A Set and Collection Lemma

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Abstract
A set $S \subseteq V(G)$ is independent if no two vertices from $S$ are adjacent. Let $\alpha(G)$ stand for the cardinality of a largest independent set.

In this paper we prove that if $\Lambda$ is a nonempty collection of maximum independent sets of a graph $G$, and $S$ is an independent set, then

- there is a matching from $S - \bigcap \Lambda$ into $\bigcup \Lambda - S$, and
- $|S| + \alpha(G) \leq |\bigcap \Lambda \cap S| + |\bigcup \Lambda \cup S|$.

Based on these findings we provide alternative proofs for a number of well-known lemmata, such as the “Maximum Stable Set Lemma” due to Claude Berge and the “Clique Collection Lemma” due to András Hajnal.

Keywords: matching; independent set; stable set; core; corona; clique

1 Introduction
Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subseteq V$, then $G[X]$ is the subgraph of $G$ spanned by $X$. By $G - W$ we mean the subgraph $G[V - W]$, if $W \subseteq V(G)$, and we use $G - w$, whenever $W = \{w\}$.

The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$, while the neighborhood of $A \subseteq V$ is $N(A) = N_G(A) = \{v \in V : N(v) \cap A \neq \emptyset\}$. By $\overline{G}$ we denote the complement of $G$.

A set $S \subseteq V(G)$ is independent (stable) if no two vertices from $S$ are adjacent, and by $\text{Ind}(G)$ we mean the set of all the independent sets of $G$. An independent set of maximum cardinality will be referred to as a maximum independent set of $G$, and the independence number of $G$ is $\alpha(G) = \max\{|S| : S \in \text{Ind}(G)\}$.
A matching (i.e., a set of non-incident edges of $G$) of maximum cardinality $\mu(G)$ is a maximum matching.

If $\alpha(G) + \mu(G) = |V(G)|$, then $G$ is called a König-Egerváry graph [5, 14].

**Lemma 1** *(Maximum Stable Set Lemma).* [1], [2] An independent set $X$ is maximum if and only if every independent set $S$ disjoint from $X$ can be matched into $X$.

Let $\Omega(G)$ denote the family of all maximum independent sets of $G$ and

$$\text{core}(G) = \bigcap\{ S : S \in \Omega(G) \} \ [11], \ \text{while}$$

$$\text{corona}(G) = \bigcup\{ S : S \in \Omega(G) \} \ [3].$$

A set $A \subseteq V(G)$ is a clique in $G$ if $A$ is independent in $\overline{G}$, and $\omega(G) = \alpha(\overline{G})$.

Our main motivation has been the “Clique Collection Lemma” due to Hajnal [8]. Some recent applications may be found in [4, 9, 13].

**Lemma 2** *(Clique Collection Lemma).* [8] If $\Gamma$ is a collection of maximum cliques in $G$, then

$$|\bigcap\Gamma| \geq 2 \cdot \omega(G) - |\bigcup\Gamma|.$$

In this paper we introduce the “Matching Lemma”. It is both a generalization and strengthening of a number of observations including the “Maximum Stable Set Lemma” due to Berge, and the “Clique Collection Lemma” due to Hajnal.

### 2 Results

It is clear that the statement “there exists a matching from a set $A$ into a set $B$” is stronger than just saying that $|A| \leq |B|$. The “Matching Lemma” offers a tool validating existence of matchings and their corresponding inequalities.

**Lemma 3** *(Matching Lemma).* Let $S \in \text{Ind}(G)$, $X \in \Lambda \subseteq \Omega(G)$, and $|\Lambda| \geq 1$. Then the following assertions are true:

(i) there exists a matching from $S - \bigcap \Lambda$ into $\bigcup \Lambda - S$;

(ii) there exists a matching from $S \cap X - \bigcap \Lambda$ into $\bigcup \Lambda - (X \cup S)$.

**Proof.** (i) In order to prove that there is a matching from $S - \bigcap \Lambda$ into $\bigcup \Lambda - S$, we use Hall’s Theorem, i.e., we show that for every $A \subseteq S - \bigcap \Lambda$ we must have

$$|A| \leq \left| N(A) \cap \left( \bigcup \Lambda \right) \right| = \left| N(A) \cap \left( \bigcup \Lambda - S \right) \right|.$$

Assume, by way of contradiction, that Hall’s condition is not satisfied. Let us choose a minimal subset $\tilde{A} \subseteq S - \bigcap \Lambda$, for which $|\tilde{A}| > \left| N(\tilde{A}) \cap \left( \bigcup \Lambda \right) \right|$.

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There exists some $W \in \Lambda$ such that $\tilde{A} \not\subseteq W$, because $\tilde{A} \subseteq S - \bigcap \Lambda$. Further, the inequality $|\tilde{A} \cap W| < |\tilde{A}|$ and the inclusion

$$N(\tilde{A} \cap W) \cap \left( \bigcup \Lambda \right) \subseteq N(\tilde{A}) \cap \left( \bigcup \Lambda \right) - W$$

imply

$$|\tilde{A} \cap W| \leq |N(\tilde{A} \cap W) \cap \left( \bigcup \Lambda \right)| \leq |N(\tilde{A}) \cap \left( \bigcup \Lambda \right) - W|,$$

because we have selected $\tilde{A}$ as a minimal subset satisfying $|\tilde{A}| > |N(\tilde{A}) \cap \left( \bigcup \Lambda \right)|$.

On the other hand,

$$|\tilde{A} \cap W| + |\tilde{A} - W| = |\tilde{A}| > |N(\tilde{A}) \cap \left( \bigcup \Lambda \right)| = |N(\tilde{A}) \cap \left( \bigcup \Lambda \right) - W| + |N(\tilde{A}) \cap W|.$$

Consequently, since $|\tilde{A} \cap W| \leq |N(\tilde{A}) \cap \left( \bigcup \Lambda \right) - W|$, we can infer that $|\tilde{A} - W| > |N(\tilde{A}) \cap W|$. Therefore,

$$\tilde{A} \cup \left( W - N(\tilde{A}) \right) = W \cup (\tilde{A} - W) - \left( N(\tilde{A}) \cap W \right)$$

is an independent set of size greater than $|W| = \alpha(G)$, which is a contradiction that proves the claim.

(ii) By part (i), there exists a matching from $S - \bigcap \Lambda$ into $\bigcup \Lambda - S$. Since $X$ is independent, there are no edges between

$$\left( S - \bigcap \Lambda \right) - (S - X) = (S \cap X) - \bigcap \Lambda \text{ and } X - S.$$ 

Therefore, there exists a matching

$$\text{from } (S \cap X) - \bigcap \Lambda \text{ into } \left( \bigcup \Lambda - S \right) - (X - S) = \bigcup \Lambda - (X \cup S),$$

as claimed. \qed

Figure 1: $\{v_1, v_2, v_3, v_5, v_8, v_{10}, v_{12}\}$, $\{v_1, v_2, v_4, v_6, v_7, v_{10}, v_{13}\}$, $\{v_1, v_2, v_4, v_6, v_7, v_{10}, v_{12}\}$ are maximum independent sets.
Example 4. Let us consider the graph $G$ from Figure 1 and $S = \{v_1, v_4, v_7\} \in \mathrm{Ind}(G)$, $\Lambda = \{S_1, S_2\}$, where $S_1 = \{v_1, v_2, v_3, v_6, v_8, v_{10}, v_{12}\}$ and $S_2 = \{v_1, v_2, v_4, v_6, v_7, v_{10}, v_{13}\}$. Then there is a matching from $S \setminus \Lambda = \{v_4, v_7\}$ into $\bigcup \Lambda - S = \{v_2, v_3, v_6, v_8, v_{10}, v_{12}, v_{13}\}$, namely, $M = \{v_3 v_4, v_7 v_8\}$.

Remark 5. The conclusions of the Matching Lemma may be false, if the family $\Lambda$ is not included in $\Omega(G)$. Note that in Figure 1, if $S = \{v_1, v_2, v_4, v_7, v_9, v_{12}\} \in \mathrm{Ind}(G)$, $\Lambda = \{S_1, S_2\}$, where $S_1 = \{v_2, v_3, v_7\}$ and $S_2 = \{v_1, v_2, v_4, v_6, v_7, v_{10}, v_{12}\}$, then there is no matching from $S \setminus \Lambda = \{v_1, v_4, v_9, v_{12}\}$ into $\bigcup \Lambda - S = \{v_3, v_6, v_{10}\}$.

The Matching Lemma allows us to give an alternative proof of the following result due to Berge.

Lemma 6 (Maximum Stable Set Lemma). [1, 2] An independent set $X$ is maximum if and only if every independent set $S$ disjoint from $X$ can be matched into $X$.

Proof. The “only if” part follows from the Matching Lemma (i), by taking $\Lambda = \{X\}$.

For the “if” part we proceed as follows. According to the hypothesis, there is a matching from $S - \Lambda = S - S \cap X$ into $X$, in fact, into $X - S \cap X$, for each $S \in \mathrm{Ind}(G)$. Let $S \in \Omega(G)$. Hence, we obtain

$$\alpha(G) = |S| = |S - X| + |S \cap X| \leq |X - S \cap X| + |S \cap X| = |X| \leq \alpha(G),$$

which clearly implies $X \in \Omega(G)$.

Applying the Matching Lemma (i) to $\Lambda = \Omega(G)$ we immediately obtain the following.

Corollary 7. [3] For every $S \in \Omega(G)$, there is a matching from $S - \mathrm{core}(G)$ into corona($G$) $- S$.

The following inequality is a numerical interpretation of the Matching Lemma.

Lemma 8 (Set and Collection Lemma). If $S \in \mathrm{Ind}(G)$, $\Lambda \subseteq \Omega(G)$, and $|\Lambda| \geq 1$, then

$$|S| + \alpha(G) \leq |\bigcap \Lambda \cap S| + \big| \bigcup \Lambda \cup S \big|.$$ 

Proof. Let $X \in \Lambda$. By the Matching Lemma (ii), there is a matching from $S \cap X - \bigcap \Lambda$ into $\bigcup \Lambda - (X \cup S)$. Hence we infer that

$$|S \cap X| - |\bigcap \Lambda \cap S| = |S \cap X| - \big| \bigcap \Lambda \cap S \cap X \big|$$

$$= \big| S \cap X - \bigcap \Lambda \big| \leq \big| \bigcup \Lambda - (X \cup S) \big|$$

$$= \big| \bigcup \Lambda \cup (X \cup S) \big| - |X \cup S| = \big| \bigcup \Lambda \cup S \big| - |X \cup S|. $$
Therefore, we obtain
\[ |S \cap X| - |\Lambda \cap S| \leq |\Lambda \cup S| - |X \cup S|, \]
which implies
\[ |S| + \alpha(G) = |S| + |X| = |S \cap X| + |X \cup S| \leq |\Lambda \cap S| + |\Lambda \cup S|, \]
as claimed.

The conclusions of the Set and Collection Lemma may be false, if the family \( \Lambda \) is not included in \( \Omega(G) \). For instance, the graph \( G \) of Figure 1 has \( \alpha(G) = 7 \), and if \( S = \{v_1, v_2, v_4, v_7, v_9, v_{12}\} \in \text{Ind}(G) \), \( \Lambda = \{S_1, S_2\} \), where \( S_1 = \{v_2, v_3, v_7\} \) and \( S_2 = \{v_1, v_2, v_4, v_6, v_7, v_{10}, v_{12}\} \), then
\[ 13 = |S| + \alpha(G) \leq |\Lambda \cap S| + |\Lambda \cup S| = 11. \]

**Corollary 9.** If \( \Lambda \subseteq \Omega(G) \), \( |\Lambda| \geq 1 \), then \( 2 \cdot \alpha(G) \leq |\Lambda| + |\Lambda| \).

**Proof.** Let \( S \in \Lambda \). Using the Set and Collection Lemma, we obtain
\[ 2 \cdot \alpha(G) = |S| + \alpha(G) \leq |\Lambda \cap S| + |\Lambda \cup S| = |\Lambda| + |\Lambda|, \]
as required. \( \square \)

Since every maximum clique of \( G \) is a maximum independent set of \( \overline{G} \), Corollary 9 is equivalent to the following result, due to Hajnal.

**Lemma 10 (Clique Collection Lemma).** [8] If \( \Gamma \) is a collection of maximum cliques in \( \Gamma \), then
\[ |\bigcap \Gamma| \geq 2 \cdot \omega(G) - |\bigcup \Gamma|. \]

If \( \Lambda = \Omega(G) \), then Corollary 9 implies the following.

**Corollary 11.** For every graph \( G \), it is true that
\[ 2 \cdot \alpha(G) \leq |\text{core}(G)| + |\text{corona}(G)|. \]

The graph \( G_1 \) from Figure 2 satisfies \( 2 \cdot \alpha(G_1) < |\text{core}(G_1)| + |\text{corona}(G_1)| \), because \( \alpha(G_1) = 4 \), \( \text{core}(G_1) = \{v_8, v_9\} \), and \( \text{corona}(G_1) = \{v_1, v_3, v_4, v_5, v_7, v_8, v_9\} \).

The vertex covering number of \( G \), denoted by \( \tau(G) \), is the number of vertices in a minimum vertex cover in \( G \), that is, the size of any smallest vertex cover in \( G \). Thus we have \( \alpha(G) + \tau(G) = |V(G)| \). Since
\[ |V(G)| - \bigcup \{S : S \in \Omega(G)\} = |\bigcap \{V(G) - S : S \in \Omega(G)\}|, \]
Corollary 11 immediately implies the following.
Corollary 12. [7] If \( G \) is a graph, then
\[
\alpha(G) - |\text{core}(G)| \leq \tau(G) - \left|\bigcap \{V(G) - S : S \in \Omega(G)\}\right|.
\]

It is clear that \(|\text{core}(G)| + |\text{corona}(G)| \leq \alpha(G) + |V(G)|\).

Proposition 13. If \( G \) is a graph with a nonempty edge set, then
\[
|\text{core}(G)| + |\text{corona}(G)| \leq \alpha(G) + |V(G)| - 1.
\]

Proof. Assume, to the contrary, that \(|\text{core}(G)| + |\text{corona}(G)| \geq \alpha(G) + |V(G)|\).

If \( S \in \Omega(G) \), then
\[
|\text{corona}(G) - S| = |\text{corona}(G)| - \alpha(G) \geq |V(G)| - |\text{core}(G)| = |V(G) - \text{core}(G)|.
\]

Since, clearly, \( \text{corona}(G) - S \subseteq V(G) - \text{core}(G) \), we obtain \( V(G) = \text{corona}(G) \) and \( \text{core}(G) = S \). It follows that \( N(\text{core}(G)) = \emptyset \), since \( \text{corona}(G) \cap N(\text{core}(G)) = \emptyset \).

On the other hand, since \( G \) has a nonempty edge set and \( S \) is a maximum independent set, we have \( \emptyset \neq N(S) = N(\text{core}(G)) \).

This contradiction proves the claimed inequality. \( \square \)

Remark 14. The complete bipartite graph \( K_{1,n-1} \) satisfies \( \alpha(K_{1,n-1}) = n - 1 \), and hence
\[
|\text{core}(K_{1,n-1})| + |\text{corona}(K_{1,n-1})| = 2(n - 1) = \alpha(G) + |V(K_{1,n-1})| - 1.
\]
In other words, the bound in Proposition 13 is tight.

It has been shown in [12] that
\[
\alpha(G) + \left|\bigcap \{V - S : S \in \Omega(G)\}\right| = \mu(G) + |\text{core}(G)|
\]
is satisfied by every König-Egerváry graph \( G \), and taking into account that
\[
\left|\bigcap \{V - S : S \in \Omega(G)\}\right| = |V(G)| - \left|\bigcup \{S : S \in \Omega(G)\}\right|,
\]
we infer that the König-Egerváry graphs enjoy the following.

Proposition 15. If \( G \) is a König-Egerváry graph, then
\[
2 \cdot \alpha(G) = |\text{core}(G)| + |\text{corona}(G)|.
\]

The converse of Proposition 15 is not true. For instance, see the graph \( G_2 \) from Figure 2, which has \( \alpha(G_2) = 3 \), \( \text{corona}(G_2) = \{u_2, u_4, u_6, u_7\} \), and \( \text{core}(G_2) = \{u_2, u_4\} \).
3 Conclusions

In this paper we have proved the “Set and Collection Lemma”, which has been employed in order to obtain a number of alternative proofs and/or strengthenings of some known results.

By Proposition 15 we know that \(2 \cdot \alpha(G) = |\text{core}(G)| + |\text{corona}(G)|\) holds for every König-Egerváry graph \(G\). Therefore, it is true for each very well-covered graph \(G\) [10]. Recall that \(G\) is a very well-covered graph if it has no isolated vertices, \(2\alpha(G) = |V(G)|\), and all its maximal independent sets are of the same cardinality [6]. It is worth noting that there are other graphs enjoying this equality, e.g., every graph \(G\) having a unique maximum independent set, because, in this case, \(\alpha(G) = |\text{core}(G)| = |\text{corona}(G)|\).

Problem 16. Characterize graphs satisfying \(2 \cdot \alpha(G) = |\text{core}(G)| + |\text{corona}(G)|\).

Let us consider a dual problem. It is clear that for every graph \(G\) there exists a collection of maximum independent sets \(\Lambda\) such that \(2 \cdot \alpha(G) = \left| \bigcup \Lambda \right| + \left| \bigcap \Lambda \right|\). Just take \(\Lambda = \{X\}\) for some maximum independent set \(X\).

Problem 17. For a given graph \(G\) find the cardinality of a largest collection of maximum independent sets \(\Lambda\) such that \(2 \cdot \alpha(G) = \left| \bigcup \Lambda \right| + \left| \bigcap \Lambda \right|\).

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References


