On the Positive Moments of Ranks of Partitions

William Y.C. Chen\textsuperscript{1}, Kathy Q. Ji\textsuperscript{2}, and Erin Y.Y. Shen\textsuperscript{3}

Center for Combinatorics, LPMC-TJKLC
Nankai University
Tianjin 300071, P.R. China
\textsuperscript{1}chen@nankai.edu.cn, \textsuperscript{2}ji@nankai.edu.cn, \textsuperscript{3}shenyiying@mail.nankai.edu.cn

Submitted: Nov 1, 2013; Accepted: Jan 12, 2014; Published: Feb 7, 2014
Mathematics Subject Classifications: 05A17, 11P83, 05A30

Abstract

By introducing $k$-marked Durfee symbols, Andrews found a combinatorial interpretation of the $2k$-th symmetrized moment $\eta_{2k}(n)$ of ranks of partitions of $n$ in terms of $(k+1)$-marked Durfee symbols of $n$. In this paper, we consider the $k$-th symmetrized positive moment $\bar{\eta}_k(n)$ of ranks of partitions of $n$ which is defined as the truncated sum over positive ranks of partitions of $n$. As combinatorial interpretations of $\bar{\eta}_{2k}(n)$ and $\bar{\eta}_{2k-1}(n)$, we show that for given $k$ and $i$ with $1 \leq i \leq k+1$, $\bar{\eta}_{2k-1}(n)$ equals the number of $(k+1)$-marked Durfee symbols of $n$ with the $i$-th rank being zero and $\bar{\eta}_{2k}(n)$ equals the number of $(k+1)$-marked Durfee symbols of $n$ with the $i$-th rank being positive. The interpretations of $\bar{\eta}_{2k-1}(n)$ and $\bar{\eta}_{2k}(n)$ are independent of $i$, and they imply the interpretation of $\eta_{2k}(n)$ given by Andrews since $\eta_{2k}(n)$ equals $\bar{\eta}_{2k-1}(n)$ plus twice of $\bar{\eta}_{2k}(n)$. Moreover, we obtain the generating functions of $\bar{\eta}_{2k}(n)$ and $\bar{\eta}_{2k-1}(n)$.

Keywords: rank of a partition; $k$-marked Durfee symbol; moment of ranks

1 Introduction

This paper is concerned with a combinatorial study of the symmetrized positive moments of ranks of partitions. The notion of symmetrized moments was introduced by Andrews [1]. Any odd symmetrized moment is zero because of the symmetry of ranks. For an even symmetrized moment, Andrews found a combinatorial interpretation by introducing $k$-marked Durfee symbols. It is natural to investigate the combinatorial interpretation of an odd symmetrized moment which is defined as a truncated sum over positive ranks of
partitions of $n$. We give combinatorial interpretations of both the even and the odd positive moments in terms of $k$-marked Durfee symbols, which also lead to the combinatorial interpretation of an even symmetrized moment of ranks given by Andrews.

The rank of a partition $\lambda$ introduced by Dyson [6] is defined as the largest part minus the number of parts. Let $N(m,n)$ denote the number of partitions of $n$ with rank $m$. The following generating function of $N(m,n)$ was conjectured by Dyson [6] in 1944 and proved by Atkin and Swinnerton-Dyer [3] in 1954. A combinatorial proof was found by Dyson [7] in 1969.

**Theorem 1.1.** For given integer $m$, we have

$$\sum_{n=0}^{+\infty} N(m,n)q^n = \frac{1}{(q; q)_{\infty}} \sum_{n=1}^{+\infty} (-1)^{n-1}q^{n(3n-1)/2+|m|n}(1 - q^n). \quad (1.1)$$

Recently, Andrews [1] introduced the $k$-th symmetrized moment $\eta_k(n)$ of ranks of partitions of $n$ as given by

$$\eta_k(n) = \sum_{m=-\infty}^{+\infty} \left( m + \left\lfloor \frac{k+1}{2} \right\rfloor \right) N(m,n). \quad (1.2)$$

It can be easily seen that for any $k$, $\eta_k(n)$ is a linear combination of the moments $N_j(n)$ of ranks given by Atkin and Garvan [4]

$$N_j(n) = \sum_{m=-\infty}^{+\infty} m^j N(m,n).$$

For example,

$$\eta_6(n) = \frac{1}{720}N_6(n) - \frac{1}{144}N_4(n) + \frac{1}{180}N_2(n).$$

In view of the symmetry $N(-m,n) = N(m,n)$, we have $\eta_{2k+1}(n) = 0$. As for an even symmetrized moment $\eta_{2k}(n)$, Andrews gave the following combinatorial interpretation by introducing $k$-marked Durfee symbols. For the definition of $k$-marked Durfee symbols, see Section 2.

**Theorem 1.2** (Andrews [1]). For any $k \geq 1$, $\eta_{2k}(n)$ is equal to the number of $(k+1)$-marked Durfee symbols of $n$.

Andrews [1] proved the above theorem by using the $k$-fold generalization of Watson’s $q$-analog of Whipple’s theorem. Ji [9] found a combinatorial proof of Theorem 1.2 by establishing a map from $k$-marked Durfee symbols to ordinary partitions. Kursungoz [10] gave another proof of Theorem 1.2 by using an alternative representation of $k$-marked Durfee symbols.

In this paper, we introduce the $k$-th symmetrized positive moment $\bar{\eta}_k(n)$ of ranks as given by

$$\bar{\eta}_k(n) = \sum_{m=1}^{\infty} \left( m + \left\lfloor \frac{k-1}{2} \right\rfloor \right) N(m,n),$$

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or equivalently,
\[
\bar{\eta}_{2k-1}(n) = \sum_{m=1}^{\infty} \left( \frac{m + k - 1}{2k - 1} \right) N(m, n)
\]  
(1.3)
and
\[
\bar{\eta}_{2k}(n) = \sum_{m=1}^{\infty} \left( \frac{m + k - 1}{2k} \right) N(m, n).
\]  
(1.4)

Furthermore, it is easy to see that for any \(k\), \(\bar{\eta}_k(n)\) is a linear combination of the positive moments \(\bar{N}_j(n)\) of ranks introduced by Andrews, Chan and Kim [2] as given by
\[
\bar{N}_j(n) = \sum_{m=1}^{\infty} m^j N(m, n).
\]

For example,
\[
\bar{\eta}_4(n) = \frac{1}{24} \bar{N}_4(n) - \frac{1}{12} \bar{N}_3(n) - \frac{1}{24} \bar{N}_2(n) + \frac{1}{12} \bar{N}_1(n),
\]
\[
\bar{\eta}_5(n) = \frac{1}{120} \bar{N}_5(n) - \frac{1}{24} \bar{N}_3(n) + \frac{1}{30} \bar{N}_1(n).
\]

By the symmetry \(N(-m, n) = N(m, n)\), it is readily seen that
\[
\eta_{2k}(n) = 2\bar{\eta}_{2k}(n) + \bar{\eta}_{2k-1}(n).
\]  
(1.5)

The main objective of this paper is to give combinatorial interpretations of \(\bar{\eta}_{2k}(n)\) and \(\bar{\eta}_{2k-1}(n)\). We show that for given \(k\) and \(i\) with \(1 \leq i \leq k + 1\), \(\bar{\eta}_{2k-1}(n)\) equals the number of \((k + 1)\)-marked Durfee symbols of \(n\) with the \(i\)-th rank being zero and \(\bar{\eta}_{2k}(n)\) equals the number of \((k + 1)\)-marked Durfee symbols of \(n\) with the \(i\)-th rank being positive. It should be noted that \(\bar{\eta}_{2k-1}(n)\) and \(\bar{\eta}_{2k}(n)\) are independent of \(i\) since the ranks of \(k\)-marked Durfee symbols are symmetric, see Andrews [1, Corollary 12].

With the aid of Theorem 2.1 and Theorem 2.2 together with the generating function (1.1) of \(N(m, n)\), we obtain the generating functions of \(\bar{\eta}_{2k}(n)\) and \(\bar{\eta}_{2k-1}(n)\).

### 2 Combinatorial interpretations

In this section, we give combinatorial interpretations of \(\bar{\eta}_{2k-1}(n)\) and \(\bar{\eta}_{2k}(n)\) in terms of \(k\)-marked Durfee symbols. For a partition \(\lambda\) of \(n\), we write \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)\) with the entries \(\lambda_i\) in nonincreasing order such that \(\lambda_1 + \lambda_2 + \cdots + \lambda_s = n\). We assume that all the parts of \(\lambda\) are positive. The number of parts of \(\lambda\) is called the length of \(\lambda\), denoted by \(\ell(\lambda)\). The weight of \(\lambda\) is the sum of parts, denoted \(|\lambda|\).

Recall that a \(k\)-marked Durfee symbol of \(n\) introduced by Andrews [1] is a two-line array composed of \(k\) pairs of partitions \((\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^k, \beta^k)\) along with a positive integer \(D\) which is represented in the following form:
\[
\tau = \left( \begin{array}{cccc}
\alpha^k, & \alpha^{k-1}, & \ldots, & \alpha^1 \\
\beta^k, & \beta^{k-1}, & \ldots, & \beta^1
\end{array} \right)_D,
\]
where the partitions $\alpha^i = (\alpha^i_1, \alpha^i_2, \ldots, \alpha^i_s)$ and $\beta^i = (\beta^i_1, \beta^i_2, \ldots, \beta^i_s)$ satisfy the following four conditions:

1. The partitions $\alpha^i \ (1 \leq i < k)$ are nonempty, while $\alpha^k$ and $\beta^i \ (1 \leq i \leq k)$ are allowed to be empty;
2. $\beta^{i-1}_1 \leq \alpha^{i-1}_1 \leq \min\{\alpha_s, \beta_s\}$ for $2 \leq i \leq k$;
3. $\alpha^k_1, \beta^k_1 \leq D$;
4. $\sum_{i=1}^k (|\alpha^i| + |\beta^i|) + D^2 = n$.

Let $\tau = \left( \begin{array}{c} \alpha^k, \alpha^{k-1}, \ldots, \alpha^1 \\ \beta^k, \beta^{k-1}, \ldots, \beta^1 \end{array} \right)_D$ be a $k$-marked Durfee symbol. The pair $(\alpha^i, \beta^i)$ of partitions is called the $i$-th vector of $\tau$. Andrews defined the $i$-th rank $\rho_i(\tau)$ of $\tau$ as follows

$$\rho_i(\tau) = \begin{cases} \ell(\alpha^i) - \ell(\beta^i) - 1, & \text{for } 1 \leq i < k, \\ \ell(\alpha^k) - \ell(\beta^k), & \text{for } i = k. \end{cases}$$

For example, consider the following 3-marked Durfee symbol

$$\tau = \left( \begin{array}{c} 3, \ 4, \ 3 \\ 4, \ 2, \ 3, \ 2, \ 2 \ \\ \beta^3, \ 3, \ 2, \ 2 \ \\ \beta^2, \ 2, \ 2, \ 2 \ \\ \beta^1, \ 2, \ 1, \ 1 \ \\ \beta^1 \end{array} \right).$$

We have $\rho_1(\tau) = -2$, $\rho_2(\tau) = 0$, and $\rho_3(\tau) = 1$.

For an odd symmetrized moment $\eta_{2k-1}(n)$, we have the following combinatorial interpretation.

**Theorem 2.1.** For given positive integers $k$ and $i$ with $1 \leq i \leq k + 1$, $\eta_{2k-1}(n)$ is equal to the number of $(k + 1)$-marked Durfee symbols of $n$ with the $i$-th rank equal to zero.

For the even case, we have the following interpretation.

**Theorem 2.2.** For given positive integers $k$ and $i$ with $1 \leq i \leq k + 1$, $\eta_{2k}(n)$ is equal to the number of $(k + 1)$-marked Durfee symbols of $n$ with the $i$-th rank being positive.

The proofs of the above two interpretations are based on the following partition identity obtained by Ji [9]. We shall adopt the notation $D_k(m_1, m_2, \ldots, m_k; n)$ as used by Andrews [1] to denote the number of $k$-marked Durfee symbols of $n$ with the $i$-th rank equal to $m_i$ for $1 \leq i \leq k$. 


Theorem 2.3. For \( k \geq 2 \) and \( n \geq 1 \), we have

\[
D_k(m_1, \ldots, m_k; n) = \sum_{t_1, \ldots, t_{k-1}=0}^{\infty} N \left( \sum_{i=1}^{k} |m_i| + 2 \sum_{i=1}^{k-1} t_i + k - 1, n \right). \tag{2.1}
\]

To derive the above interpretations of \( \bar{\eta}_{2k-1}(n) \) and \( \bar{\eta}_{2k}(n) \), we also need the following symmetric property given by Andrews [1]. Boulet and Kursungöz [5] found a combinatorial proof of this fact.

Theorem 2.4. For \( k \geq 2 \) and \( n \geq 1 \), \( D_k(m_1, \ldots, m_k; n) \) is symmetric in \( m_1, m_2, \ldots, m_k \).

We are now in a position to prove Theorem 2.1 and Theorem 2.2.

Proof of Theorem 2.1. By Theorem 2.4, it suffices to show that

\[
\sum_{m_2, m_3, \ldots, m_{k+1}=-\infty}^{\infty} D_{k+1}(0, m_2, m_3, \ldots, m_{k+1}; n) = \bar{\eta}_{2k-1}(n). \tag{2.2}
\]

Using Theorem 2.3, we get

\[
\sum_{m_2, m_3, \ldots, m_{k+1}=-\infty}^{\infty} D_{k+1}(0, m_2, m_3, \ldots, m_{k+1}; n)
= \sum_{m_2, m_3, \ldots, m_{k+1}=-\infty}^{\infty} \sum_{t_1, \ldots, t_k=0}^{\infty} N \left( \sum_{i=2}^{k} |m_i| + 2 \sum_{i=1}^{k-1} t_i + k, n \right). \tag{2.3}
\]

For \( k \geq 1 \) and \( m \geq k \), let \( c_k(m) \) denote the number of integer solutions to the equation

\[
|m_2| + \cdots + |m_{k+1}| + 2t_1 + \cdots + 2t_k = m - k,
\]

where \( m_i \) are integers and \( t_i \) are nonnegative integers. It is easy to see that the generating function of \( c_k(m) \) is equal to

\[
\sum_{m=k}^{\infty} c_k(m) q^{m-k} = (1 + 2q + 2q^2 + 2q^3 + \cdots)^k (1 + q^2 + q^4 + q^6 + \cdots)^k
= \left( \frac{1+q}{1-q} \right)^k \left( \frac{1}{1-q^2} \right)^k
= \frac{1}{(1-q)^{2k}}
= \sum_{m=k}^{\infty} \binom{m + k - 1}{2k - 1} q^{m-k}. \tag{2.4}
\]

Hence

\[
c_k(m) = \binom{m + k - 1}{2k - 1},
\]
and (2.3) can be written as
\[
\sum_{m_2, m_3, \ldots, m_{k+1} = -\infty}^{\infty} D_{k+1}(0, m_2, m_3, \ldots, m_{k+1}; n) = \sum_{m=1}^{\infty} \binom{m + k - 1}{2k - 1} N(m, n),
\]
which is the defining expression of $\bar{\eta}_{2k-1}(n)$. This completes the proof. \(\square\)

**Proof of Theorem 2.2.** Similarly, by Theorem 2.4, it is sufficient to show that
\[
\sum_{m_1 > 0, m_2, m_3, \ldots, m_{k+1} = -\infty}^{\infty} D_{k+1}(m_1, m_2, \ldots, m_{k+1}; n) = \bar{\eta}_{2k}(n). \tag{2.5}
\]
Invoking Theorem 2.3, we get
\[
\sum_{m_1 > 0, m_2, m_3, \ldots, m_{k+1} = -\infty}^{\infty} D_{k+1}(m_1, m_2, \ldots, m_{k+1}; n) = \sum_{m_1 > 0}^{\infty} \sum_{m_2, m_3, \ldots, m_{k+1} = -\infty}^{\infty} N \left( m_1 + \sum_{i=2}^{k+1} |m_i| + 2 \sum_{i=1}^{k} t_i + k, n \right). \tag{2.6}
\]
For $k \geq 1$ and $m \geq k + 1$, let $\tilde{c}_k(m)$ denote the number of integer solutions to the equation
\[
m_1 + |m_2| + \cdots + |m_{k+1}| + 2t_1 + \cdots + 2t_k = m - k,
\]
where $m_1$ is a positive integer, $m_i$ ($2 \leq i \leq k + 1$) are integers and $t_i$ are nonnegative integers. An easy computation shows that
\[
\sum_{m=k+1}^{\infty} \tilde{c}_k(m) q^{m-k} = \frac{q}{(1 - q)^{2k+1}}, \tag{2.7}
\]
so that
\[
\tilde{c}_k(m) = \binom{m + k - 1}{2k}.
\]
Thus, the sum on the right hand side of (2.6) becomes
\[
\sum_{m=1}^{\infty} \binom{m + k - 1}{2k} N(m, n),
\]
which is in accordance with the definition of $\bar{\eta}_{2k}(n)$, and hence the proof is complete. \(\square\)

Note that the number $D_k(m_1, \ldots, m_k; n)$ has the mirror symmetry with respect to each $m_i$, that is, for $1 \leq i \leq k$, we have
\[
D_k(m_1, \ldots, m_i, \ldots, m_k; n) = D_k(m_1, \ldots, -m_i, \ldots, m_k; n).
\]
Using this symmetry property, Theorem 2.2 can be restated as follows.
Theorem 2.5. For given positive integers $k$ and $i$ with $1 \leq i \leq k + 1$, $\bar{\eta}_{2k}(n)$ is also equal to the number of $(k + 1)$-marked Durfee symbols of $n$ with the $i$-th rank being negative.

$$
\begin{array}{ccc}
\bar{\eta}_1(5) & \bar{\eta}_2(5) & \bar{\eta}_2(5) \\
\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix} \\
\begin{pmatrix} 1 & 2 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \end{pmatrix} \\
\begin{pmatrix} 1 & 2 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \end{pmatrix} \\
\begin{pmatrix} 1 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 1 \end{pmatrix} & \begin{pmatrix} 1 \end{pmatrix} \\
\begin{pmatrix} 1 & 2 \end{pmatrix} & \begin{pmatrix} 2 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 1 \end{pmatrix} \\
\begin{pmatrix} 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 \end{pmatrix} \\
\end{array}
$$

Table 2.1: 2-Marked Durfee Symbols of 5.

For example, for $n = 5$, $k = 1$ and $i = 1$, there are twenty-one 2-marked Durfee symbols of 5 as listed in Table 2.1. The first column in Table 2.1 gives seven 2-marked Durfee symbols $\tau$ with $\rho_1(\tau) = 0$, the second column contains seven 2-marked Durfee symbols $\tau$ with $\rho_1(\tau) > 0$ and the third column contains seven 2-marked Durfee symbols $\tau$ with $\rho_1(\tau) < 0$. It can be verified that $\bar{\eta}_1(5) = 7$, $\bar{\eta}_2(5) = 7$ and $\eta_2(5) = \bar{\eta}_1(5) + 2\bar{\eta}_2(5) = 21$.

3 The generating functions of $\bar{\eta}_{2k-1}(n)$ and $\bar{\eta}_{2k}(n)$

In this section, we obtain the generating functions of $\bar{\eta}_{2k-1}(n)$ and $\bar{\eta}_{2k}(n)$ with the aid of Theorem 2.1 and Theorem 2.2. In doing so, we use the generating function of $N(m, n)$ to derive the generating functions of $D_{k+1}(0, m_2, \ldots, m_{k+1}; n)$ and $D_{k+1}(m_1, m_2, \ldots, m_{k+1}; n)$. 

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Theorem 3.1. For \( k \geq 1 \), we have

\[ \sum_{m_2, \ldots, m_{k+1} = -\infty}^{\infty} D_{k+1}(0, m_2, \ldots, m_{k+1}; n) x_1^{m_2} \cdots x_k^{m_{k+1}} q^n \]

\[ = \frac{1}{(q; q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2 + k n} \prod_{j=1}^{k} (1 - x_j q^n) (1 - x_j^{-1} q^n). \] \hspace{1cm} (3.1)

Proof. Let

\[ G_k(x_1, \ldots, x_k; q) = \sum_{m_2, \ldots, m_{k+1} = -\infty}^{\infty} D_{k+1}(0, m_2, \ldots, m_{k+1}; n) x_1^{m_2} \cdots x_k^{m_{k+1}} q^n. \]

By Theorem 2.3, we have

\[ G_k(x_1, \ldots, x_k; q) = \sum_{m_2, \ldots, m_{k+1} = -\infty}^{\infty} \sum_{t_1, \ldots, t_k = 0}^{\infty} x_1^{m_2} \cdots x_k^{m_{k+1}} \sum_{n=0}^{\infty} N \left( \sum_{i=2}^{k+1} |m_i| + 2 \sum_{i=1}^{k} t_i + k, n \right) q^n. \] \hspace{1cm} (3.2)

Using the generating function (1.1) of \( N(m, n) \) with \( m \) replaced by \( \sum_{i=2}^{k+1} |m_i| + 2 \sum_{i=1}^{k} t_i + k \), we find that

\[ \sum_{n=0}^{\infty} N \left( \sum_{i=2}^{k+1} |m_i| + 2 \sum_{i=1}^{k} t_i + k, n \right) q^n \]

\[ = \frac{1}{(q; q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2 + n(\sum_{i=2}^{k+1} |m_i| + 2 \sum_{i=1}^{k} t_i + k)} (1 - q^n). \] \hspace{1cm} (3.3)

Substituting (3.3) into (3.2), we get

\[ G_k(x_1, \ldots, x_k; q) = \sum_{m_2, \ldots, m_{k+1} = -\infty}^{\infty} \sum_{t_1, \ldots, t_k = 0}^{\infty} x_1^{m_2} \cdots x_k^{m_{k+1}} \times \frac{1}{(q; q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2 + n(\sum_{i=2}^{k+1} |m_i| + 2 \sum_{i=1}^{k} t_i + k)} (1 - q^n). \] \hspace{1cm} (3.4)

Write (3.4) in the following form

\[ G_k(x_1, \ldots, x_k; q) = \frac{1}{(q; q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2 + k n} (1 - q^n) \]

\[ \times \sum_{m_2, \ldots, m_{k+1} = -\infty}^{\infty} \sum_{t_1, \ldots, t_k = 0}^{\infty} x_1^{m_2} \cdots x_k^{m_{k+1}} q^{n(\sum_{i=2}^{k+1} |m_i| + 2 \sum_{i=1}^{k} t_i)}. \] \hspace{1cm} (3.5)
Notice that
\[
\sum_{a=-\infty}^{+\infty} \sum_{b=0}^{+\infty} x^a q^{n|a|+2b} = \frac{1}{(1-xq^n)(1-x^{-1}q^n)}.
\] (3.6)

Applying the above formula (3.6) repeatedly to (3.5), we deduce that
\[
G_k(x_1, \ldots, x_k; q) = \frac{1}{(q; q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2+kn} \prod_{j=1}^{k} (1-x_j q^n)(1-x_j^{-1} q^n),
\]
as required. \(\square\)

Setting \(x_j = 1\) for \(1 \leq j \leq k\) in Theorem 3.1 and applying Theorem 2.1, we arrive at the following generating function of \(\bar{\eta}_{2k-1}(n)\).

**Corollary 3.2.** For \(k \geq 1\), we have
\[
\sum_{n=1}^{\infty} \bar{\eta}_{2k-1}(n) q^n = \frac{1}{(q; q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2+kn} \prod_{j=2}^{k} (1-x_j q^n)(1-x_j^{-1} q^n). \tag{3.7}
\]

Since \(\bar{\eta}_1(n) = \overline{N}_1(n)\), when taking \(k = 1\) in (3.7), we are led to the generating function for \(\overline{N}_1(n)\) as given by Andrews, Chan and Kim in [2, Theorem 1].

The following generating function can be derived by using the same reasoning as in the proof of Theorem 3.1.

**Theorem 3.3.** For \(k \geq 1\), we have
\[
\sum_{m_2, \ldots, m_{k+1} = -\infty}^{m_1 > 0} \sum_{n=1}^{\infty} D_{k+1}(m_1, m_2, \ldots, m_{k+1}; n) x_1^{m_1} \cdots x_{k+1}^{m_{k+1}} q^n
= \frac{1}{(q; q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n+1)/2+kn} \frac{x_1(1-q^n)}{(1-x_1 q^n) \prod_{j=2}^{k+1} (1-x_j q^n)(1-x_j^{-1} q^n)}. \tag{3.8}
\]

Setting \(x_j = 1\) for \(1 \leq j \leq k+1\) in Theorem 3.3 and using Theorem 2.2, we come to the following generating function of \(\bar{\eta}_{2k}(n)\).

**Corollary 3.4.** For \(k \geq 1\), we have
\[
\sum_{n=1}^{\infty} \bar{\eta}_{2k}(n) q^n = \frac{1}{(q; q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n+1)/2+kn} \frac{1}{(1-q^n)^{2k}}. \tag{3.9}
\]

**Acknowledgments.** We wish to thank the referee for helpful suggestions. This work was supported by the 973 Project and the National Science Foundation of China.
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