Distance-regular graphs with an eigenvalue $-k < \theta \leqslant 2 - k$

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Abstract

It is known that bipartite distance-regular graphs with diameter $D \geqslant 3$, valency $k \geqslant 3$, intersection number $c_2 \geqslant 2$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$ satisfy $\theta_1 \leqslant k-2$ and thus $\theta_{D-1} \geqslant 2-k$. In this paper we classify non-complete distance-regular graphs with valency $k \geqslant 2$, intersection number $c_2 \geqslant 2$ and an eigenvalue θ satisfying $-k < \theta \leqslant 2-k$. Moreover, we give a lower bound for valency k which implies $\theta_D \geqslant 2-k$ for distance-regular graphs with girth $g \geqslant 5$ satisfying $g = 5$ or $g \equiv 3 \pmod 4$.

Keywords: Distance-regular graph; Girth; Smallest eigenvalue; Folded $(2D+1)$-cube

1 Introduction

Let Γ be a distance-regular graph with diameter $D \geqslant 3$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. It is shown in [2, Theorem 4.4.3 (ii)] that if $c_2 \geqslant 2$ then either Γ is the icosahedron or Γ satisfies $\theta_1 \leqslant b_1 - 1$. Distance-regular graphs with $c_2 \geqslant 2$ and $\theta_1 = b_1 - 1$ are classified (see [2, Theorem 4.4.11]). In particular, any non-complete bipartite distance-regular graph Γ with valency $k \geqslant 2$, intersection number $c_2 \geqslant 2$ and an eigenvalue θ with $-k < \theta \leqslant 2-k$ satisfies $\theta = 2-k$ and Γ is either the cycle of length four or the Hamming D-cube by $2-k \leqslant -\theta_1 \leqslant \theta \leqslant 2-k$ and [2, Theorem 4.4.11].

In the following theorem we classify non-complete distance-regular graphs with valency $k \geqslant 2$, intersection number $c_2 \geqslant 2$ and an eigenvalue θ satisfying $-k < \theta \leqslant 2-k$.

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Theorem 1. *Let* Γ *be a distance-regular graph with diameter* $D \geqslant 2$*, valency* $k \geqslant 2$ *and intersection number* $c_2 \geqslant 2$*. If there exists an eigenvalue* θ *of* Γ *satisfying* $-k < \theta \leqslant 2-k$ *then* $\theta = 2-k$ *and* Γ *is one of the following:*
(i) the cycle of length four,
(ii) the Johnson graph $J(4,2)$*,*
(iii) the 3×3*-grid,*
(iv) the Hamming D*-cube* $H(D,2)$*, or*
(v) the folded $(2D+1)$*-cube.*

The folded n-cube ($n \neq 6$) is uniquely characterized by its intersection array (cf. [2, Theorem 9.2.7]). It follows by Theorem 1 that a distance-regular graph with $D \geqslant 3$, $k \geqslant 3$, $c_2 \geqslant 2$ and an eigenvalue θ satisfying $-k < \theta \leqslant 2-k$ is either the Hamming D-cube or the folded $(2D+1)$-cube.
A distance-regular graph Γ with diameter $D \geqslant 3$ and girth $g = 3$ is either the icosahedron or Γ satisfies $\theta_D \geqslant -\frac{b_1}{2} - 1$ (cf. [2, Theorem 4.4.3 (iii)]). Distance-regular graphs with $a_1 \geqslant 2$ and $\theta_D = -\frac{b_1}{2} - 1$ are classified in [4] (see also [5]). There are non-complete distance-regular graphs with girth $g \geqslant 4$ and an eigenvalue θ satisfying $-k < \theta < -\frac{b_1}{2} - 1$, such as the Hamming D-cube ($D \geqslant 6$) and the folded $(2D+1)$-cube ($D \geqslant 3$) which have $2-k$ as an eigenvalue. If $g = 4$ then any eigenvalue $\theta \neq -k$ satisfies $\theta \geqslant 2-k$ (see Theorem 1). In Theorem 2 and Theorem 3 we study distance-regular graphs with girth $g \geqslant 5$ satisfying either $g = 5$ or $g \equiv 3 \pmod 4$, and give a lower bound for valency k which implies $\theta_D \geqslant 2-k$ by considering a lower bound for $\frac{\theta_D}{k}$.

Theorem 2. *Let* Γ *be a distance-regular graph with diameter* $D \geqslant 2$*, valency* $k \geqslant 3$ *and girth* $g = 5$*. Then the smallest eigenvalue* θ_D *of* Γ *satisfies*

$$\theta_D \geqslant \left(\frac{1-\sqrt{73}}{9}\right) k. \tag{1}$$

In particular, if $k \geqslant 10$ *then* $\theta_D > 2-k$*.*

Theorem 3. *Let* Γ *be a distance-regular graph with diameter* $D \geqslant 3$*, valency* $k \geqslant 3$ *and girth* $g > 3$ *satisfying* $g \equiv 3 \pmod 4$*. Then there exist real numbers* $C(g) \geqslant 3$ *and* $\gamma(g) \in (-1, -0.64)$ *(depending only on* g*) such that if* $k \geqslant C(g)$ *then the smallest eigenvalue* θ_D *satisfies*

$$\theta_D \geqslant \gamma(g)k.$$

In particular, if $k \geqslant \max\left\{C(g), \frac{2}{\gamma(g)+1}\right\}$ *then* $\theta_D \geqslant 2-k$*.*

The paper is organized as follows. In Section 2 we review some definitions and basic concepts. In Section 3 we prove Theorem 1. In the last section we prove Theorem 2 and Theorem 3. As an example of Theorem 3, we will consider the case $g = 7$ (see Example 11).

2 Preliminaries

All the graphs considered in this paper are finite, undirected and simple. The reader is referred to [2] for more background information. For a connected graph $\Gamma = (V(\Gamma), E(\Gamma))$, the distance $d(x, y)$ between two vertices x, y of Γ is the length of a shortest path between x and y in Γ, and the diameter D is the maximum distance between any two vertices of Γ. For any vertex $x \in V(\Gamma)$, let $\Gamma_i(x)$ be the set of vertices in Γ at distance i from x $(0 \leqslant i \leqslant D)$. The *adjacency matrix* A is the $|V(\Gamma)| \times |V(\Gamma)|$-matrix with rows and columns indexed by $V(\Gamma)$, where the (x, y)-entry of A equals 1 whenever $d(x, y) = 1$ and 0 otherwise. The eigenvalues of Γ are the eigenvalues of A. The *girth* of Γ, denoted by g, is the length of a shortest cycle in Γ. A connected graph Γ is called a *distance-regular graph* if there exist integers b_i, c_i, $i = 0, 1, \ldots, D$, such that for any two vertices x, y at distance $i = d(x, y)$, there are precisely c_i neighbors of y in $\Gamma_{i-1}(x)$ and b_i neighbors of y in $\Gamma_{i+1}(x)$. In particular, Γ is regular with valency $k := b_0$. The numbers b_i, c_i and $a_i := k - b_i - c_i$ $(0 \leqslant i \leqslant D)$ are called the *intersection numbers* of Γ. Set $c_0 = b_D = 0$. We observe $a_0 = 0$ and $c_1 = 1$. The array

$$\iota(\Gamma) = \{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$$

is called the *intersection array* of Γ. The intersection numbers satisfy the following restrictions

$$1 = c_1 \leqslant c_2 \leqslant \cdots \leqslant c_D \leqslant k \quad \text{and} \quad k = b_0 \geqslant b_1 \geqslant \cdots \geqslant b_{D-1} \geqslant 1$$

(cf. [2, Proposition 4.1.6]). The following inequalities for intersection numbers of a distance-regular graph will be used in Section 3.

Lemma 4. *([2, Proposition 5.5.4 (ii)]) Let Γ be a distance-regular graph with diameter $D \geqslant 2$. Suppose that $a_i \neq 0$ for some $1 \leqslant i \leqslant D$, and define $a_{D+1} = 0$. If $i > 1$ then $c_i \leqslant a_i + \frac{a_{i-1}c_i}{a_i}$, and for $i < D$ equality implies $a_{i+1} = a_i$.*

Suppose that Γ is a distance-regular graph with diameter $D \geqslant 2$ and valency $k \geqslant 2$. We define $k_i := |\Gamma_i(x)|$ for any vertex x and $i = 0, 1, \ldots, D$. Then we have

$$k_0 = 1, \quad k_1 = b_0, \quad k_{i+1} = \frac{k_i b_i}{c_{i+1}} \quad (i = 0, 1, \ldots, D-1).$$

It is known that Γ has exactly $D + 1$ distinct eigenvalues which are the eigenvalues of the tridiagonal matrix

$$L_1(\Gamma) := \begin{pmatrix} 0 & b_0 & & & & & & \\ c_1 & a_1 & b_1 & & & & & \\ & c_2 & a_2 & b_2 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & c_i & a_i & b_i & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & c_{D-1} & a_{D-1} & b_{D-1} \\ & & & & & & c_D & a_D \end{pmatrix}$$

(cf. [2, p.128]). Let $k = \theta_0 > \theta_1 > \cdots > \theta_D$ be the $D+1$ distinct eigenvalues of Γ. A *clique* is a set of pairwise adjacent vertices. Any clique C in Γ satisfies

$$|C| \leqslant 1 - \frac{k}{\theta_D} \tag{2}$$

(see [2, Proposition 4.4.6 (i)]). The *standard sequence* $u_i = u_i(\theta)$ $(0 \leqslant i \leqslant D)$ corresponding to an eigenvalue θ is a sequence satisfying $u_0 = 1$, $u_1 = \frac{\theta}{k}$ and

$$c_i u_{i-1} + a_i u_i + b_i u_{i+1} = \theta u_i \quad (1 \leqslant i \leqslant D) \tag{3}$$

(cf. [2, p. 128]). The multiplicity of eigenvalue θ is given by

$$m(\theta) = \frac{|V(\Gamma)|}{\sum_{i=0}^{D} k_i u_i^2(\theta)}$$

which is known as *Biggs' formula* (cf. [1, Theorem 21.4], [2, Theorem 4.1.4]). Let $\theta \neq k$ be an eigenvalue of Γ with multiplicity $m = m(\theta)$. Then there exists a map $\rho : V(\Gamma) \to \mathbb{R}^m$ such that
(i) $\sum_{x \in V(\Gamma)} \rho(x) = 0$ and
(ii) for any two vertices x, y with $d(x, y) = i$, the inner product satisfies $\langle \rho(x), \rho(y) \rangle = u_i(\theta)$ where $\mathbb{R}$ is the real numbers (see [2, Proposition 4.4.1]). The map ρ is called the *standard representation* of Γ corresponding to θ.

3 Proof of Theorem 1

In this section we classify distance-regular graphs with intersection numbers a_1 and c_2 satisfying $(a_1, c_2) \neq (0, 1)$ and an eigenvalue θ satisfying $-k < \theta \leqslant 2 - k$. We first consider distance-regular graphs with $a_1 \geqslant 1$. Using the classification of distance-regular graphs with valency four by Brouwer and Koolen [3], we obtain the following lemma.

Lemma 5. *Let Γ be a distance-regular graph with diameter $D \geqslant 2$, valency $k \geqslant 3$ and intersection number $a_1 \geqslant 1$. If the smallest eigenvalue θ_D satisfies $\theta_D \leqslant 2 - k$ then $k = 4$ and Γ is one of the following:*
(i) the Johnson graph $J(4,2)$ with $\iota(\Gamma) = \{4, 1; 1, 4\}$,
(ii) the 3×3-grid with $\iota(\Gamma) = \{4, 2; 1, 2\}$,
(iii) the line graph of Petersen graph with $\iota(\Gamma) = \{4, 2, 1; 1, 1, 4\}$,
(iv) the flag graph of PG(2,2) with $\iota(\Gamma) = \{4, 2, 2; 1, 1, 2\}$,
(v) the flag graph of GQ(2,2) with $\iota(\Gamma) = \{4, 2, 2, 2; 1, 1, 1, 2\}$, or
(vi) the flag graph of GH(2,2) with $\iota(\Gamma) = \{4, 2, 2, 2, 2, 2; 1, 1, 1, 1, 1, 2\}$.

Proof. Suppose that θ_D satisfies $\theta_D \leqslant 2 - k$. By $a_1 \geqslant 1$ and (2), each clique C in Γ satisfies

$$3 \leqslant |C| \leqslant 1 - \frac{k}{\theta_D} \leqslant 1 + \frac{k}{k-2}.$$

Hence we find $k \leqslant 4$. Since there are no distance-regular graphs satisfying $D \geqslant 2$, $k = 3$ and $a_1 \geqslant 1$, we obtain $k = 4$ and thus the result follows by [3, Theorem 1.1]. □

Using [2, Proposition 4.4.9 (i)], we obtain Lemma 6 (i). The result Lemma 6 (ii) is shown by Terwilliger ([6], cf. [2, Theorem 5.2.1]).

Lemma 6. *Let Γ be a distance-regular graph with diameter $D \geqslant 2$. If Γ contains an induced quadrangle then the following hold.*
(i) For any eigenvalue θ, $u_0(\theta) + 2u_1(\theta) + u_2(\theta) \geqslant 0$.
(ii) For each $i = 1, 2, \dots, D$, $c_i - b_i \geqslant c_{i-1} - b_{i-1} + a_1 + 2$.

Proof. Suppose that Γ contains an induced quadrangle, say $Q = x_0x_1x_2x_3$ where

$$d(x_i, x_{i+1}) = 1 = d(x_0, x_3) \quad i = 0, 1, 2.$$

(i): Let ρ be the standard representation of Γ corresponding to an eigenvalue θ, and put $\alpha := \rho(x_0) + \rho(x_2)$ and $\beta := \rho(x_1) + \rho(x_3)$. Then the result (i) follows from

$$0 \leqslant \langle \alpha + \beta, \alpha + \beta \rangle = \langle \alpha, \alpha \rangle + 2\langle \alpha, \beta \rangle + \langle \beta, \beta \rangle = 4(u_0(\theta) + 2u_1(\theta) + u_2(\theta)).$$

(ii): This result is shown by Terwilliger ([6], cf. [2, Theorem 5.2.1]). □

To complete the proof of Theorem 1, we consider triangle-free distance-regular graphs in Lemma 7 and Lemma 8. If Γ contains an induced quadrangle then the inequality $u_0(\theta) - 2u_1(\theta) + u_2(\theta) \geqslant 0$ in [2, Proposition 4.4.9 (i)] is equivalent to either $\theta = k$ or $\theta \leqslant b_1 - 1$. In the following lemma we consider an equivalent condition to the inequality $u_0(\theta) + 2u_1(\theta) + u_2(\theta) \geqslant 0$ of Lemma 6 (i) when Γ is a non-complete triangle-free distance-regular graph.

Lemma 7. *Let Γ be a triangle-free distance-regular graph with diameter $D \geqslant 2$ and valency $k \geqslant 2$. For an eigenvalue θ of Γ, $u_0(\theta) + 2u_1(\theta) + u_2(\theta) \geqslant 0$ holds if and only if $\theta = -k$ or $\theta \geqslant 2 - k$.*

Proof. Let θ be an eigenvalue of Γ. It follows by $(c_1, a_1, b_1) = (1, 0, k-1)$ and (3) that $u_0(\theta) + 2u_1(\theta) + u_2(\theta) = \frac{(\theta+k)(\theta+k-2)}{k(k-1)}$, from which the result follows as $\theta \geqslant -k$. □

Lemma 8. *Let Γ be a triangle-free distance-regular graph with diameter $D \geqslant 2$ and valency $k \geqslant 2$. If Γ contains an induced quadrangle and $2 - k$ is an eigenvalue of Γ then the following hold.*
(i) $u_i(2-k) = (-1)^i \left(1 - \frac{2i}{k}\right) \quad (0 \leqslant i \leqslant D)$.
(ii) $(k - 1 - 2i)\ a_i = 2(c_i - i) \quad (1 \leqslant i \leqslant D)$.

Proof. Suppose that Γ contains an induced quadrangle and $2 - k$ is an eigenvalue of Γ. Let ρ be the standard representation of Γ corresponding to eigenvalue $2 - k$, and let $Q = x_0x_1x_2x_3$ be an induced quadrangle where $d(x_i, x_{i+1}) = 1 = d(x_0, x_3)$ $i = 0, 1, 2$. Put $\alpha := \rho(x_0) + \rho(x_2)$ and $\beta := \rho(x_1) + \rho(x_3)$.
(i): Using (3) with $\theta = 2 - k$ and $(c_1, a_1, b_1) = (1, 0, k-1)$, we find $u_0(2-k) + 2u_1(2-k) + u_2(2-k) = 0$ and thus $\alpha + \beta = 0$ follows from

$$\langle \alpha + \beta, \alpha + \beta \rangle = \langle \alpha, \alpha \rangle + 2\langle \alpha, \beta \rangle + \langle \beta, \beta \rangle = 4(u_0(\theta) + 2u_1(\theta) + u_2(\theta)) = 0.$$

As $\alpha+\beta = 0$ and $\Gamma_{i+2}(x_0)\cap\Gamma_i(x_2) \subseteq \Gamma_{i+1}(x_1)\cap\Gamma_{i+1}(x_3)$ $(i = 0, 1, \ldots, D-2)$, the following holds for each vertex $v \in \Gamma_{i+2}(x_0) \cap \Gamma_i(x_2)$ $(i = 0, 1, \ldots, D-2)$:

$$0 = \langle \alpha + \beta, \rho(v)\rangle = u_i(2-k) + 2u_{i+1}(2-k) + u_{i+2}(2-k). \tag{4}$$

As $u_0(2-k) = 1$ and $u_1(2-k) = \frac{2-k}{k}$, (i) follows from (4).
(ii): It follows by (3) with $\theta = 2-k$ that $c_i u_{i-1}(2-k) + (a_i - 2 + k)u_i(2-k) + (k - a_i - c_i)u_{i+1}(2-k) = 0$, and this shows the result by Lemma 8 (i). □

We now classify non-complete distance-regular graphs with $k \geqslant 2$, $c_2 \geqslant 2$ and an eigenvalue θ satisfying $-k < \theta \leqslant 2-k$.

Proof of Theorem 1. Suppose that θ is an eigenvalue of Γ satisfying $-k < \theta \leqslant 2-k$. If Γ is bipartite then there is an induced quadrangle as $c_2 \geqslant 2$. By Lemma 6 (i) and Lemma 7, $\theta = 2-k = 1-b_1 = \theta_{D-1} = -\theta_1$. By [2, Theorem 4.4.11], Γ is either (i) or (iv).
In the rest of the proof, we assume that Γ is not bipartite and put

$$m := \min\{i \mid a_i \geqslant 1,\ 1 \leqslant i \leqslant D\}.$$

Then $1 \leqslant m \leqslant D$. If $m = 1$ then it follows by Lemma 5 that $\theta = 2-k = -2$ and Γ is either (ii) or (iii). Now suppose $2 \leqslant m \leqslant D$. As $c_2 \geqslant 2$ and $m \geqslant 2$, Γ contains an induced quadrangle. By Lemma 6 (i) and Lemma 7,

$$\theta = 2-k. \tag{5}$$

We first show the following claim.

Claim 9. $m = D$

Proof of Claim 9. Assume $2 \leqslant m \leqslant D-1$. Then by Lemma 4,

$$c_m \leqslant a_m \text{ and the equality implies } \ a_{m+1} = a_m. \tag{6}$$

By (5), $m \geqslant 2$ and Lemma 8 (ii), we find

$$(k-1-2m)a_m = 2(c_m - m). \tag{7}$$

Using Lemma 6 (ii) with $a_i = 0$ $(1 \leqslant i \leqslant m-1)$ we have $c_i \geqslant c_{i-1} + 1$ $(1 \leqslant i \leqslant m-1)$ and thus $c_m \geqslant c_{m-1} \geqslant m-1$ follows. If $c_m = m-1$ then it follows by (6) and (7) that $2 \leqslant c_2 \leqslant m-1 = c_m \leqslant a_m \leqslant 2$ and thus $m = 3$, $k = 6$ and $a_{m+1} = a_m = 2$. The case $i = m+1 = 4$ of Lemma 8 (ii) implies $c_4 = c_{m+1} = 1$ which is impossible as $c_4 \geqslant c_2 \geqslant 2$. Hence we find $c_m \geqslant m$ and thus $k \geqslant 2m+1$ from (7). On the other hand, $2(c_m - m) = a_m(k-1-2m) \geqslant c_m(k-1-2m)$ holds by (6) and (7). Hence we find $2c_m \leqslant c_m + a_m \leqslant k \leqslant 2m+2$ and thus $c_m \in \{m, m+1\}$. If $c_m = m+1$ then $(c_m, a_m, b_m) = (m+1, m+1, 0)$, which contradicts to $m \leqslant D-1$. Hence $c_m = m$ and thus $k = 2m+1$ and $m = c_m = a_m = a_{m+1}$ by (6) and (7). The equation of Lemma 8 (ii) with $i = m+1$ yields $c_{m+1} = 1$. This is also impossible as $c_{m+1} \geqslant c_2 \geqslant 2$. Hence $m = D$. □

By Lemma 8 (ii), Claim 9 and (5), $a_i = 0$ and $c_i = i$ for all $i = 1, 2, \ldots, D-1$, i.e.,

$$\iota(\Gamma) = \{k, k-1, \ldots, k-D+2, k-D+1; 1, 2, \ldots, D-1, c_D\}.$$

Note here that $k \neq 2D-1$ otherwise we have $D = a_D + c_D = k = 2D-1$ by Lemma 8 (ii) with $i = D$, which contradicts to the condition $D \geqslant 2$. Applying Lemma 8 (ii) with $i = D$, we have

$$c_D = \frac{(k-2D)(k-1)}{k-2D+1}. \tag{8}$$

Since we have $a_D \geqslant c_D \geqslant c_{D-1} = D-1$ by Lemma 4, we find $\max\{2, D-1\} \leqslant c_D \leqslant \frac{k}{2}$ which implies $k \geqslant 4$ and $2(D-1) \leqslant k \leqslant 2D+2$ by (8). Moreover, it follows by $a_D \geqslant c_D$, Lemma 8 (ii) and (8) that $k = 2D+1$ and $c_D = D$. Therefore Γ has the same intersection array with the folded $(2D+1)$-cube,

$$\iota(\Gamma) = \{2D+1, 2D, \ldots, D+3, D+2; 1, 2, \ldots, D\}.$$

As the folded $(2D+1)$-cube is uniquely determined by its intersection numbers (cf. [2, Theorem 9.2.7]), Γ is the folded $(2D+1)$-cube. This completes the proof of Theorem 1. □

4 Proofs of Theorem 2 and Theorem 3

In this section we consider lower bounds for the smallest eigenvalue of a distance-regular graph with girth $g \in \{5, 4s-1 \mid s \geqslant 2\}$.

Let Γ be a distance-regular graph with diameter $D \geqslant 3$ and girth $g = 3$. Then by [2, Theorem 4.4.3 (iii)], Γ is either the icosahedron or Γ satisfies $\theta_D \geqslant -\frac{b_1}{2} - 1$. For both cases, the smallest eigenvalue θ_D satisfies $\theta_D \geqslant -\frac{1}{2}k$. In Theorem 2 and Theorem 3 we consider a lower bound for $\frac{\theta_D}{k}$ using girth g if $g > 3$ satisfies $g = 5$ or $g \equiv 3 \pmod 4$, and give a lower bound for valency k which implies $\theta_D \geqslant 2 - k$.

We first consider distance-regular graphs with girth $g = 5$ and prove Theorem 2.

Proof of Theorem 2. Let $P = x_0x_1x_2x_3x_4$ be an induced pentagon in Γ where $d(x_i, x_{i+1}) = 1 = d(x_0, x_4), i = 0, 1, 2, 3$. For the smallest eigenvalue $\theta = \theta_D$, let ρ be the corresponding standard representation and put $\alpha := (\rho(x_0) + \rho(x_1) + \rho(x_4)) - (\rho(x_2) + \rho(x_3))$. Then

$$\frac{k-1-\sqrt{31k^2+4k+1}}{6} \leqslant \theta \leqslant k < \frac{k-1+\sqrt{31k^2+4k+1}}{6} \tag{9}$$

follows by

$$0 \leqslant \langle \alpha, \alpha \rangle = 5u_0(\theta) + 2u_1(\theta) - 6u_2(\theta) = \frac{-1}{k(k-1)} \left\{6\theta^2 + 2(-k+1)\theta - k(5k+1)\right\}.$$

As the function $C(k) := \frac{7k-1-\sqrt{31k^2+4k+1}}{6k}$ is an increasing function on $k \geqslant 3$ and $C(3) = \frac{10-\sqrt{73}}{9}$, Inequality (1) follows by (9) and

$$\theta \geqslant \frac{k-1-\sqrt{31k^2+4k+1}}{6} = -k + C(k)k \geqslant -k + C(3)k = \left(\frac{1-\sqrt{73}}{9}\right) k \ .$$

In particular, $\theta \geqslant \frac{k-1-\sqrt{31k^2+4k+1}}{6} > 2-k$ holds for all $k \geqslant 10$. This completes the proof. □

To prove Theorem 3, we first need the following lemma.

Lemma 10. *For each integer $s \geqslant 2$, let $F_s(x) = 2x^{2s-1} + 2x^{2s-2} + \cdots + 2x^2 + 2x + 1$ and let z_s be the smallest zero of the function $F_s(x)$. Then*
(i) $-0.65 < z_2 < -0.64$.
(ii) $F_s(-1) = -1$ and $F_s(0) = 1$ for each $s \geqslant 2$.
(iii) $-1 < z_{s+1} < z_s < -0.64$ for each $s \geqslant 2$.

Proof. (i)-(ii): It is straightforward.
(iii): Let $s \geqslant 2$ be an integer. As $F_{s+1}(x) = 2x + 1 + \sum_{i=1}^{s} 2x^{2i}(x+1)$,

$$F_{s+1}(-1-\epsilon) < 2(-1-\epsilon) + 1 = -1 - 2\epsilon < 0$$

holds for any $\epsilon > 0$. Hence $-1 < z_{s+1} < 0$ follows by (ii). On the other hand, we find $F_{s+1}(z_s) = z_s^2 F_s(z_s) + (z_s+1)^2 = (z_s+1)^2 > 0$ as $F_{s+1}(x) = x^2 F_s(x) + (x+1)^2$. This shows $z_{s+1} < z_s$ and thus (iii) follows by (i). □

Let Γ be a distance-regular graph with girth $g > 3$. Then $(c_i, a_i, b_i) = (1, 0, k-1)$ for all $i = 1, \ldots, \lfloor \frac{g}{2} \rfloor - 1$. For an eigenvalue θ of Γ, it follows by (3) that

$$k(k-1)^{i-1} u_i(\theta) = \theta^i + \sum_{0 \leqslant \ell + n \leqslant i-1} t_{(\ell,n)} k^\ell \theta^n \quad \left(1 \leqslant i \leqslant \left\lfloor \frac{g}{2} \right\rfloor\right) \tag{10}$$

where $t_{(\ell,n)} \in \mathbb{R}$ for all $0 \leqslant \ell + n \leqslant i - 1 \leqslant \lfloor \frac{g}{2} \rfloor - 2$.

Proof of Theorem 3. Let ρ be the standard representation of Γ corresponding to the smallest eigenvalue $\theta = \theta_D$. As $g \equiv 3 \pmod 4$ and $g > 3$, let $g = 4s - 1$ for some $s \geqslant 2$. Suppose that $P = x_0 x_1 \ldots x_{4s-2}$ is an induced polygon of length $4s-1$, where $d(x_i, x_{i+1}) = 1 = d(x_0, x_{4s-2})$ $i = 0, 1, \ldots, 4s-3$. Put $\alpha := \sum_{i=0}^{4s-2} \rho(x_i)$. Then we have

$$0 \leqslant \frac{k(k-1)^{2s-2}}{(4s-1)k^{2s-1}} \langle \alpha, \alpha \rangle = \frac{k(k-1)^{2s-2}}{k^{2s-1}} \left(u_0(\theta) + 2 \sum_{i=1}^{2s-1} u_i(\theta) \right). \tag{11}$$

Using (10), Inequality (11) is equivalent to

$$F_s\left(\frac{\theta}{k}\right) \geqslant \frac{1}{k^{2s-1}} G_s(k, \theta) \tag{12}$$

where $F_s(x) = 2x^{2s-1} + 2x^{2s-2} + \cdots + 2x^2 + 2x + 1$ and $G_s(k, \theta) = \sum_{0 \leqslant i+j \leqslant 2s-2} c_{(i,j)} k^i \theta^j$ for some real numbers $c_{(i,j)}$. Hence it follows by $|\theta| < k$ that

$$\lim_{k \to \infty} \frac{G_s(k, \theta)}{k^{2s-1}} = 0.$$

Thus there exists a positive integer $C(g) \geqslant 3$ such that if $k \geqslant C(g)$ then $\frac{G_s(k,\theta)}{k^{2s-1}} \geqslant -\frac{1}{2}$ holds. Note here that for any real number x,

$$F_s'(x) = 2sx^{2s-2} + 2(x+1)^2 \sum_{i=1}^{s-1} (s-i) x^{2(s-1-i)} > 0.$$

Hence it follows by Lemma 10 (ii)-(iii) and Equation (12) that $F_s(\frac{\theta}{k}) \geqslant -\frac{1}{2}$ and there exists a real number $\gamma(g) \in (-1, z_s)$ satisfying $\frac{\theta}{k} \geqslant \gamma(g)$. As $-1 < \gamma(g) < z_s \leqslant z_2 < -0.64$ holds by Lemma 10 (iii), the result follows. $\square$

As an example of Theorem 3, we will give a lower bound -0.86 for $\frac{\theta_D}{k}$ if $g = 7$.

Example 11. Let Γ be a distance-regular graph with diameter $D \geqslant 3$, valency $k \geqslant 3$ and girth $g = 7$. Then the smallest eigenvalue θ_D of Γ satisfies

$$\theta_D > -0.86k.$$

In particular, if $k \geqslant 15$ then $\theta_D > 2 - k$.

Proof. As $g = 7$, we have $g = 4s - 1 = 7$ with $s = 2$. Suppose that $P = x_0 x_1 \ldots x_6$ is an induced polygon of length 7, where $d(x_i, x_{i+1}) = 1 = d(x_0, x_6)$ $i = 0, 1, \ldots, 5$. For the smallest eigenvalue $\theta = \theta_D$, let ρ be the corresponding standard representation and let $\alpha := \sum_{i=0}^{6} \rho(x_i)$. It follows by (3) and (10) that

$$\begin{aligned} 0 &\leqslant \frac{k(k-1)^2}{7k^3} \langle \alpha, \alpha \rangle = \frac{k(k-1)^2}{k^3} \left(u_0(\theta) + 2u_1(\theta) + 2u_2(\theta) + 2u_3(\theta)\right) \\ &= \frac{1}{k^3} \{k^3 + 2k^2(\theta - 2) + k(2\theta^2 - 8\theta + 3) + 2\theta(\theta^2 - \theta + 2)\} \end{aligned}$$

which is equivalent to

$$F_2\left(\frac{\theta}{k}\right) \geqslant \frac{1}{k^3}(2\theta^2 + 8k\theta - 4\theta + 4k^2 - 3k) := \frac{1}{k^3} G_2(k, \theta)$$

where $F_2(x) = 2x^3 + 2x^2 + 2x + 1$. In particular, if $k \geqslant 4$ then $\frac{G_2(k,\theta)}{k^3} > -\frac{1}{2}$ as $|\theta| < k$ and

$$2G_2(k, \theta) + k^3 = 4\theta^2 + (16k - 8)\theta + (k^3 + 8k^2 - 6k) > k(k^2 - 4k + 2) > 0.$$

Since $x > -0.86$ follows by $F_2(x) = 2x^3 + 2x^2 + 2x + 1 \geqslant -\frac{1}{2}$, this shows that if $k \geqslant 4$ then $\theta > -0.86k$. If $k = 3$ then

$$0 \leqslant \frac{4\langle \alpha, \alpha \rangle}{63} = \frac{4}{9}\left(u_0(\theta) + 2u_1(\theta) + 2u_2(\theta) + 2u_3(\theta)\right) = \frac{2\theta(\theta + 1 + \sqrt{2})(\theta + 1 - \sqrt{2})}{27},$$

which shows $\theta \geqslant -1 - \sqrt{2} > -0.86 \times 3$. In particular, $\theta > -0.86k \geqslant 2 - k$ holds for all $k \geqslant 15$. This completes the proof. $\square$

Remark 12. There are distance-regular graphs Γ with girth $g \geqslant 6$ satisfying $g \equiv 1 \pmod 4$ or $g \equiv 0 \pmod 2$ and an eigenvalue θ satisfying $-k < \theta \leqslant 2 - k$, such as the Biggs-Smith graph with intersection array $\iota(\Gamma) = \{3, 2, 2, 2, 1, 1, 1; 1, 1, 1, 1, 1, 1, 3\}$, the odd graph on 13 points with $\iota(\Gamma) = \{7, 6, 6, 5, 5, 4; 1, 1, 2, 2, 3, 3\}$ and the Foster graph with $\iota(\Gamma) = \{3, 2, 2, 2, 2, 1, 1, 1; 1, 1, 1, 1, 2, 2, 2, 3\}$.

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