Semiarcs with long secants

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Abstract

In a projective plane \( \Pi_q \) of order \( q \), a non-empty point set \( S_t \) is a \( t \)-semiarc if the number of tangent lines to \( S_t \) at each of its points is \( t \). If \( S_t \) is a \( t \)-semiarc in \( \Pi_q \), \( t < q \), then each line intersects \( S_t \) in at most \( q + 1 - t \) points. Dover proved that semiovals (semiarcs with \( t = 1 \)) containing \( q \) collinear points exist in \( \Pi_q \) only if \( q \leq 3 \). We show that if \( t > 1 \), then \( t \)-semiarcs with \( q + 1 - t \) collinear points exist only if \( t \geq \sqrt{q - 1} \). In PG(2, q) we prove the lower bound \( t \geq (q - 1)/2 \), with equality only if \( S_t \) is a blocking set of Rédei type of size \( 3(q + 1)/2 \).

We call the symmetric difference of two lines, with \( t \) further points removed from each line, a \( V_t \)-configuration. We give conditions ensuring a \( t \)-semiarc to contain a \( V_t \)-configuration and give the complete characterization of such \( t \)-semiarcs in PG(2, q).

Keywords: collineation group; blocking set; semioval

1 Introduction

Semiarcs are natural generalizations of arcs. Let \( \Pi_q \) be a projective plane of order \( q \). A non-empty point set \( S_t \subset \Pi_q \) is called a \( t \)-semiarc if for every point \( P \in S_t \) there exist exactly \( t \) lines \( \ell_1, \ell_2, \ldots, \ell_t \) such that \( S_t \cap \ell_i = \{P\} \) for \( i = 1, 2, \ldots, t \). These lines are called the tangents to \( S_t \) at \( P \). If a line \( \ell \) meets \( S_t \) in \( k > 1 \) points, then \( \ell \) is called a \( k \)-secant of \( S_t \). The classical examples of semiarcs are the semiovals (semiarcs with \( t = 1 \)) and point sets of type \((0, 1, n)\) (i.e. point sets meeting each line in either 0, or 1, or \( n \) points, in this case \( t = q + 1 - (s - 1)/(n - 1) \), where \( s \) denotes the size of the point set). Arcs, unitals, and subplanes are semiarcs of the latter type. For more examples, see [1], [5] and [10].

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Because of the huge diversity of the geometry of semi-arcs, their complete classification is hopeless. In [7] Dover investigated semi-ovals with a $q$-secant and semi-ovals with more than one $(q-1)$-secant. The aim of this paper is to generalize these results and characterize $t$-semi-arcs with long secants.

Many of the known $t$-semi-arcs contain the symmetric difference of two lines, with $t$ further points removed from each line. We will call this set of $2(q-t)$ points a $V_t$-configuration. Recently in [5] it was proved that in $\text{PG}(2,q)$ small semi-arcs with a long secant necessarily contain a $V_t$-configuration or can be obtained from a blocking set of Rédei type. Here we give another condition ensuring a $t$-semiarc to contain a $V_t$-configuration and we give the complete characterization of such $t$-semi-arcs in $\text{PG}(2,q)$. To do this we use the classification of perspective point sets in $\text{PG}(2,q)$. This is a result due to Korchmáros and Mazzoca [11] and it is related to Dickson’s classification of the subgroups of the affine group on the line $\text{AG}(1,q)$.

Using a result of Weiner and Szönyi, that was conjectured by Metsch, we prove that $t$-semi-arcs in $\text{PG}(2,q)$ with $q+1-t$ collinear points exist if and only if $t \geq (q-1)/2$. The case of equality is strongly related to blocking sets of Rédei type, we also discuss these connections.

If $t = q+1, q$ or $q-1$, then $S_t$ is single point, a subset of a line or three non-collinear points respectively. To avoid trivial cases, we may assume for the rest of this paper that $t < q-1$.

## 2 Semi-arcs with one long secant

If $S_t$ is a $t$-semiarc in $\Pi_q$, $t < q$, then each line intersects $S_t$ in at most $q+1-t$ points. In this section we study $t$-semi-arcs containing $q+1-t$ collinear points. The following lemma gives an upper bound for the size of such $t$-semi-arcs.

**Lemma 1.** If $S_t$ is a $t$-semiarc in $\Pi_q$ and $\ell$ is a $(q+1-t)$-secant of $S_t$, then $|S_t \setminus \ell| \leq q$.

**Proof.** Let $U = S_t \setminus \ell$ and let $D = \ell \setminus S_t$. Through each point of $U$ there pass exactly $t$ tangents to $S_t$ and each of them intersects $\ell$ in $D$. This implies $t|U| \leq q|D|$. Since $|D| = t$, we have $|U| \leq q$. \hfill \Box

In [7] Dover proved that semi-ovals with a $q$-secant exist in $\Pi_q$ if and only if $q \leq 3$. Our first theorem generalizes this result and shows that if $S_t$ has a $(q+1-t)$-secant, then $t$ cannot be arbitrary. For related ideas of the proof, see the survey paper by Blokhuis et. al. [3], Theorem 3.2.

**Theorem 2.** If $S_t$ is a $t$-semiarc in $\Pi_q$ with a $(q+1-t)$-secant, then $t = 1$ and $q \leq 3$ or $t \geq \sqrt{q-1}$.

**Proof.** Let $\ell$ be a line that satisfies $|S_t \cap \ell| = q+1-t$ and let $U = S_t \setminus \ell$. The size of $U$ has to be at least $q-t$, otherwise the points of $\ell \cap S_t$ would have more than $t$ tangents. This and Lemma 1 together yield:

$$q-t \leq |U| \leq q.$$  \hfill (1)
Let \( q - t + k \) be the size of \( U \), where \( 0 \leq k \leq t \). Let \( \delta \) be the number of lines that do not meet \( U \) and denote by \( L_1, L_2, \ldots, L_{q^2+q+1-\delta} \) the lines that meet \( U \). For these lines let \( e_i = |L_i \cap U| \). The standard double counting argument gives:

\[
\sum_{i=1}^{q^2+q+1-\delta} e_i = (q-t+k)(q+1), \tag{2}
\]

\[
\sum_{i=1}^{q^2+q+1-\delta} e_i(e_i-1) = (q-t+k)(q-t+k-1). \tag{3}
\]

If a line \( \ell' \) intersects \( U \) in more than one point, then \( Q := \ell' \cap \ell \) is in \( S_t \), otherwise the points of \( \ell' \cap U \) would have at most \( t - 1 \) tangents. The point \( Q \in S_t \) has at least \( q - 1 - (q-t+k-|\ell' \cap U|) = t-1-k+|\ell' \cap U| \) tangents, hence \( |\ell' \cap U| \leq k+1 \). This implies \( e_i \leq k+1 \), for \( i = 1, 2, \ldots, q^2 + q + 1 - \delta \), thus the following holds:

\[
\sum_{i=1}^{q^2+q+1-\delta} e_i(e_i-1) \leq (k+1) \sum_{i=1}^{q^2+q+1-\delta} (e_i-1) = (k+1)((q-t+k)(q+1)-(q^2+q+1-\delta)). \tag{4}
\]

The line \( \ell \) does not meet \( U \) and the other lines that do not meet \( U \) fall into two classes: there are \((q+1-t)t\) of them passing through \( \ell \cap S_t \) (the tangents to \( S_t \) through the points of \( \ell \cap S_t \)) and there are \( tq - (q-t+k)t \) of them passing through \( \ell \setminus S_t \) (the lines intersecting \( \ell \setminus S_t \) minus the tangents to \( S_t \) through the points of \( U \)). This implies \( \delta = t(q+1-k)+1 \), hence we can write (4) as:

\[
(q-t+k)(q-t+k-1) \leq (k+1)((q-t+k)(q+1)-(q^2+q)+t(q+1-k)). \tag{5}
\]

Rearranging this inequality we obtain:

\[
q^2 - q(2t + 1 - k + k^2) + k^2t - kt - 2k + t^2 + t \leq 0.
\]

The discriminant of the left-hand side polynomial is \( k^4 - 2k^3 + 3k^2 + 6k + 1 \). If \( k = 0, 1, 2 \), then we get \( q \leq t + 1, t + 2, t + 4 \) respectively. Otherwise, we have \( k^4 - 2k^3 + 3k^2 + 6k + 1 < (k^2 - k + 3)^2 \), which yields \( q \leq t + k^2 - k + 1 \). The maximum value of \( k \) is \( t \), therefore \( q \leq t^2 + 1 \) follows for \( k \geq 3 \). If \( t = 1 \), then \( k \leq 1 \), hence \( q \leq t + 2 = 3 \). If \( t = 2 \), then \( k \leq 2 \), hence \( q \leq t + 4 = 6 \). Since there is no projective plane of order 6, in this case we get \( q \leq 5 \). If \( t \geq 3 \) and \( k < 3 \), then \( q \leq t + 4 < t^2 + 1 \) and this completes the proof. \( \Box \)

Before we go further we need some definitions about blocking sets. A blocking set of a projective plane is a point set \( B \) that intersects every line in the plane. A blocking set is minimal if it does not contain a smaller blocking set and it is non-trivial if it does not contain a line. If \( B \) is a non-trivial blocking set, then we have \(|\ell \cap B| \leq |B| - q \) for every line \( \ell \). If there is a line \( \ell \) such that \(|\ell \cap B| = |B| - q \), then \( B \) is a blocking set of Rédei type and the line \( \ell \) is a Rédei line of \( B \).

In PG(2, q) we can improve the bound in Theorem 2. To do this we use the following result, conjectured by Metsch [13] and proven by Weiner and Szőnyi in [15, 16].

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Theorem 3 ([15, 16]). Let $U$ be a point set in PG(2, $q$), $P$ a point not in $U$ and assume that there pass exactly $r$ lines through $P$ meeting $U$. Then the total number of lines meeting $U$ is at most $1 + rq + (|U| - r)(q + 1 - r)$.

Theorem 4. Let $S_t$ be a $t$-semiarc in PG(2, $q$). If $S_t$ has a $(q + 1 - t)$-secant, then $t \geq (q - 1)/2$. In the case of equality, $S_t$ is a blocking set of Rédei type and its $(q + 1 - t)$-secants are Rédei lines.

Proof. Let $\ell$ be a $(q + 1 - t)$-secant of $S_t$ and let $U = S_t \setminus \ell$. From Lemma 1, we have:
$$|U| \leq q. \quad (6)$$

The following statements are easy to check:
- the lines intersecting $U$ in more than one point intersect $\ell$ in $\ell \cap S_t$;
- through each point of $\ell \cap S_t$ there pass exactly $r = q - t$ lines meeting $U$;
- the total number of lines meeting $U$ is $\delta = |U|t + (q + 1 - t)(q - t)$.

Applying Theorem 3 for the point set $U$ and for a point $P \in \ell \cap S_t$, we obtain:
$$\delta = |U|t + (q + 1 - t)(q - t) \leq 1 + (q - t)q + (|U| - q + t)(t + 1). \quad (7)$$

After rearranging, we get:
$$2q - 2t - 1 \leq |U|. \quad (8)$$

Equations (6) and (8) together imply $t \geq (q - 1)/2$. If $t = (q - 1)/2$, then $|U| = q$ and there are $\delta = (3q^2 + 2q + 3)/4$ lines meeting $U$ and $(q + 1 - t)t = (q^2 + 2q - 3)/4$ lines meeting $\ell \cap S_t$ but not $U$. Together with the line $\ell$ we get the total number of lines in PG(2, $q$), thus $S_t$ is a blocking set of Rédei type and $\ell$ is a Rédei line of $S_t$.

The following result by Blokhuis yields another connection between blocking sets and semiarcs.

Theorem 5 ([2]). If $B$ is a minimal non-trivial blocking set in PG(2, $p$), $p > 2$ prime, then $|B| \geq 3(p + 1)/2$. In the case of equality there pass exactly $(p - 1)/2$ tangent lines through each point of $B$.

Example 6 ([9], Lemma 13.6). Denote by $C$ the set of non-zero squares in GF($q$), $q$ odd, and let $S_t = \{(c, 0, 1), (0, -c, 1), (c, 1, 0) : c \in C\} \cup \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. This point set is called projective triangle and it is a $t$-semiarc with three $(q + 1 - t)$-secants, where $t = (q - 1)/2$. This example shows the sharpness of Theorems 4 and 5.

In PG(2, $q$), $q$ prime, Lovász and Schrijver proved that blocking sets of Rédei type of size $3(q + 1)/2$ are projectively equivalent to the projective triangle, see [12]. Gács, Lovász, and Szőnyi proved the same if $q$ is a square of a prime, see [8]. These results and Theorem 4 together yield the following:

Corollary 7 ([8, 12]). Let $S_t$ be a $t$-semiarc in PG(2, $q$) with a $(q + 1 - t)$-secant. If $t = (q - 1)/2$ and $q = p$ or $q = p^2$, $p$ prime, then $S_t$ is projectively equivalent to the projective triangle.
3 Semiars with two long secants

Throughout the paper, if \(A\) and \(B\) are two point sets in \(\Pi_q\), then \(A\Delta B\) denotes their symmetric difference, that is \((A \setminus B) \cup (B \setminus A)\).

**Definition 8.** A \(V_t\)-configuration is the symmetric difference of two lines, with \(t\) further points removed from both lines. Semiars containing a \(V_t\)-configuration fall into two types. Let \(S_t\) be a \(t\)-semiarc and suppose that there are two lines, \(\ell_1\) and \(\ell_2\), such that \((\ell_1 \triangle \ell_2) \cap S_t\) is a \(V_t\)-configuration, then:

- \(S_t\) is of \(V_t^o\) type if \(\ell_1 \cap \ell_2 \notin S_t\),
- \(S_t\) is of \(V_t^*\) type if \(\ell_1 \cap \ell_2 \in S_t\).

For semiiovals, Dover proved the following characterization:

**Theorem 9** ([7], Lemma 4.1, Theorem 4.2). Let \(S_1\) be a semioval in \(\Pi_q\). If \(S_1\) is of \(V_1^o\) type, then it is contained in a vertexless triangle. If \(q > 5\) and \(S_1\) has at least two \((q - 1)\)-secants, then \(S_1\) is of \(V_1^o\) type.

As the above result suggests, the characterization of \(t\)-semiars with two \((q - t)\)-secants works nicely only for semiars of \(V_1^o\) type. In Proposition 11 we generalize the last statement of the above result, but the characterization of \(V_t^*\) type semiars seems to be hopeless in general. In Proposition 12 we consider the case when \(t = 2\), but for larger values of \(t\) we deal only with the Desarguesian case, see Section 4.

**Lemma 10.** Let \(S_t\) be a \(t\)-semiarc in \(\Pi_q\), \(t < q\), and suppose that there exist two lines, \(\ell_1\) and \(\ell_2\), with their common point in \(S_t\) such that \(|\ell_1 \setminus (S_t \cup \ell_1)| = n\) and \(|\ell_2 \setminus (S_t \cup \ell_1)| = m\). Then \(q \leq t + 1 + nm/t\) and \(|S_t \setminus (\ell_1 \cup \ell_2)| = q - 1 - t\) in the case of equality.

**Proof.** Since \(S_t\) is not contained in a line, we have \(n, m \geq t\). If one of \(n\) or \(m\) is equal to \(q\), then \(q < q + t + 1 \leq t + 1 + nm/t\) and the assertion follows. Thus we can assume that \(\ell_1\) and \(\ell_2\) are not tangents to \(S_t\). Let \(X = S_t \setminus (\ell_1 \cup \ell_2)\). Through the point \(\ell_1 \cap \ell_2\) there pass exactly \(t\) tangents to \(S_t\), hence \(q - 1 - t \leq |X|\). Through the points of \(X\) there pass \(|X|t\) tangents to \(S_t\), each of them intersects \(\ell_1\) and \(\ell_2\) off \(S_t\), hence \(|X|t \leq nm\). These two inequalities imply \(q \leq t + 1 + nm/t\) and \(|X| = q - 1 - t\) in the case of equality.

**Proposition 11.** Let \(S_t\) be a \(t\)-semiarc in \(\Pi_q\). If \(S_t\) has at least two \((q - t)\)-secants and \(q > 2t + 3\), then \(S_t\) is of \(V_t^o\) type. If \(S_t\) has at least two \((q - t + 1)\)-secants, then \(S_t\) is of \(V_t^*\) type.

**Proof.** If \(S_t\) has at least two \((q - t)\)-secants with their common point in \(S_t\), then Lemma 10 implies \(q \leq t + 1 + (t + 1)^2/t = 2t + 3 + 1/t\). If \(q > 2t + 3\), then this is only possible when \(t = 1\) and \(q = 6\), but there is no projective plane of order 6. Hence the common point of the \((q - t)\)-secants is not contained in \(S_t\), which means that \(S_t\) is of \(V_t^o\) type. The proof of the second statement is straightforward.

**Proposition 12.** Let \(S_t\) be a \(t\)-semiarc of \(V_t^o\) type in \(\Pi_q\). Then the following hold.

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(a) \(|S_t| \neq 2q - 2t + 1\).

(b) If \(t = 2\), then \(S_t\) is a \(V_2\)-configuration or \(|S_t| = 2q - 2\) and \(S_t = (\ell_1 \cup \ell_2) \triangle \Pi_2\), where \(\ell_1\) and \(\ell_2\) are two lines in \(\Pi_2\), that is a Fano subplane contained in \(\Pi_q\).

(c) If \(t > 1\), then \(|S_t| \leq 2q - t\).

\textbf{Proof.} Let \(S_t\) be a \(t\)-semiarc of \(V_t^o\) type and let \(\ell_1\) and \(\ell_2\) be two \((q - t)\)-secants of \(S_t\) such that \(P := \ell_1 \cap \ell_2\) is not contained in \(S_t\). Denote the points of \(\ell_1 \setminus (S_t \cup P)\) by \(A_1, \ldots, A_t\), the points of \(\ell_2 \setminus (S_t \cup P)\) by \(B_1, \ldots, B_t\). Let \(X = S_t \setminus (\ell_1 \cup \ell_2)\) and define the line set \(\mathcal{L} := \{A_iB_j : 1 \leq i, j \leq t\}\) of size \(t^2\). Through each point \(Q \in X\) there pass exactly \(t\) lines of \(\mathcal{L}\), otherwise there would be an index \(i \in \{1, 2, \ldots, t\}\) for which the line \(QA_i\) meets \(\ell_2\) in \(S_t\). But then there would be at most \(t - 1\) tangents to \(S_t\) through the point \(QA_i \cap \ell_2\), a contradiction.

Suppose, contrary to our claim, that \(X\) consists of a unique point denoted by \(Q\). Then \(Q\) would have \(t + 1\) tangents: the \(t\) lines of \(\mathcal{L}\) that pass through \(Q\) and the line \(PQ\).

If \(t = 2\), then exactly two of the points of \(\Pi_q \setminus (\ell_1 \cup \ell_2)\) are contained in two lines of \(\mathcal{L}\). These are \(Q_1 := A_1B_1 \cap A_2B_2\) and \(Q_2 := A_1B_2 \cap A_2B_1\). Since \(|X| > 1\), we have \(X = Q_1, Q_2\). If \(P\) were not collinear with \(Q_1\) and \(Q_2\), then \(PQ_t\) would be a third tangent to \(S_t\) at \(Q_i\), for \(i = 1, 2\). It follows that the point set \(\Pi_t := \{P, A_1, A_2, B_1, B_2, Q_1, Q_2\}\) is a Fano subplane in \(\Pi_q\).

To prove (c), define \(Y \subseteq X\) as \(Y := \{A : A \in X, |AP \cap S_t| = 1\}\). The line set \(\mathcal{L}\) contains \(|Y|(t - 1)\) tangents through the points of \(Y\) and \((|X| - |Y|)t\) tangents through the points of \(X \setminus Y\), hence

\[|X|(t - 1) \leq |X|t - |Y| = |\mathcal{L}| - \delta \leq t^2,\]

where \(\delta\) denotes the number of non-tangent lines in \(\mathcal{L}\). Because of (b), we may assume \(t > 2\), hence \(|X| \leq t^2/(t - 1) < t + 2\) follows. To obtain a contradiction, suppose that \(|X| = t + 1\). If this is the case, then (9) implies \(t \leq |Y|\). If \(|Y| = t\), then \(X \setminus Y\) consists of a unique point, but this contradicts the definition of \(Y\). If \(|Y| = t + 1\), then \(X = Y\) and through each point of \(X\) there pass a non-tangent line, which is in \(\mathcal{L}\). Thus if \(\delta = 1\), then the points of \(X\) are contained in a line \(\ell \in \mathcal{L}\). We may assume that \(\ell = A_1B_t\). Then we can find \(2(t - 1)\) other non-tangent lines in \(\mathcal{L}\), these are \(A_tB_t\) and \(B_iA_i\) for \(i = 1, 2, \ldots, t - 1\). On the other hand \(\delta > 1\) contradicts (9) and this contradiction proves \(|X| < t\). \(\square\)

The following result shows some kind of stability of semiars containing a \(V_t\)-configuration.

\textbf{Theorem 13.} Let \(S_t\) be a \(t\)-semiarc in \(\Pi_q\), \(t < q\), and suppose that there exist two lines, \(\ell_1\) and \(\ell_2\), such that \(|\ell_1 \setminus (S_t \cup \ell_2)| = n\) and \(|\ell_2 \setminus (S_t \cup \ell_1)| = m\).

1. If \(\ell_1 \cap \ell_2 \notin S_t\), \(t > 1\) and \(q > \min\{n, m\} + 2nm/(t - 1)\), then \(S_t\) is of \(V_t^o\) type.

2. If \(\ell_1 \cap \ell_2 \in S_t\) and \(q > \min\{n, m\} + nm/t\), then \(t = (q - 1)/2\), \(|S_t| = 3(q + 1)/2\) and \(S_t\) is of \(V_t^\bullet\) type.
We have \( n = m = t \) in both cases.

**Proof.** We may assume \( m \geq n \). In part 1, we have \( n \geq t - 1 \), with equality only if \( \ell_2 \) is not a secant of \( S_t \), i.e. when \( m \in \{q - 1, q\} \). The assumption \( q > \min\{n, m\} + 2nm/(t - 1) \) implies \( n, m < q - 1 \), hence this is not the case. It follows that \( n, m \geq t \) holds. In part 2, we have \( n, m \geq t \), hence the assumption implies \( n, m < q \) or, equivalently, the lines \( \ell_1 \) and \( \ell_2 \) are secants of \( S_t \). First we show \( n = m = t \) in both cases. From this, part 1 follows immediately.

Suppose, contrary to our claim, that \( m \geq t + 1 \). Denote by \( P \) the intersection of \( \ell_1 \) and \( \ell_2 \). Let \( N = \{N_1, N_2, \ldots, N_{q-n}\} \) be the set of points of \((\ell_1 \cap P) \cap S_t \) and \( M = \{M_1, M_2, \ldots, M_m\} \) be the set of points of \( \ell_2 \setminus (S_t \cup P) \). Let \( X = S_t \setminus (\ell_1 \cup \ell_2) \). Through each point \( N_j \in N \) there pass exactly \( m - t \) non-tangent lines that intersect \( \ell_2 \) in \( M \). Each of these lines contains at least one point of \( X \). Denote the set of these points by \( X(N_j) \). Then we have the following:

- \( |X(N_i)| \geq m - t \), for \( i = 1, 2, \ldots, q - n \),
- \( X \supseteq \cup_{i=1}^{q-n} X(N_i) \),
- if \( P \not\in S_t \), then each point of \( X \) is contained in at most \( m - t + 1 \) point sets of \( \{X(N_1), \ldots, X(N_{q-n})\} \),
- if \( P \in S_t \), then each point of \( X \) is contained in at most \( m - t \) point sets of \( \{X(N_1), \ldots, X(N_{q-n})\} \).

In part 1, we have the following lower bound for the size of \( X \):

\[
\frac{(q - n)(m - t)}{m - t + 1} \leq |X|.
\]  

(10)

On the other hand, through each point of \( X \) there pass at least \( t - 1 \) tangents that intersect both \( \ell_1 \setminus (S_t \cup P) \) and \( M \). Hence we have:

\[
|X| \leq \frac{nm}{t - 1}.
\]  

(11)

Summarizing these two inequalities we get:

\[
q \leq n + \frac{nm}{t - 1} + \frac{nm}{(m - t)(t - 1)} \leq n + \frac{2nm}{t - 1},
\]

that is a contradiction.

In part 2, observe that Lemma 10 and \( q > \min\{m, n\} + nm/t \) together imply \( n = t \). If \( m \geq t + 1 \), then similarly to (10) and (11), we get \( (q - t)(m - t)/(m - t) \leq |X| \) and \( |X| \leq mt/t \) respectively. These two inequalities imply \( q \leq t + m \), contradicting our assumption \( q > \min\{n, m\} + nm/t = t + m \), hence \( m = t \) follows. If \( n = m = t \), then Lemma 10 implies \( q \leq 2t + 1 \) while our assumption yields \( q > 2t \), thus \( t = (q - 1)/2 \). Since in this case there is equality in Lemma 10, we have \( |S_t| = 3q - 3t = 3(q + 1)/2 \).  \( \square \)
Let $\alpha_{n,m}$ and $\beta_{n,m}$ denote the lower bounds on $q$ in part 1 and in part 2 of Theorem 13, respectively. The following example shows that the weaker assumptions $\alpha_{n,m} < 3q$ and $\beta_{n,m} < 2q$, respectively, do not imply the existence of a $V_t$-configuration contained in the semiarc.

**Example 14.** We give two examples for $t$-semiarcs, $S_t$, such that they do not contain a $V_t$-configuration and there exist two lines, $\ell_1$ and $\ell_2$, with $\ell_1 \setminus (\ell_2 \cup S_t) = t$ and $\ell_1 \setminus (\ell_2 \cup S_t) = t + 1$. To do this, choose a conic $C$ in $\Pi_q$, that is a projective plane of order $s > 3$. Let $Q_1$ and $Q_2$ be two points of $C$ and proceed as follows.

1. Let $\ell_i$ be the tangent of $C$ at the point $Q_i$, for $i = 1, 2$, and denote $\ell_1 \cap \ell_2$ by $P$. Take a point $Z \in Q_1Q_2$ such that $PZ$ is a secant of $C$. Then $S_0 := (\ell_1 \cup \ell_2 \cup C \cup \{Z\}) \setminus \{P, Q_2\}$ is a point set without tangents. Now, if $\Pi_s$ is a $t$-semiarc in $\Pi_q$, then $S_0 \subset \Pi_s$ is a $t$-semiarc in $\Pi_q$, with $t = q - s$. We have $\ell_1 \cap \ell_2 \notin S_t$ and

$$\alpha_{t,t+1} = (q - s) + 2 \frac{(q - s + 1)(q - s)}{q - s - 1} < 3q.$$

2. Let $\ell_1$ be the tangent of $C$ at $Q_1$ and let $\ell_2$ be the line $Q_1Q_2$. Take a point $Z \in \ell \setminus (\ell_1 \cup \ell_2)$, where $\ell$ denotes the tangent of $C$ at $Q_2$. Then $S_0 := (\ell_1 \cup \ell_2 \cup C \cup \{Z\}) \setminus \{Q_2\}$ is a point set without tangents. As before, if $\Pi_s$ is contained in $\Pi_q$, then $S_0 \subset \Pi_s$ is a $t$-semiarc in $\Pi_q$, with $t = q - s$. We have $\ell_1 \cap \ell_2 \in S_t$ and

$$\beta_{t,t+1} = (q - s) + \frac{(q - s + 1)(q - s)}{q - s} < 2q.$$

The next example is due to Sugetake and it shows that when $t = 1$, then there is no analogous result for part 1 of Theorem 13.

**Example 15 ([14], Example 3.3).** Let $A$ be a proper, not empty subset of $\text{GF}(q) \setminus \{0\}$, such that $A = -A := \{-a : a \in A\}$ and $|A| \geq 2$. Let $B = \text{GF}(q) \setminus (A \cup \{0\})$ and define the following set of points in $\text{PG}(2,q)$:

$$S_1 := \{(0, a, 1), (b, 0, 1), (c, c, 1), (m, 1, 0) : a \in A, b \in B, c, m \in \text{GF}(q) \setminus \{0\}, m \neq 1\}.$$

Then $S_1$ is a semiarc with a $(q - 1)$-secant, $X = Y$, and a $(q - 2)$-secant, $Z = 0$, intersecting each other not in $S_1$. Also, $S_1$ is not of $V_1^\circ$ type.

When $A = \text{GF}(q) \setminus \{0\}$ in the above example, then $S_1$ is a vertexless triangle with one point deleted from one of its sides. This example exists also in non-Desarguesian planes, but it is a semiarc of $V_1^\circ$ type.

Semiarcs that properly contain a $V_t$-configuration exist in $\Pi_q$ whenever $\Pi_q$ contains a subplane. Some of the following examples were motivated by an example due to Korchmáros and Mazzocca (see [11], pg. 64).
Example 16. Let $\Pi^0, \Pi^1, \ldots, \Pi^{s-1}$ be subplanes of $\Pi^s := \Pi_q$ such that $\Pi^{i-1} \subset \Pi^i$ for $i = 1, \ldots, s$. Denote by $r$ the order of $\Pi^0$ and let $\ell_1$ and $\ell_2$ be two lines in this plane. Let $P = \ell_1 \cap \ell_2$ and set

$$S(0) := (\ell_1 \cup \ell_2) \cap (\Pi^0 \setminus P), \quad S(j) := (\ell_1 \cup \ell_2) \cap (\Pi^j \setminus \Pi^{j-1}),$$

for $j = 1, \ldots, s$.

By $I$ we denote a subset of $\{1, 2, \ldots, s\}$. We give four examples.

1. Let $\ell$ be a line in $\Pi^0$ passing through $P$ and let $Z$ be a subset of $(\ell \cap \Pi^0) \setminus \{P\}$ of size at least two. If $I$ is not empty, then $S_I := \cup_{j \in I} S(j) \cup Z$ is a $t$-semiarc of $V_t^0$ type with $t = q - \frac{1}{2} \sum_{j \in I} |S(j)|$.

2. Let $\ell$ be a line in $\Pi^0$ that does not pass through $P$ and let $Z$ be a subset of $(\ell \cap \Pi^0) \setminus (\ell_1 \cup \ell_2)$ of size at least two. If $I$ is not empty, then $S_I := \cup_{j \in I} S(j) \cup Z$ is a $t$-semiarc of $V_t^0$ type with $t = q - r - \frac{1}{2} \sum_{j \in I} |S(j)|$.

3. Let $Z$ be a subset of $\Pi^0 \setminus (\ell_1 \cup \ell_2)$ such that there is no line in $\Pi^0$ passing through $P$ and meeting $Z$ in exactly one point. If $I$ is a proper subset of $\{1, 2, \ldots, s\}$, then $S_I := \cup_{j \in I} S(j) \cup Z \cup S(0)$ is a $t$-semiarc of $V_t^0$ type with $t = q - r - \frac{1}{2} \sum_{j \in I} |S(j)|$.

4. Let $Z$ be a subset of $\Pi^0 \setminus (\ell_1 \cup \ell_2)$ such that for each line $\ell \neq \ell_1, \ell_2$ through $P$, $\ell$ is a line in $\Pi^0$, we have $|\ell \cap Z| \geq 1$. Then $S_I := \{P\} \cup S(0) \cup Z$ is a $t$-semiarc of $V_t^*$ type with $t = q - r$.

4 Semiarcs containing a $V_t$-configuration in $PG(2, q)$

In this section our aim is to characterize $t$-semiarcs containing a $V_t$-configuration in $PG(2, q)$. We will need the following definition.

Definition 17. Let $\ell_1$ and $\ell_2$ be two lines in a projective plane and let $P$ denote their common point. We say that $X_1 \subseteq \ell_1 \setminus P$ and $X_2 \subseteq \ell_2 \setminus P$ are two perspective point sets if there is a point $Q$ such that each line through $Q$ intersects both $X_1$ and $X_2$ or intersects none of them. In other words, there is a perspectivity which maps $X_1$ onto $X_2$.

Lemma 18. Let $S_I$ be a $t$-semiarc in $\Pi_q$ and suppose that $(\ell_1 \cup \ell_2) \cap S_I$ is a $V_t$-configuration for some lines $\ell_1$ and $\ell_2$. If $S_I \not\subseteq \ell_1 \cup \ell_2$, then $S_I \cap (\ell_1 \setminus \ell_2)$ and $S_I \cap (\ell_2 \setminus \ell_1)$ are perspective point sets and each point of $S_I \setminus (\ell_1 \cup \ell_2)$ is the centre of a perspectivity which maps $S_I \cap (\ell_1 \setminus \ell_2)$ onto $S_I \cap (\ell_2 \setminus \ell_1)$.

Proof. Let $X = S_I \setminus (\ell_1 \cup \ell_2)$ and $X_i = S_I \cap (\ell_i \setminus \ell_j)$, for $\{i, j\} = \{1, 2\}$. For each $Q \in X$, if there were a line $\ell$ through $Q$ intersecting $X_i$ but not $X_j$, then the point $\ell \cap X_i \in S_I$ would have at most $t - 1$ tangents. This shows that each point of $X$ is the centre of a perspectivity which maps $X_1$ onto $X_2$. If $S_I \not\subseteq \ell_1 \cup \ell_2$, then $X$ is not empty, hence $X_1$ and $X_2$ are perspective point sets. \hfill $\Box$
The following theorem characterizes perspective point sets in PG(2, q). This result was first published by Korchmáros and Mazzoca in [11] but we will use the notation of [4] by Bruen, Mazzocca and Polverino.

**Theorem 19** ([4], Result 2.2, Result 2.3, Result 2.4, see also [11]). Let \( \ell_1 \) and \( \ell_2 \) be two lines in PG(2, q), \( q = p^r \), and let \( P \) denote their common point. Let \( X_1 \subseteq \ell_1 \setminus P \) and \( X_2 \subseteq \ell_2 \setminus P \) be two perspective point sets. Denote by \( U \) the set of all points which are centres of a perspectivity mapping \( X_1 \) onto \( X_2 \). Using a suitable projective frame in PG(2, q), there exist an additive subgroup \( B \) of GF(q) and a multiplicative subgroup \( A \) of GF(q) such that:

(a) \( B \) is a subspace of GF(q) of dimension \( h_1 \) considered as a vectorspace over a subfield GF(q\(_1\)) of GF(q) with \( q_1 = p^d \) and \( d \mid r \). This implies that \( B \) is an additive subgroup of GF(q) of order \( p^h \) with \( h = dh_1 \).

(b) \( A \) is a multiplicative subgroup of GF(q\(_1\)) of order \( n \), where \( n \mid (p^d - 1) \). In this way, \( B \) is invariant under \( A \), i.e. \( B = AB := \{ab : a \in A, b \in B\} \).

(c) If \( G_i \) denotes the full group of affinities of \( \ell_i \setminus P \) preserving the set \( X_i \), \( i = 1, 2 \), then \( G_1 \cong G_2 \cong G = G(A,B) = \{g : g(y) = ay + b, a \in A, b \in B\} \leq \Sigma \), where \( \Sigma \) is the full affine group on the line AG(1, q).

(d) \( X_i \) is a union of orbits of \( G_i \) on \( \ell_i \setminus P \), \( i = 1, 2 \), and \( |U| = |G| = np^h \).

(e) For every two integers \( n \), \( h \), such that \( n \mid (p^d - 1) \) and \( d \mid \gcd(r, h) \), there exists in \( \Sigma \) a subgroup of type \( G = G(A,B) \) of order \( np^h \), where \( A \) and \( B \) are multiplicative and additive subgroups of GF(q) of order \( n \) and \( p^h \), respectively.

(f) \( G \) has one orbit of length \( p^h \) on AG(1, q), namely \( B \), and \( G \) acts regularly on the remaining orbits, say \( O_1, O_2, \ldots, O_m \), where

\[
m = \frac{q - p^h}{np^h} = \frac{p^{r-h} - 1}{n}.\]

In the sequel we denote by \( B^i \) the orbit of \( G_i \) on \( \ell_i \setminus P \) corresponding to \( B \) and by \( O^i_1, O^i_2, \ldots, O^i_m \) the remaining orbits, for \( i = 1, 2 \). With this notation \( B^1 \) is the image of \( B^2 \) under the perspectivities with centre in \( U \) and also \( O^2_j \) is the image of \( O^1_j \) for \( j = 1, 2, \ldots, m \) and vice versa.

(g) \( B^1 \subseteq X_1 \) if and only if \( B^2 \subseteq X_2 \) and the same holds for the other orbits, i.e. \( O^1_j \subseteq X_1 \) if and only if \( O^2_j \subseteq X_2 \), for \( j = 1, 2, \ldots, m \).

(h) If a line \( \ell \) not through \( P \) meets \( U \) in at least two points, then \( \ell \) intersects both \( B^1 \) and \( B^2 \).

Exactly one of the following cases must occur:
1. Both $A$ and $B$ are trivial. Then $U$ consists of a singleton.

2. $A$ is trivial and $B$ is not trivial. Then $U$ is a set of $p^h$ points all collinear with the point $P$.

3. $B$ is trivial and $A$ is not trivial. Then $U$ is a set of $n$ points on a line not through $P$.

4. $A$ and $B$ are the multiplicative and the additive group, respectively, of a subfield $GF(p^h)$ of $GF(q)$. Then

\[ U \cup B^1 \cup B^2 \cup \{P\} = PG(2, p^h). \]

5. None of the previous cases occur. Then $U$ is a point set of size $np^h$ and of type $(0, 1, n, p^h)$, i.e. $0, 1, n, p^h$ are the only intersection numbers of $U$ with respect to the lines in $PG(2, q)$. In addition, using the fact that $|U| = np^h$,

- there are exactly $n$ lines intersecting $U$ in exactly $p^h$ points and they are all concurrent at the common point $P$ of $\ell_1$ and $\ell_2$,
- each line intersecting $U$ in exactly $n$ points meets both $B^1$ and $B^2$.

Lemma 20 ([6], Proposition 3.1). If $S_t$ is a $(q - 2)$-semiarc in $\Pi_q$, then it is one of the following three configurations: four points in general position, the six vertices of a complete quadrilateral, or a Fano subplane.

In the next theorems we will use the notation of Theorem 19.

**Theorem 21.** Let $S_t$ be a $t$-semiarc in $PG(2, q)$, $q = p^r$, and suppose that $(\ell_1 \triangle \ell_2) \cap S_t$ is a $V_t$-configuration for some lines $\ell_1$ and $\ell_2$. To avoid trivial cases, suppose that $S_t \nsubseteq \ell_1 \cup \ell_2$.

Let $X_i = \ell_i \cap S_t$, for $i = 1, 2$, and let $X = S_t \setminus (\ell_1 \cup \ell_2)$. Also let $P = \ell_1 \cap \ell_2$. Because of Lemma 18 we have that $X_1$ and $X_2$ are perspective point sets and $X \subseteq U$, where $U$ is the set of all points which are centres of a perspectivity mapping $X_1$ onto $X_2$. Choose a suitable coordinate system as in Theorem 19 and suppose that the size of $G = G(A, B)$ is $np^h$, i.e. $|A| = n$ and $|B| = p^h$, where $A$ and $B$ are the multiplicative and the additive subgroup of $GF(q)$ associated to the perspective point sets $X_1$ and $X_2$.

(I) If $P \notin S_t$, i.e. $S_t$ is of $V_t^\circ$ type, then one of the following holds.

(i) $X$ is contained in a line through $P$ that meets $U$ in $p^h$ points, $h \geq 1$, and we have $2 \leq |X| \leq p^h$,

(ii) $X$ is contained in a line not through $P$ that meets $U$ in $n \geq 2$ points and we have $2 \leq |X| \leq n$,

(iii) $|X| \geq 2$ and $X$ is a subset of $U$ such that there is no line through $P$ that meets $X$ in exactly one point.
In the first two cases \( X_i = \bigcup_{j \in I} O^i_j \) for some not empty subset \( I \subseteq \{1, 2, \ldots, m\} \) and for \( i = 1, 2 \). We have \( t = q - knp^h \), where \( k = |I| \) and \( 1 \leq k \leq m \), where \( m = (p^{r-h} - 1)/n \).

In the third case \( X_i = \bigcup_{j \in I} O^i_j \cup B^i \) for some proper subset \( I \subset \{1, 2, \ldots, m\} \) and for \( i = 1, 2 \). We have \( t = q - knp^h - p^h \), where \( k = |I|, h \geq 1 \) and \( 0 \leq k \leq m - 1 \).

\[(II)\] If \( P \in S_t \), i.e. \( S_t \) is of \( V_t^\bullet \) type, then one of the following holds.

\[(i)\] \( S_t \) consists of the six vertices of a complete quadrilateral or \( S_t \) is a Fano subplane. We have \( t = q - 2 \) in both cases.

\[(ii)\] \( \ell_1 \) and \( \ell_2 \) are lines in the subplane \( \text{PG}(2,p^h) \) and

\[ S_t = \text{PG}(2,p^h) \cap (\ell_1 \cup \ell_2) \cup X, \]

where \( X \) is a subset of \( \text{PG}(2,p^h) \setminus (\ell_1 \cup \ell_2) \) such that for each line \( \ell \neq \ell_1, \ell_2 \) through \( P \), \( \ell \) is a line in \( \text{PG}(2,p^h) \), we have \( |\ell \cap X| \geq 1 \). In this case \( t = q - p^h \).

\[(iii)\] \( S_t \) is projectively equivalent to the following set of \( 3(n + 1) \) points:

\[ S_t := \{(a, 0, 1), (0, -a, 1), (a, 1, 0) : a \in A\} \cup \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}, \]

In this case \( t = q - 1 - n \), where \( n \mid q - 1 \).

The converse is also true, if \( X_1 \) and \( X_2 \) are perspective point sets and \( X \) is as in one of the three cases in \((i)\), then \( X \cup X_1 \cup X_2 \) is a \( t \)-semiarc of \( V_t^\circ \) type. If \( S_t \) is as in one of the three cases in \((ii)\), then \( S_t \) is a \( t \)-semiarc of \( V_t^\bullet \) type.

**Proof.** We begin by proving \((i)\). First assume \( B^1 \subseteq \ell_1 \setminus X_1 \). Then Theorem 19 (g) implies \( B^2 \subseteq \ell_2 \setminus X_2 \). Suppose that there exist three non-collinear points in \( X \), say \( L, M \) and \( N \). Then between the lines \( LM, LN \) and \( MN \) there are at least two, say \( LM \) and \( LN \), not through \( P \). Theorem 19 (h) and \( X \subseteq U \) imply that these two lines intersect both \( B^1 \) and \( B^2 \). But then through \( L \) there pass at most \( t - 1 \) tangents, a contradiction. It follows that \( X \) is contained in a line and hence it is as in one of our first two cases. The condition \( |X| \geq 2 \) comes from Proposition 12 (a).

Now assume \( B^1 \subseteq X_1 \) and hence \( B^2 \subseteq X_2 \). In this case for every two points \( M, N \in X \), the line \( MN \) intersects \( \ell_i \) in \( X_i \), for \( i = 1, 2 \). Thus the number of tangents through a point \( L \in X \) is \( t \) if and only if the line \( LP \) contains at least one other point of \( X \). Case 3 of Theorem 19 shows that this is not possible when \( B \) is trivial, i.e. when \( h = 0 \). Hence \( X \) is as in our third case.

Now we prove \((ii)\). First assume \( B^1 \subseteq \ell_1 \setminus X_1 \) and hence \( B^2 \subseteq \ell_2 \setminus X_2 \). Suppose that there exist two points in \( X \), say \( M \) and \( N \), not collinear with \( P \). Then the line \( MN \) intersects \( \ell_1 \) and \( \ell_2 \) not in \( S_t \). But then the number of tangents through \( M \) is at most \( t - 1 \), a contradiction. Thus \( X \) is contained in a line through \( P \) and through \( P \) there pass exactly \( q - 2 \) tangents. So \( S_t \) is a \((q - 2)\)-semiarc. According to Lemma 20, \( S_t \) is as in \((ii)(i)\).

Now assume \( B^1 \subseteq X_1 \) and hence \( B^2 \subseteq X_2 \). In this case \( t = q - knp^h - p^h \) for some \( k \in \{0, 1, \ldots, m - 1\} \), where \( m \) is the number of orbits of \( G \) of size \( np^h \) on \( AG(1,q) \setminus B \).
Since $P$ has exactly $t$ tangents, there are $q + 1 - t$ non-tangent lines through $P$. According to Theorem 19, we have $q - 1 - t \leq n$ and hence $kn^p^h + p^h - 1 \leq n$. We distinguish two subcases.

If $h > 0$, then $n|p^h - 1$ implies $n \leq p^h - 1$ and hence $kn^p^h = 0$. This occurs only if $k = 0$ and $n = p^h - 1$. But $n$ divides $p^d - 1$, where $d|h$ and $B$ is a subspace over the field GF($p^d$). This implies $d = h$, thus $B$ is a subfield and $U$ is as in case 4 of Theorem 19. This is only possible if $S_t$ is as in our second case.

If $h = 0$, then $kn \leq n$ and $U$ is as in case 3 of Theorem 19. If $k = 0$, then $t = q - 1$, which we excluded. Thus we have $k = 1$ and $t = q - n - 1$. This occurs only if $S_t$ is as in our third case (see [4], pg. 56–57).

**Theorem 22.** Let $S_t$ be a $t$-semiarc of $V_t^\circ$ type in $PG(2,q)$, $q = p^r$. Then the following hold.

(a) If $\gcd(q,t) = 1$ and $\gcd(q - 1, t - 1) = 1$, then $S_t$ is a $V_t$-configuration.

(b) If $\gcd(q,t) = 1$, then $S_t$ is contained in a vertexless triangle.

(c) If $\gcd(q - 1, t) = 1$, then $S_t$ is contained in a vertexless triangle or in the union of three concurrent lines without their common point.

**Proof.** We have $p^h|t$ in all three cases of Theorem 21 (I), where $p^h$ is the size of $B$. Hence $\gcd(q,t) = 1$ implies $p^h = 1$, i.e. $h = 0$. This occurs only in the second case of Theorem 21 (I) and this proves (b).

In the first two cases of Theorem 21 (I) we have $n|(t - 1)$ and hence also $n|\gcd(q - 1, t - 1)$, where $n$ is the size of $A$. We have seen previously that $\gcd(q,t) = 1$ can hold only in the second case of Theorem 21 (I). But in that case we have $n \geq 2$, which is a contradiction when $\gcd(q - 1, t - 1) = 1$. This proves (a).

If $S_t$ is as in one of the first two cases of Theorem 21 (I), then we are done. So to prove (c), it is enough to consider Theorem 21 (I)(iii). In this case $t = (q - 1) - nkp^h - (p^h - 1)$ and hence $n|\gcd(q - 1, t)$. If $\gcd(q - 1, t) = 1$, then $n = 1$, i.e. $A$ is trivial. If this happens, then case 2 of Theorem 19 implies that $S_t$ is contained in the union of three concurrent lines without their common point.

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**References**


