The gap structure of a family of integer subsets

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Abstract

In this paper we investigate the gap structure of a certain family of subsets of $\mathbb{N}$ which produces counterexamples both to the "density version" and the "canonical version" of Brown's lemma. This family includes the members of all complementing pairs of $\mathbb{N}$. We will also relate the asymptotical gap structure of subsets of $\mathbb{N}$ with their density and investigate the asymptotical gap structure of monochromatic and rainbow sets with respect to arbitrary infinite colorings of $\mathbb{N}$.

Keywords: piecewise syndetic; complementing pairs; Brown's lemma; Ramsey theory.

1 Introduction

Let $\mathbb{N}$ be the set of all nonnegative integers. The *gap* of a finite subset $A = \{a_1, \ldots, a_k\}$ of $\mathbb{N}$ is the number $\operatorname{gap}(A) := \max\{a_{i+1} - a_i : 1 \leqslant i \leqslant k-1\}$. An infinite subset X of $\mathbb{N}$ is *piecewise syndetic* if it contains arbitrarily large subsets with uniformly bounded gaps. This means that the sequence in $k \in \mathbb{N}$ defined by

$$d_k(X) := \min\{\operatorname{gap}(A) : A \subset X \text{ and } |A| = k+1\}, \tag{1}$$

is bounded. An induction argument in the number of colors shows [2, 3] that any finite coloring of $\mathbb{N}$ admits a monochromatic piecewise syndetic set. This result is known as *Brown's lemma*.

Brown's lemma does not admit a density version analogous to Szemerédi's theorem [8], that is, there are subsets X of $\mathbb{N}$ with positive density which are not piecewise syndetic. An example of such a subset is given in [1], Theorem 2.8.

Brown's lemma also does not admit a canonical version analogous to the Erdős-Graham canonical version of van der Waerden's theorem [6]. In fact, T. Brown [4, 5] showed that there is an infinite coloring $\tau : \mathbb{N} \to \mathbb{N}$ for which the sequence in $k \in \mathbb{N}$ defined by

$$d_k(\tau) := \min\{\text{gap}(A) : |A| = k+1 \text{ and either } |\tau(A)| = 1 \text{ or } |\tau(A)| = k+1\} \qquad (2)$$

is not bounded. The infinite coloring used by T. Brown consists of infinitely many translates of an infinite set, that is, it is a coloring associated to a certain *complementing pair* of $\mathbb{N}$. Two infinite subsets X_1 and X_2 of $\mathbb{N}$ are a complementing pair of $\mathbb{N}$, and we write $\mathbb{N} = X_1 \oplus X_2$, if for each $n \in \mathbb{N}$ there exist unique $n_1 \in X_1$ and $n_2 \in X_2$ such that $n = n_1 + n_2$. In this case we can define an infinite coloring $\tau : \mathbb{N} \to \mathbb{N}$ by $\tau(n) = n_2$.

In this paper we will investigate the gap structure of a certain family of subsets of $\mathbb{N}$ which produces counterexamples both to the "density version" and the "canonical version" of Brown's lemma. This family includes the members of all complementing pairs of $\mathbb{N}$. We will also investigate the asymptotical upper bounds of $d_k(X)$ and $d_k(\tau)$ when X is a subset of $\mathbb{N}$ with positive upper density and τ is an infinite coloring of $\mathbb{N}$.

2 A familiy of non-piecewise syndetic sets with positive density

We will denote by $\overline{\sigma}(X)$ and $\underline{\sigma}(X)$, respectively, the *upper density* and the *lower density* of X:

$$\overline{\sigma}(X) := \limsup_n \frac{|X \cap [0,n]|}{n}, \text{ and } \underline{\sigma}(X) := \liminf_n \frac{|X \cap [0,n]|}{n}.$$

If $\overline{\sigma}(X) = \underline{\sigma}(X)$, the *density* of X is equal to this common value and is denoted by $\sigma(X)$. Consider two infinite sequences a_n and d_n of positive integers, with $a_0 = 1$. Assume that a_n is strictly increasing, d_n is nondecreasing and $\frac{a_{n+1}}{a_n}$ is an integer for each $n \in \mathbb{N}$. Fix an integer $K > 0$. We define recursively an increasing sequence of finite subsets $I_n := I_n(a_n, d_n, K)$ of $\mathbb{N}$, with $\beta_n := \max I_n$, as follows: $I_0 = [0, K]$ and

$$I_n = I_{n-1} \cup \{\beta_{n-1} + d_n + I_{n-1}\} \cup \ldots \cup \Big\{ \big(\frac{a_n}{a_{n-1}} - 1\big)\beta_{n-1} + \big(\frac{a_n}{a_{n-1}} - 1\big) d_n + I_{n-1} \Big\}. \quad (3)$$

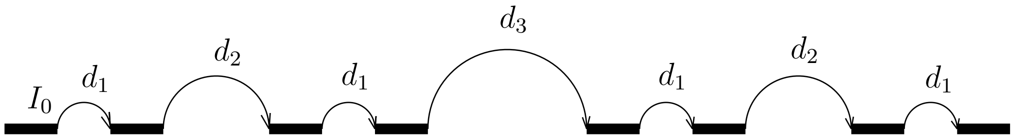
Set $\mathcal{I} = \bigcup_{n \in \mathbb{N}} I_n$. Observe that $|I_n| = \frac{a_n}{a_{n-1}} |I_{n-1}|$ and

$$\beta_n = \frac{a_n}{a_{n-1}} \beta_{n-1} + \big(\frac{a_n}{a_{n-1}} - 1\big) d_n.$$

Hence

$$|I_n| = a_n(K+1) \qquad \text{and} \qquad \beta_n = a_nK + a_n\sum_{i=1}^{n}\Big(\frac{1}{a_{i-1}} - \frac{1}{a_i}\Big)d_i. \tag{4}$$

Example 1. If $a_n = 2^n$, then $I_0 = [0, K]$, $I_1 = I_0 \cup \{d_1 + K + I_0\}$, $I_2 = I_1 \cup \{d_2 + d_1 + 2K + I_1\}$, and the structure of I_3 is illustrated by the following figure.



Lemma 2. *The subset $\mathcal{I}(a_n, d_n, K) := \mathcal{I} = \bigcup_{n\in\mathbb{N}} I_n$ of $\mathbb{N}$ has positive upper density if and only if the positive series*

$$\sum_{i=1}^{\infty}\Big(\frac{1}{a_{i-1}} - \frac{1}{a_i}\Big)d_i \tag{5}$$

converges. Moreover, $\overline{\sigma}(\mathcal{I}) = \underline{\sigma}(\mathcal{I})$.

Proof. Taking into account (4) we have

$$x_n := \frac{|I_n|}{\beta_n} = \frac{K+1}{K + \sum_{i=1}^{n}\big(\frac{1}{a_{i-1}} - \frac{1}{a_i}\big)d_i}.$$

This sequence is always convergent and

$$x_n = \sup\Big\{\frac{|\mathcal{I}\cap[0,N]|}{N} \,:\, N \geqslant \beta_n\Big\}.$$

This means that the largest limit of subsequences of $\frac{|\mathcal{I}\cap[0,n]|}{n}$ is attained by x_n. Hence

$$\overline{\sigma}(\mathcal{I}) = \lim_n \frac{K+1}{K + \sum_{i=1}^{n}\big(\frac{1}{a_{i-1}} - \frac{1}{a_i}\big)d_i},$$

which means that $\overline{\sigma}(\mathcal{I}) > 0$ if and only if the series (5) converges. If the series (5) diverges, then it is clear that $\overline{\sigma}(\mathcal{I}) = \underline{\sigma}(\mathcal{I}) = 0$.

Assume now that the series (5) converges. In this case

$$0 = \lim_n\Big(\frac{1}{a_{n-1}} - \frac{1}{a_n}\Big)d_n = \lim_n \frac{1}{a_n}\Big(\frac{a_n}{a_{n-1}} - 1\Big)d_n \geqslant \lim_n \frac{d_n}{a_n},$$

that is $\lim_n \frac{d_n}{a_n} = 0$. On the other hand,

$$y_n := \frac{|I_n|}{\beta_n + d_n - 1} = \min\Big\{\frac{|\mathcal{I}\cap[0,N]|}{N} \,:\, N \leqslant \beta_n + d_n - 1\Big\}.$$

Since $\lim_n \frac{d_n}{a_n} = 0$, we have $\lim x_n = \lim y_n$, that is the smallest limit of subsequences of $\frac{|\mathcal{I}\cap[0,n]|}{n}$ is attained by y_n and it is equal to $\overline{\sigma}(\mathcal{I})$. □

Remark 3. Given sequences a_n and d_n for which (5) converges, we can make $\overline{\sigma}(\mathcal{I})$ arbitrarily close to 1 by taking $K \to \infty$.

Remark 4. Taking into account its construction, if $\lim d_n = \infty$ the subset $\mathcal{I}$ is not piecewise syndetic. For example, if $a_n = 2^n$ and $d_n = n$, $\mathcal{I}$ is not piecewise syndetic but it has positive density $\frac{K+1}{K+2}$.

This family of subsets is optimal in the following sense.

Lemma 5. *For each $n \in \mathbb{N}$, we have $d_{a_n(K+1)}(\mathcal{I}) = d_{n+1}$. Moreover, given $X \subset \mathbb{N}$, then $\overline{\sigma}(X) \leqslant \sigma(\mathcal{I})$ if $d_{a_n(K+1)}(X) \geqslant d_{n+1}$ for each $n \in \mathbb{N}$.*

Proof. The first assertion follows directly from the definitions of $\mathcal{I}$ and $d_k(\mathcal{I})$. With the respect to the second assertion, observe that, for each $k \in \mathbb{N}$, we have $d_k(\mathcal{I}) = d_{a_{n_k}(K+1)}(\mathcal{I})$, where

$$n_k = \max\{n : a_n(K+1) \leqslant k\}.$$

This means that, if $d_{a_n(K+1)}(X) \geqslant d_{n+1}$, then $d_k(X) \geqslant d_k(\mathcal{I})$ for all k, and consequently $\overline{\sigma}(X) \leqslant \overline{\sigma}(\mathcal{I}) = \sigma(\mathcal{I})$. □

3 Complementing pairs of $\mathbb{N}$

Complementing pairs of $\mathbb{N}$ admit the following characterization (see [9] and the references therein). Given two infinite subsets X_1 and X_2 of $\mathbb{N}$, we have $\mathbb{N} = X_1 \oplus X_2$ if and only if there exists a sequence m_i, with $m_i \geqslant 2$ for all $i \in \mathbb{N}$, such that X_1 is the set of all finite sums $\sum_{i \geqslant 0} x_{2i} M_{2i}$ and X_2 is the set of all finite sums $\sum_{i \geqslant 0} x_{2i+1} M_{2i+1}$, where $M_0 = 1$, $M_i = \prod_{j=1}^{i} m_j$ and $0 \leqslant x_i < m_{i+1}$. Let

$$M_i^+ = \prod_{j=1,\, j \text{ even}}^{i} m_j, \quad M_i^- = \prod_{j=1,\, j \text{ odd}}^{i} m_j,$$

so that $M_i = M_i^+ M_i^-$.

Example 6. Take $m_i = 2$ for all $i \in \mathbb{N}$. Set $I_n = \left\{ \sum_{i=0}^{2n} x_{2i} M_{2i} : 0 \leqslant x_i \leqslant 1 \right\}$, with $M_i = 2^i$:

$$I_0 = [0,1],\ I_1 = [0,1] \cup [4,5],\ I_2 = \{[0,1] \cup [4,5]\} \cup \{[16,17] \cup [20,21]\},\ \ldots$$

For $K = 1$, $a_n = 2^n$, and $d_n = \frac{2^{2n+1}+1}{3}$, we have $X_1 = \mathcal{I}(a_n, d_n, K)$.

More generally, given a complementing pair $\mathbb{N} = X_1 \oplus X_2$, take $K = m_1 - 1$, $a_n = \frac{M_{2n+1}^-}{m_1}$ and

$$d_n = M_{2n} - \{(m_{2n-1} - 1)M_{2n-2} + (m_{2n-3} - 1)M_{2n-4} + \ldots + (m_3 - 1)M_2 + (m_1 - 1)\}. \quad (6)$$

With respect to these choices, the sets I_n in (3) are given by $I_0 = \{x_0 : 0 \leqslant x_0 < m_1\}$ and

$$I_n = \Big\{ \sum_{i=0}^{2n} x_{2i} M_{2i} : 0 \leqslant x_i < m_{i+1} \Big\}.$$

Hence $X_1 = \mathcal{I}(a_n, d_n, K)$.

Proposition 7. *If $\mathbb{N} = X_1 \oplus X_2$, then X_1 is not piecewise syndetic and $\sigma(X_1) = 0$.*

Proof. To see that X_1 is not piecewise syndetic we only have to check that $\lim d_n = \infty$. We can rewrite (6) as

$$d_n = (M_{2n} - M_{2n-1}) + (M_{2n-2} - M_{2n-3}) + \ldots + (M_2 - M_1) + 1.$$

Since $m_i \geqslant 2$ for all $i \geqslant 1$, we have $M_{2i} - M_{2i-1} \geqslant 1$, which means that d_n is strictly increasing. □

We say that $A \subset \mathbb{N}$ is a *rainbow set* with respect to a coloring $\tau : \mathbb{N} \to \mathbb{N}$ if $|\tau(A)| = |A|$.

Theorem 8. *Given a complementing pair $\mathbb{N} = X_1 \oplus X_2$, consider the associated infinite coloring τ, as defined in the Introduction section. If*

$$\lim_n \frac{m_{2n}}{M^-_{2(n-1)}} = 0, \tag{7}$$

then there does not exist $d \in \mathbb{N}$ and arbitrarily large sets A such that gap$(A) \leqslant d$ and A is either monochromatic or rainbow..

Proof. Observe that the number of colors in each interval of the form $J_i^k = [kM_{2i}, (k+1)M_{2i}]$ is precisely the cardinality of the set $\big\{ \sum_{j=0}^{2i-1} x_{2j+1} M_{2j+1} : 0 \leqslant x_j < m_{j+1} \big\}$. Hence, each interval $J_i^k = [kM_{2i}, (k+1)M_{2i}]$ has exactly M^+_{2i} colors and each color appears exactly M^-_{2i} times. Let $A = \{b_1, \ldots, b_n\}$ be a finite subset of $\mathbb{N}$ and choose s minimal so that $A \subseteq J_s^{k-1} \cup J_s^k$. We have $2M_{2(s-1)} \leqslant b_n - b_1 \leqslant \mathrm{gap}(A) n$. On the other hand, $|\tau(A)| \leqslant 2M^+_{2s}$. Then

$$|\tau(A)| \leqslant \frac{\mathrm{gap}(A)|A| m_{2s}}{M^-_{2(s-1)}}. \tag{8}$$

Hence, if $\mathrm{gap}(A) \leqslant d$ for some fixed d and $|A|$ is large enough, from condition (7) we get $|\tau(A)| < |A|$, that is, we can not have arbitrarily large rainbow sequences with bounded gaps.

On the other hand, τ does not admit arbitrarily large monochromatic sequences with uniformly bounded gaps because X_1 is not piecewise syndetic and, for each color n_0, the monochromatic subset $\tau^{-1}(n_0)$ is just the translation copy of X_1 by n_0. □

Remark 9. The infinite coloring used in [5] is the one defined by the complementing pair $\mathbb{N} = X_1 \oplus X_2$ with X_1 the set of all finite sums $\sum_{i\,\mathrm{even}} 2^i$ and X_2 the set of all finite sums $\sum_{i\,\mathrm{odd}} 2^i$. In this case, $m_i = 2$ for all $i \geqslant 1$, and condition (7) certainly holds.

4 Asymptotical gap structure of positive density sets

Not surprisingly, the sequence $d_k(X)$ defined by (1) grows at most linearly with k for sets X with positive density.

Proposition 10. *Let X be a subset of $\mathbb{N}$ with positive lower density $\underline{\sigma} := \underline{\sigma}(X)$. Then $d_k(X) = O(k)$ as $k \to \infty$.*

Proof. Given $0 < \epsilon < \underline{\sigma}$, for all sufficiently large n, we must have $(\underline{\sigma}-\epsilon)n+1 < |[1,n] \cap X|$. Then the gap of $[1,n] \cap X$ is at most $n - (\underline{\sigma} - \epsilon)n$. Hence $d_{\lceil(\underline{\sigma}-\epsilon)n\rceil+1}(X) \leqslant n - (\underline{\sigma} - \epsilon)n$. Taking $k = \lceil(\underline{\sigma} - \epsilon)n\rceil + 1$, we conclude that $d_k(X) = O(k)$ as $k \to \infty$. □

As the following theorem shows, this asymptotical bound is not optimal.

Theorem 11. *Let $\varpi : [0, +\infty[\to \mathbb{R}$ be a continuous increasing function so that $\varpi(x)/x^2$ decreases with x. Then, if the integral*

$$\int_1^{+\infty} \frac{\varpi(x)}{x^2}\, dx \tag{9}$$

diverges, any subset X of $\mathbb{N}$ with $\varpi(k) = O(d_k(X))$ as $k \to \infty$ has upper density zero.

Proof. Let X be a subset of $\mathbb{N}$ with $\varpi(k) = O(d_k(X))$ and consider the increasing sequences a_n and d_n defined by $a_n = 2^n$ and $d_n = d_{2^n}(X)$. Consider the subset $\mathcal{I} = \mathcal{I}(a_n, d_n, 1)$. By Lemma 5, $\overline{\sigma}(X) \leqslant \sigma(\mathcal{I})$.

Since $\varpi(k) = O(d_k(X))$, the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{a_{n-1}} - \frac{1}{a_n}\right) d_n = \sum_{n=1}^{\infty} \frac{d_{2^n}(X)}{2^n},$$

diverges if $\sum_{n=1}^{\infty} \frac{\varpi(2^n)}{2^n}$ diverges. But, taking the substitution $x = 2^y$, we get

$$\int_0^{\infty} \frac{\varpi(2^y)}{2^y}\, dy = \frac{1}{\ln 2} \int_1^{\infty} \frac{\varpi(x)}{x^2}\, dx.$$

Hence, by the integral convergence test, $\sum_{n=1}^{\infty} \frac{\varpi(2^n)}{2^n}$ diverges. By Lemma 2, we conclude that $\sigma(\mathcal{I}) = 0$, and consequently $\overline{\sigma}(X) = 0$. □

Conversely,

Theorem 12. *Let $\varpi : [0, +\infty[\to \mathbb{R}$ be a continuous increasing function so that $\varpi(x)/x^2$ decreases with x. Then, if the integral (9) converges, there exists a subset X of $\mathbb{N}$ with $\varpi(k) = O(d_k(X))$ as $k \to \infty$ and positive upper density.*

Proof. Set $a_n = 2^n$, $d_n = \lceil \varpi(2^n) \rceil$, and consider the subset $\mathcal{I} = \mathcal{I}(a_n, d_n, 1)$. If the integral (9) converges, we can apply the integral convergence test, as in the proof of Theorem 11, to conclude that the series (5) converge, and consequently $\sigma(\mathcal{I}) > 0$. Since ϖ is increasing and, for $2^n < k < 2^{n+1}$, we have $d_k(\mathcal{I}) = d_{2^{n+1}}(\mathcal{I}) = d_{n+1}$, it is clear that $\varpi(k) = O(d_k(\mathcal{I}))$. Set $X = \mathcal{I}$, and we are done. □

Remark 13. In [7], R. Salem and D.C. Spencer studied the influence of gaps in the density of integer subsets. However, a different notion of gap structure is considered there. More precisely, given an positive increasing function ω of the real nonnegative variable x, they were concerned with subsets X of $\mathbb{N}$ satisfying the following property: for any closed interval $[a, a+l]$, with $a \geqslant 0$ and $l > 0$, there exists an open interval not less than $\omega(l)$ which contains no points of X. For that purpose, they used sequences $u(n)$ defined by

$$u(n) = g_0 n + g_1 \left[\frac{n}{2}\right] + g_2 \left[\frac{n}{2^2}\right] + \ldots + g_p \left[\frac{n}{2^p}\right] + \ldots,$$

where g_p is a given sequence of positive integers. For $g_0 = 1$ and $g_p \geqslant 1$, these sequences are of the form $\mathcal{I}(a_n, d_n, 1)$, with $a_n = 2^n$ and $d_n = g_0 + g_1 + \ldots + g_n$. In spite of the different notions of gap structure, the asymptotical bounds given by Theorems 11 and 12 are the same as those given by Theorems I and II in [7].

5 Asymptotical gap structure and infinite colorings

Next we investigate the asymptotical growth with k of the sequence $d_k(\tau)$ defined by (2).

Theorem 14. *Given an infinite coloring $\tau : \mathbb{N} \to \mathbb{N}$, we have $d_k(\tau) = O(k^2)$.*

Proof. Set $\theta(n) = |\tau([1, n])|$ (the number of distinct colors occurring in the interval $[1, n]$) and define $\alpha_n = \lceil \frac{n}{\theta(n)} \rceil$. By the pigeonhole principle, there always exists a monochromatic subset A_{α_n} of $[1, n]$ with α_n elements. For each n, consider also a rainbow subset $B_{\theta(n)}$ of $[1, n]$ with $\theta(n)$ elements and $\theta(n)$ distinct colors.

Suppose first that α_n is bounded: there exists $C > 1$ such that $1 \leqslant \frac{n}{\theta(n)} \leqslant C$ for all $n \in \mathbb{N}$. In this case,

$$\mathrm{gap}(B_{\theta(n)}) \leqslant n - (\theta(n) - 1) \leqslant (C - 1)\theta(n) + 1,$$

which means that $d_k(\tau) = O(k)$.

If α_n is not bounded, then we can assume, by taking a subsequence if necessary, that $\alpha_n \to \infty$. We have

$$\mathrm{gap}(A_{\alpha_n}) \leqslant n - (\alpha_n - 1) \leqslant \lceil n/\theta(n) \rceil \theta(n) - \lceil n/\theta(n) \rceil + 1.$$

Suppose that there exists $\xi > 0$ such that $\xi \leqslant \lceil n/\theta(n) \rceil / \theta(n)$ for all n. In this case,

$$\mathrm{gap}(A_{\alpha_n}) \leqslant 1/\xi \lceil n/\theta(n) \rceil^2 - \lceil n/\theta(n) \rceil + 1,$$

and $d_k(X) = O(k^2)$. Finally, if $\lceil n/\theta(n) \rceil / \theta(n) \to 0$ (or some of its subsequences), then, for some $\eta > 0$ and n sufficiently large, we have $\mathrm{gap}(B_{\theta(n)}) \leqslant \eta \theta^2(n) - \theta(n) + 1$, and consequently $d_k(X) = O(k^2)$. □

Example 15. When τ is the infinite coloring of $\mathbb{N}$ associated to the complementing pair $\mathbb{N} = X_1 \oplus X_2$, where X_1 is the set of all finite sums $\sum x_{2i}M_{2i}$, with $0 \leqslant x_i < m_{i+1}$, we can give the following asymptotical bounds for $d_k(\tau)$. To simplify the discussion, assume further that, for some $m \geqslant 2$, we have $m_i = m$ for all $i \geqslant 1$. In this case, from (6) we can check that

$$d_n = \frac{m^{2n+1}+1}{m+1}.$$

On the other hand, $|X_1 \cap [0, M_{2n}]| = m^n + 1$ and for any other interval $[\alpha, \beta]$ with $|X_1 \cap [\alpha, \beta]| = m^n + 1$ we have

$$\text{gap}(|X_1 \cap [\alpha, \beta]|) \geqslant \text{gap}(|X_1 \cap [0, M_{2n}]|) = d_n.$$

This means that gap(A) grows asymptotically as fast as $|A|^2$ for monochromatic subsets A. From (8) we see that gap(A) is asymptotically bounded below by $|A|$ for rainbow sets A.

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