol{\nu}+\boldsymbol{\delta}) <_\ell \mathfrak{o}(\mathbf{p})$ and if $[\alpha] \in [\Sigma_n(\boldsymbol{\delta})]$ but $\beta \notin [\Sigma_n(\boldsymbol{\gamma})]$ then $\mathfrak{o}(2\alpha\boldsymbol{\nu}+\boldsymbol{\delta}) = \mathfrak{o}(\mathbf{p})$ but $\mathfrak{o}(2\beta\boldsymbol{\mu}+\boldsymbol{\gamma}) <_\ell \mathfrak{o}(\mathbf{q})$. In either case we conclude

$$\mathfrak{o}(n(\mathbf{x},\mathbf{y})) = (\mathfrak{o}(2\alpha\boldsymbol{\nu}+\boldsymbol{\delta}), \mathfrak{o}(2\beta\boldsymbol{\mu}+\boldsymbol{\gamma})) <_\ell (\mathfrak{o}(\mathbf{p}), \mathfrak{o}(\mathbf{q})) = \mathfrak{o}(m(\mathbf{x},\mathbf{y})).$$

It follows that in this case $n(\mathbf{x},\mathbf{y}) \precneqq m(\mathbf{x},\mathbf{y})$.

[1] The reader is warned that the functions $s_{2\boldsymbol{\nu}}(\mathbf{x})$ and $s_{2\boldsymbol{\mu}}(\mathbf{y})$ are symmetric functions and not Schur functions as the notation may suggest.

• Case 2. Suppose that $[\alpha] \in [\Sigma_n(\boldsymbol{\delta})]$ and $[\beta] \in [\Sigma_n(\boldsymbol{\gamma})]$. We can assume without loss of generality that $\alpha \in \Sigma_n(\boldsymbol{\delta})$ and $\beta \in \Sigma_n(\boldsymbol{\gamma})$. Then

$$x^{2\alpha\boldsymbol{\nu}+\boldsymbol{\delta}}y^{2\beta\boldsymbol{\mu}+\boldsymbol{\gamma}} = x^{\alpha(2\boldsymbol{\nu}+\boldsymbol{\delta})}y^{\beta(2\boldsymbol{\mu}+\boldsymbol{\gamma})} = x^{\alpha(\mathbf{p})}y^{\beta(\mathbf{q})}.$$

Therefore $n(\mathbf{x},\mathbf{y}) = x^{\mathbf{p}}y^{\pi\alpha^{-1}\beta(\mathbf{q})}$ for some $\pi \in \Sigma_n$. Note that the permutation π has to stabilize $\mathbf{p}$ and $\boldsymbol{\delta}$, thus in particular $\pi \in \Sigma_n(\boldsymbol{\delta})$. In this case

$$\mathfrak{o}(n(\mathbf{x},\mathbf{y})) = \mathfrak{o}(x^{\alpha(\mathbf{p})}y^{\beta(\mathbf{q})}) = \mathfrak{o}(m(\mathbf{x},\mathbf{y})).$$

To prove that $n(\mathbf{x}.y) \preccurlyeq m(\mathbf{x},\mathbf{y})$ we need to show that $(\mathbf{p},\mathfrak{s}(\pi\alpha^{-1}\beta(\mathbf{q}))) \leqslant_\ell (\mathbf{p},\mathfrak{s}(\mathbf{q}))$; that is, we need to prove that $\mathfrak{s}(\pi\alpha^{-1}\beta(\mathbf{q})) \leqslant_\ell \mathfrak{s}(\mathbf{q})$. If $\mathfrak{s}(\pi\alpha^{-1}\beta(\mathbf{q})) = \mathfrak{s}(\mathbf{q})$ there is nothing to prove. Assume that $\mathfrak{s}(\pi\alpha^{-1}\beta(\mathbf{q})) \neq \mathfrak{s}(\mathbf{q})$ and let $1 \leqslant k \leqslant n$ be the smallest integer such that $\mathfrak{s}(q_{\pi\alpha^{-1}\beta(k)}) \neq \mathfrak{s}(q_k)$. We need to show that $\mathfrak{s}(q_{\pi\alpha^{-1}\beta(k)}) < \mathfrak{s}(q_k)$. Since $\beta \in \Sigma_n(\boldsymbol{\gamma})$, then by Proposition 15 part (4) we have that if $i < \beta(i)$ then $\mathfrak{s}(q_i) \geqslant \mathfrak{s}(q_{\beta(i)})$. Similarly, since $\alpha,\pi \in \Sigma_n(\boldsymbol{\delta})$ by Proposition 17 part (4) whenever $i < \pi\alpha^{-1}(i)$ then $\mathfrak{s}(q_i) \geqslant \mathfrak{s}(q_{\pi\alpha^{-1}(i)})$. Using this we can see that $\mathfrak{s}(q_k) > \mathfrak{s}(q_{\pi\alpha^{-1}\beta(k)})$.

In either case we conclude that $n(\mathbf{x},\mathbf{y}) \preccurlyeq m(\mathbf{x},\mathbf{y})$. □

Example 19. Suppose that $n = 3$ and that $m_1(\mathbf{x},\mathbf{y}) = x_3^3y_1y_2^2$. Observe that this monomial is not ordered in the sense of Definition 10. Let $m(\mathbf{x},\mathbf{y}) = x_1^3y_1y_2$, then $m(\mathbf{x},\mathbf{y}) \in \mathcal{O}_3$ and $\rho(m(\mathbf{x},\mathbf{y})) = \rho(m_1(\mathbf{x},\mathbf{y}))$. Let $\mathbf{p} = (3,0,0)$ and $\mathbf{q} = (1,2,0)$ so that $m(\mathbf{x},\mathbf{y}) = x^{\mathbf{p}}y^{\mathbf{q}}$. In this case the signed index permutation corresponding to $m(\mathbf{x},\mathbf{y})$ is the signed permutation $\sigma = [2,-1,3]$. It follows that $\sigma^{-1} = [-2,1,3]$ and that $Des(\sigma) = \{1\}$ and $Des(\sigma^{-1}) = \emptyset$. Also

$$f_1(\sigma) = 2, f_2(\sigma) = 1, f_3(\sigma) = 0,$$
$$f_1(\sigma^{-1}) = 1, f_2(\sigma^{-1}) = 0, f_3(\sigma^{-1}) = 0.$$

In the notation of the previous theorem we have $\boldsymbol{\nu} = (1,0,0)$, $\boldsymbol{\mu} = (0,0,0)$, $\boldsymbol{\delta} = (1,0,0)$ and $\boldsymbol{\gamma} = (1,2,0)$. Thus we have $\mathbf{p} = 2\boldsymbol{\nu} + \boldsymbol{\delta}$ and $\mathbf{q} = 2\boldsymbol{\mu} + \boldsymbol{\gamma}$. In this case

$$s_{2\boldsymbol{\nu}}(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 \text{ and } s_{2\boldsymbol{\mu}}(\mathbf{y}) = 1.$$

In addition

$$c_\sigma = \rho(x_1y_1y_2^2) = \frac{1}{6}\left(x_1y_1y_2^2 + x_2y_1^2y_2 + x_3y_2^2y_3 + x_1y_1y_3^2 + x_2y_2y_3^2 + x_3y_1^2y_3\right).$$

The decomposition given by the previous theorem in this case is

$$\begin{aligned}
&s_{2\boldsymbol{\nu}}(\mathbf{x})s_{2\boldsymbol{\mu}}(\mathbf{y})\rho(c_\sigma) = s_{2\boldsymbol{\nu}}(\mathbf{x})\rho(c_\sigma)\\
&= \frac{1}{6}(x_1^2 + x_2^2 + x_3^2)\left(x_1y_1y_2^2 + x_2y_1^2y_2 + x_3y_2^2y_3 + x_1y_1y_3^2 + x_2y_2y_3^2 + x_3y_1^2y_3\right)\\
&= \rho(x_1^3y_1y_2^2) + \rho(x_1^2x_2y_1^2y_2) + \rho(x_1^2x_3y_2^2y_3).
\end{aligned}$$

Observe that $x_1^2x_2y_1^2y_2, x_1^2x_3y_2^2y_3 \in \mathcal{O}_3$. Furthermore, $x_1^3y_1y_2^2 \succcurlyeq x_1^2x_2y_1^2y_2$ and $x_1^3y_1y_2^2 \succcurlyeq x_1^2x_3y_2^2y_3$. We conclude then that

$$\rho(x_1^3y_1y_2^2) = s_{2\boldsymbol{\nu}}(\mathbf{x})\rho(c_\sigma) - \rho(x_1^2x_2y_1^2y_2) - \rho(x_1^2x_3y_2^2y_3).$$

Iterating this procedure on the monomials $x_1^2x_2y_1^2y_2$ and $x_1^2x_3y_2^2y_3$, we can write $\rho(x_1^3y_1y_2^2)$ as a linear combination of the $\rho(c_\sigma)$'s with coefficients in R^{B_n}. In the next theorem we show that this method works in general and thus $\{\rho(c_\sigma)\}_{\sigma\in B_n}$ yields a free basis of M^{B_n} as a module over R^{B_n}.

3.9 Main theorem

Finally we are ready to prove the main theorem of this article.

Theorem 20. *Suppose that $n \geqslant 1$. Then the collection $\{\rho(c_\sigma)\}_{\sigma\in B_n}$ forms a free basis of M^{B_n} as a module over R^{B_n}.*

Proof. Let's show first that $\{\rho(c_\sigma)\}_{\sigma\in B_n}$ generates M^{B_n} as a module over R^{B_n}. It suffices to show that for every $m(\mathbf{x},\mathbf{y}) \in \mathcal{O}_n$ the polynomial $\rho(m(\mathbf{x},\mathbf{y}))$ is generated by the different $\rho(c_\sigma)$. Fix $m(\mathbf{x},\mathbf{y}) \in \mathcal{O}_n$ and let σ be the corresponding signed index permutation. By the previous theorem we have

$$m(\mathbf{x},\mathbf{y}) = k_{m(\mathbf{x},\mathbf{y})}s_{2\boldsymbol{nu}}(\mathbf{x})s_{2\boldsymbol{\mu}}(\mathbf{y})\rho(c_\sigma) + \sum_{n(\mathbf{x},\mathbf{y})\precneqq m(\mathbf{x},\mathbf{y})} k_{n(\mathbf{x},\mathbf{y})}n(\mathbf{x},\mathbf{y}),$$

for some constants $k_{n(\mathbf{x},\mathbf{y})}$ and monomials $n(\mathbf{x},\mathbf{y}) \precneqq m(\mathbf{x},\mathbf{y})$ of same total degree. Iterating this process on the monomials $n(\mathbf{x},\mathbf{y}) \precneqq m(\mathbf{x},\mathbf{y})$ as many times as necessary, we see that we can write $m(\mathbf{x},\mathbf{y})$ as a linear combination of the elements $\{\rho(c_\sigma)\}_{\sigma\in B_n}$ with coefficients in R^{B_n}. (Note that this process must terminate after finitely many stages as there are only finitely many monomials $n(\mathbf{x},\mathbf{y})$ of same total degree as $m(\mathbf{x},\mathbf{y})$). This proves that $\{\rho(c_\sigma)\}_{\sigma\in B_n}$ generates M^{B_n} as a R^{B_n}-module. On the other hand, note that M^{B_n} is a bigraded ring over $\mathbb{Q}$ with

$$\operatorname{bideg}(x^{\mathbf{p}}y^{\mathbf{q}}) = (|\mathbf{p}|, |\mathbf{q}|),$$

where $|\mathbf{p}| = p_1+\cdots+p_n$. With this grading, for every $\sigma \in B_n$ we have that the polynomial $\rho(c_\sigma)$ is homogeneous and

$$\operatorname{bideg}(\rho(c_\sigma)) = (\operatorname{fmaj}(\sigma^{-1}), \operatorname{fmaj}(\sigma)).$$

Let $P_{M^{B_n}}(s,t)$ denote the bigraded Hilbert series of the bigraded ring M^{B_n}. Using [4, Theorem 3] we conclude that the series $P_{M^{B_n}}(s,t)$ is given by

$$P_{M^{B_n}}(s,t) = \frac{\left(\sum_{\sigma\in B_n} s^{\operatorname{fmaj}(\sigma^{-1})}t^{\operatorname{fmaj}(\sigma)}\right)}{\prod_{i=1}^n(1-s^{2i})(1-t^{2i})}.$$

This together with [5, Theorem 1.4] show that $\{\rho(c_\sigma)\}_{\sigma\in B_n}$ is a free basis of M^{B_n} as module over R^{B_n}. □

Example 21. Suppose that $n = 2$. Then the basis constructed in the previous theorem is:

$$c_1 = 1,\ c_2 = \frac{1}{2}\left(x_1y_1 + x_2y_2\right),\ c_3 = \frac{1}{2}\left(x_1^2y_2^2 + x_2^2y_1^2\right),$$
$$c_4 = \frac{1}{2}\left(x_1y_1y_2^2 + x_2y_1^2y_2\right),\ c_5 = x_1x_2y_1y_2,\ c_6 = \frac{1}{2}(x_1^2x_2y_2 + x_1x_2^2y_1),$$
$$c_7 = \frac{1}{2}(x_1^2x_2y_1^2y_2 + x_1x_2^2y_1y_2^2),\ c_8 = \frac{1}{2}(x_1^3x_2y_1^3y_2 + x_1x_2^3y_1y_2^3).$$

As in the previous cases this theorem also has a geometric application. Let $B_{com}Sp(n)$ be the geometric realization of the simplicial space obtained by considering the space of commuting k-tuples in $Sp(n)$. Explicitly, this simplicial space is defined by $[B_{com}Sp(n)]_k = \mathrm{Hom}(\mathbb{Z}^k, Sp(n))$. Thus $B_{com}Sp(n)$ is the classifying space for commutativity on $Sp(n)$. In [1, Section 8], it is proved that the signed diagonal descent monomials can be used to obtain an explicit basis of $H^*(B_{com}Sp(n);\mathbb{Q})$ seen as a module over $H^*(BSp(n);\mathbb{Q})$.

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