Minimal crystallizations of 3-manifolds

Biplab Basak Basudeb Datta
Department of Mathematics
Indian Institute of Science
Bangalore 560 012, India
biplab10@math.iisc.ernet.in dattab@math.iisc.ernet.in

Submitted: Dec 10, 2013; Accepted: Mar 1, 2014; Published: Mar 17, 2014
Mathematics Subject Classifications: 57Q15; 57Q05; 57N10; 05C15

Abstract

We have introduced the weight of a group which has a presentation with number of relations is at most the number of generators. We have shown that the number of facets of any contracted pseudotriangulation of a connected closed 3-manifold $M$ is at least the weight of the fundamental group of $M$. This lower bound is sharp for the 3-manifolds $\mathbb{R}P^3$, $L(3,1)$, $L(5,2)$, $S^1 \times S^1 \times S^1$, $S^2 \times S^1$, $S^2 \times S^1$ and $S^3/Q_8$, where $Q_8$ is the quaternion group. Moreover, there is a unique such facet minimal pseudotriangulation in each of these seven cases. We have also constructed contracted pseudotriangulations of $L(kq-1,q)$ with $4(q+k-1)$ facets for $q \geq 3$, $k \geq 2$ and $L(kq+1,q)$ with $4(q+k)$ facets for $q \geq 4$, $k \geq 1$. By a recent result of Swartz, our pseudotriangulations of $L(kq+1,q)$ are facet minimal when $kq+1$ are even. In 1979, Gagliardi found presentations of the fundamental group of a manifold $M$ in terms of a contracted pseudotriangulation of $M$. Our construction is the converse of this, namely, given a presentation of the fundamental group of a 3-manifold $M$, we construct a contracted pseudotriangulation of $M$. So, our construction of a contracted pseudotriangulation of a 3-manifold $M$ is based on a presentation of the fundamental group of $M$ and it is computer-free.

Keywords: Pseudotriangulations of manifolds, Crystallizations, Lens spaces, Presentations of groups.

1 Introduction and Results

A simplicial cell complex $K$ of dimension $d$ is a poset isomorphic to the face poset $\mathcal{X}$ of a $d$-dimensional simplicial CW-complex $X$. The topological space $X$ is called the geometric carrier of $K$ and is also denoted by $|K|$. If a topological space $M$ is homeomorphic to $|K|$, then $K$ is said to be a pseudotriangulation of $M$. For $d \geq 1$, a $(d+1)$-colored contracted graph $\Gamma = (V,E)$ with an edge coloring $\gamma : E \to \{1, \ldots, d+1\}$ determines a
$d$-dimensional simplicial cell complex $K(\Gamma)$ whose vertices have one to one correspondence with the colors $1, \ldots, d+1$ and the facets have one to one correspondence with the vertices in $V$. If $K(\Gamma)$ is a pseudotriangulation of a space $M$ then $(\Gamma, \gamma)$ is called a crystallization of $M$. So, if $(\Gamma, \gamma)$ is a crystallization of a $d$-manifold $M$ then the number of vertices in the pseudotriangulation $K(\Gamma)$ of $M$ is $d+1$. In [15], Pezzana showed the following.

**Proposition 1** (Pezzana). Every connected closed PL-manifold admits a crystallization.

Thus, every connected closed pl $d$-manifold has a contracted pseudotriangulation, i.e., a pseudotriangulation with $d+1$ vertices. In this article, we are interested in crystallizations of connected closed 3-manifolds with minimum number of vertices.

In [6], Epstein proved that the fundamental group of a 3-manifold has a presentation with the number of relations less than or equal to the number of generators. For such a group $G$, we define the weight $\psi(G)$ of $G$ in Definition 10 below. The weight of the trivial group is $2$ and $\psi(G) \geq 8$ for any non-trivial group $G$ as we see later.

**Definition 2.** For a connected closed 3-manifold $M$, let $\psi(M)$ be the weight $\psi(\pi(M, x))$ of the group $\pi(M, x)$ for some $x$ in $M$.

If $M$ and $N$ are homeomorphic then clearly $\psi(M) = \psi(N)$. Thus, $\psi(M)$ is a topological invariant. Clearly, $\psi(S^3) = 2$ and, in view of Perelman’s theorem (Poincaré conjecture) [14], $\psi(M) \geq 8$ for $M \neq S^3$. Here, we have the following.

**Lemma 3.** Let $\psi(M)$ be as above and let $Q_8$ be the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$. Then $\psi(\mathbb{R}P^3) = \psi(S^2 \times S^1) = 8$, $\psi(S^1 \times S^1) = 16$, $\psi(S^2 \times S^1) = 18$, $\psi(S^1 \times S^1 \times S^1) = 24$ for $1 \leq q \leq 2$.

For a $d$-dimensional simplicial cell complex $K$, let $f_j(K)$ denote the number of $j$-cells of $K$ for $0 \leq j \leq d$. Let $g_2(K) := f_1(K) - (d+1)f_0(K) + \binom{d+2}{2}$ and $h_2(K) := f_1(K) - df_0(K) + \binom{d+1}{2}$. For a connected simplicial cell complex $K$, let $m(K)$ be the minimal number of generators of $\pi(\{K\}, \ast)$. For a connected closed pl $d$-manifold $M$, let

$$\Psi(M) = \min\{m : M \text{ has a crystallization with } m \text{ vertices}\} = \min\{f_d(K) : K \text{ is a contracted pseudotriangulation of } M\}.$$  

In [11], Klee proved that $h_2(K) \geq \binom{d+1}{2} m(K)$ for any $d$-dimensional normal pseudomanifold $K$ whose edge graph is $(d+1)$-colorable. Here we have the following.

**Theorem 4.** Let $M$ be a connected closed 3-manifold. If $(\Gamma, \gamma)$ is a crystallization of $M$ then $\Gamma$ has at least $\psi(M)$ vertices. Equivalently, if $X$ is a contracted pseudotriangulation of $M$ then $f_3(X) \geq \psi(M)$.

**Corollary 5.** Let $M$ be a connected closed 3-manifold $M$ and $F$ be a field. If $X$ is a contracted pseudotriangulation of $M$ then $g_2(X) = h_2(X) \geq \Psi(M) - 2 \geq \psi(M) - 2 \geq 6m(M) \geq 6\beta_1(M; F)$. 

**THE ELECTRONIC JOURNAL OF COMBINATORICS** 21(1) (2014), #P1.61
Consider the contracted pseudotriangulation \( K_1 := \mathcal{K}(\mathcal{J}_1) \) of \( S^2 \times S^1 \) corresponding to the crystallization \( \mathcal{J}_1 \) in Fig. 2 below. Since \( f_3(K_1) = 8 \), it follows that \( f_2(K_1) = 16 \) and hence \( f_1(K_1) = 12 \). Therefore, \( g_2(K_1) = 12 - 16 + 10 = 6 = 6\beta_1(S^2 \times S^1; \mathbb{Q}) \). Thus, the inequalities in Corollary 5 are equalities and (hence) the lower bound is sharp.

From the complete enumeration (obtained by using high-powered computers) of crystallizations of prime 3-manifolds with at most 30 vertices, we know \( \Psi(M) \) for all closed prime 3-manifolds \( M \) with \( \Psi(M) \leq 30 \) (cf. [3, 12]). In particular, we know that the minimal crystallizations of several 3-manifolds are unique and there are 3-manifolds which have more than one minimal crystallizations (see Remark 25 below). We have proved the existence and the uniqueness of some crystallizations using presentations of the fundamental groups. Consider a group \( G \) which has a presentation with number of relations is at most the number of generators. From Theorem 4 we know that the number of vertices in any crystallization \((\Gamma, \gamma)\) of a closed connected 3-manifold \( M \), whose fundamental group is \( G \), is at least \( \psi(G) \). We have constructed crystallizations on \( \psi(G) \) vertices which yield presentations of \( G \) as mentioned at the end of Section 2.4. We have considered the groups \( \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_7 \) and \( Q_8 \) and have obtained such crystallizations. Generalizing some of these constructions, we have constructed two infinite families of crystallizations of lens spaces. More explicitly, we have the following.

**Theorem 6.** (i) If \( M = \mathbb{RP}^3, S^2 \times S^1, S^2 \times S^1, L(3, 1), L(5, 2), S^3/Q_8 \) or \( S^1 \times S^1 \times S^1 \) then \( \Psi(M) = \psi(M) \) and \( M \) has a unique contracted pseudotriangulation with \( \psi(M) \) facets.

(ii) Let \( X \) be a contracted pseudotriangulation of a connected closed 3-manifold \( M \). If \( f_3(X) \leq 8 \) then \( M \) is (homeomorphic to) \( S^3, \mathbb{RP}^3, S^2 \times S^1 \) or \( S^2 \times S^1 \).

**Corollary 7.** Let \( X \) be a contracted pseudotriangulation of a closed 3-manifold \( M \). If \( M \) is \( S^3/Q_8, S^1 \times S^1 \times S^1 \) or \( L(p, q) \) for some \( p \geq 3 \) then \( h_2(X) > 6m(M) \).

**Theorem 8.** (i) \( \Psi(L(kq - 1, q)) \leq 4(k + q - 1) \) for \( k, q \geq 2 \) and

(ii) \( \Psi(L(kq + 1, q)) \leq 4(k + q) \) for \( k, q \geq 1 \).

**Remark 9.** Recently, Swartz proved that \( \Psi(L(kq + 1, q)) \geq 4(k + q) \) whenever \( k, q \) are odd ([16]). Thus, \( \Psi(L(kq + 1, q)) = 4(k + q) \) for odd positive integers \( k, q \). We found that \( \Psi(L(5, 1)) = 20 = \Psi(L(7, 2)) \). So, Swartz’s bound is also valid for \( L(5, 1) \) and \( L(7, 2) \). We also found that \( \psi(\mathbb{Z}_4) = 14 \) and \( \psi(\mathbb{Z}_6) = \psi(\mathbb{Z}_7) = 18 \). Proofs of these are in earlier versions of this article in the arXiv (arXiv:1308.6137). We have omitted these proofs from this version for the sake of brevity.

## 2 Preliminaries

### 2.1 Colored Graphs

All graphs considered here are finite multigraphs without loops. If \( \Gamma = (V, E) \) is a graph and \( U \subseteq V \) then the *induced* subgraph \( \Gamma[U] \) is the subgraph of \( \Gamma \) whose vertex set is \( U \) and
edges are those edges of \( \Gamma \) whose end points are in \( U \). For \( n \geq 2 \), an \( n \)-cycle is a closed path with \( n \) distinct vertices and \( n \) edges. If vertices \( a_i \) and \( a_{i+1} \) are adjacent in an \( n \)-cycle for \( 1 \leq i \leq n \) (addition is modulo \( n \)) then the \( n \)-cycle is denoted by \( C_n(a_1, a_2, \ldots , a_n) \). A graph \( \Gamma \) is called \( h \)-regular if the number of edges adjacent to each vertex is \( h \).

An edge coloring of a graph \( \Gamma = (V, E) \) is a map \( \gamma : E \to C \) such that \( \gamma(e) \neq \gamma(f) \) whenever \( e \) and \( f \) are adjacent (i.e., \( e \) and \( f \) are adjacent to a common vertex). The elements of the set \( C \) are called the colors. If \( C \) has \( h \) elements then \( (\Gamma, \gamma) \) is said to be an \( h \)-colored graph.

Let \( (\Gamma, \gamma) \) be an \( h \)-colored graph with color set \( C \). If \( B \subseteq C \) with \( k \) elements then the graph \( (V(\Gamma), \gamma^{-1}(B)) \) is a \( k \)-colored graph with coloring \( \gamma|_{\gamma^{-1}(B)} \). This colored graph is denoted by \( \Gamma_B \). Let \( (\Gamma, \gamma) \) be an \( h \)-colored connected graph with color set \( C \). If \( \Gamma_{C \setminus \{c\}} \) is connected for all \( c \in C \) then \( (\Gamma, \gamma) \) is called contracted.

Let \( \Gamma_1 = (V_1, E_1) \) and \( \Gamma_2 = (V_2, E_2) \) be two disjoint \( h \)-regular \( h \)-colored graphs with same color set \( \{1, \ldots , h\} \). For \( 1 \leq i \leq 2 \), let \( v_i \in V_i \). Consider the graph \( \Gamma \) which is obtained from \( (\Gamma_1 \setminus \{v_1\}) \sqcup (\Gamma_2 \setminus \{v_2\}) \) by adding \( h \) new edges \( e_1, \ldots , e_h \) with colors \( 1, \ldots , h \) respectively such that the end points of \( e_j \) are \( u_{j, 1} \) and \( u_{j, 2} \), where \( v_i \) and \( u_{j, i} \) are joined in \( \Gamma_i \) with an edge of color \( j \) for \( 1 \leq j \leq h \), \( 1 \leq i \leq 2 \). (Here \( \Gamma_1 \setminus \{v_1\} = \Gamma_1[V_1 \setminus \{v_1\}] \)). The colored graph \( \Gamma \) is called the connected sum of \( \Gamma_1 \) and \( \Gamma_2 \) and is denoted by \( \Gamma_1 \#_{v_1v_2} \Gamma_2 \).

Let \( \Gamma = (V, E) \) be a \((d + 1)\)-regular graph with a \((d + 1)\)-coloring \( \gamma : E \to C \). Let \( x, y \in V \) be joined by \( k \) edges \( e_1, \ldots , e_k \), where \( 1 \leq k \leq d \). Let \( B = C \setminus \gamma(\{e_1, \ldots , e_k\}) \). Let \( X \) (resp., \( Y \)) be the components of \( \Gamma_B \) containing \( x \) (resp., \( y \)). If \( X \neq Y \) then \( \Gamma[\{x, y\}] \) is called a \( d \)-dimensional dipole of type \( k \). Dipoles of types \( 1 \) and \( d \) are called degenerate dipoles.

Let \( \Gamma = (V, E) \) be a \((d + 1)\)-regular graph with a \((d + 1)\)-coloring \( \gamma : E \to C \) and a dipole \( \Gamma[\{x, y\}] \) of type \( k \). Let \( B, X \) and \( Y \) be as above. A \((d + 1)\)-regular graph \( (\Gamma', \gamma') \) with same color set \( C \) is said to be obtained from \( \Gamma \) by cancelling the dipole \( \Gamma[\{x, y\}] \) if (i) \( \Gamma'_B \) is obtained from \( \Gamma_B \) by replacing \( X \sqcup Y \) by \( X \#_{xy} Y \), and (ii) two vertices \( u, v \) of \( \Gamma' \) are joined by an edge of color \( c \in B \) if and only if the corresponding vertices of \( \Gamma \) are so (cf. [7]). For standard terminology on graphs see [2].

### 2.2 Presentation of Groups

Given a set \( S \), let \( F(S) \) denote the free group generated by \( S \). So, any element \( w \) of \( F(S) \) is of the form \( w = x_1^{\varepsilon_1} \cdots x_m^{\varepsilon_m} \), where \( x_1, \ldots , x_m \in S \) and \( \varepsilon_i = \pm 1 \) for \( 1 \leq i \leq m \) and \( (x_{j+1}, \varepsilon_{j+1}) \neq (x_j, -\varepsilon_j) \) for \( 1 \leq j \leq m - 1 \). For \( R \subseteq F(S) \), let \( N(R) \) be the smallest normal subgroup of \( F(S) \) containing \( R \). Then the quotient group \( F(S)/N(R) \) is denoted by \( \langle S \mid R \rangle \). So, \( \langle S \mid T \rangle = \langle S \mid R \rangle \) if \( N(T) = N(R) \). We write \( \langle S_1 \mid R_1 \rangle = \langle S_2 \mid R_2 \rangle \) only when \( F(S_1) = F(S_2) \) and \( N(R_1) = N(R_2) \). For \( w_1, w_2 \in F(S) \), if \( w_1N(R) = w_2N(R) \in (S \mid R) \) then we write \( w_1 \equiv w_2 \mod R \). Two elements \( w_1, w_2 \in F(S) \) are said to be independent (resp., dependent) if \( N(\{w_1\}) \neq N(\{w_2\}) \) (resp., \( N(\{w_1\}) = N(\{w_2\}) \)).

For a finite subset \( R \) of \( F(S) \), let

\[
\overline{R} := \{ w \in N(R) : N((R \setminus \{r\}) \cup \{w\}) = N(R) \text{ for each } r \in R \}. \quad (2.1)
\]

THE ELECTRONIC JOURNAL OF COMBINATORICS 21(1) (2014), #P1.61
Observe that \( \emptyset = \emptyset \) and if \( R \neq \emptyset \) is a finite set then \( w := \prod_{r \in R} r \in \mathbb{R} \) and hence \( \mathbb{R} \neq \emptyset \). Also, \( \{ wrw^{-1}, wr^{-1}w^{-1} : w \in F(S) \} \subseteq \{ r \} \) for \( r \in F(S) \).

For \( w = x_1^{\varepsilon_1} \cdots x_m^{\varepsilon_m} \in F(S), m \geq 1, \) let

\[
\varepsilon(w) := \begin{cases} 
0 & \text{if } m = 1, \\
|\varepsilon_1 - \varepsilon_2| + \cdots + |\varepsilon_{m-1} - \varepsilon_m| + |\varepsilon_m - \varepsilon_1| & \text{if } m \geq 2.
\end{cases}
\]

Consider the map \( \lambda : F(S) \to \mathbb{Z}^+ \) define inductively as follows.

\[
\lambda(w) := \begin{cases} 
2 & \text{if } w = \emptyset, \\
2m - \varepsilon(w) & \text{if } w = x_1^{\varepsilon_1} \cdots x_m^{\varepsilon_m}, (x_m, \varepsilon_m) \neq (x_1, -\varepsilon_1), \\
\lambda(w') & \text{if } w = x_1^{\varepsilon_1}w'x_1^{-\varepsilon_1}.
\end{cases}
\] (2.2)

Since \( |\varepsilon_i - \varepsilon_j| \leq 0 \) or 2, \( \varepsilon(w) \) is an even integer and hence \( \lambda(w) \) is also even. For \( w \in F(S) \), \( \lambda(w) \) is said to be the weight of \( w \). Observe that \( \lambda(w_1w_2) = \lambda(w_2w_1) \) for \( w_1, w_2 \in F(S) \).

Let \( S = \{ x_1, \ldots, x_s \} \) and \( R = \{ r_1, \ldots, r_t \} \subseteq F(S) \), where \( t \leq s \). Let \( r_{t+1} \) be an element in \( R \) of minimum weight. Let

\[
\varphi(S, R) := \lambda(r_1) + \cdots + \lambda(r_t) + \lambda(r_{t+1}) + 2(s - t). \quad (2.3)
\]

For a finitely presented group \( G \) and a non-negative integer \( q \), we define

\[
\mathcal{P}_q(G) := \{ \langle S | R \rangle \cong G : \#(R) \leq \#(S) \leq q \}.
\]

For a finitely presented group \( G \), let \( m(G) \) be the minimum number of generators of \( G \). Here, we are interested on those groups \( G \) for which \( \mathcal{P}_q(G) \neq \emptyset \) for some \( q \). Let

\[
\mu(G) := \min\{ q : \mathcal{P}_q(G) \neq \emptyset \}, \quad (2.4a)
\]
\[
\psi(G; q) := \min\{ \varphi(S, R) : \langle S | R \rangle \in \mathcal{P}_q(G) \} \text{ for } q \geq \mu(G). \quad (2.4b)
\]

Clearly, \( \mu(G) \geq m(G) \) and \( \psi(G, q) \leq \psi(G, \mu(G)) \) for all \( q \geq \mu(G) \). Let

\[
\rho(G) := \min\{ q \geq \mu(G) : \psi(G; q) \leq 6(q + 1) \}. \quad (2.5)
\]

So, \( \rho(G) \) is the smallest integer \( q \) such that \( \psi(G; q) \leq 6(q + 1) \).

**Definition 10.** Let \( G \) be a group which has a presentation with the number of relations less than or equal to the number of generators. Let \( \mu(G), \psi(G; q) \) and \( \rho(G) \) be as above. Then \( \psi(G) = \max\{ \psi(G; \rho(G)), 6\mu(G) + 2 \} \) is a positive even integer. The integer \( \psi(G) \) is said to be the weight of the group \( G \).

**Remark 11.** Observe that \( \min\{ \varphi(S, R) : \langle S | R \rangle \cong G, \#(R) \leq \#(S) < \infty \} = 4 = \psi(\mathbb{Z}, \rho(\mathbb{Z})) < 8 = \psi(\mathbb{Z}) \) (see the proof of Lemma 3). In general, we have \( \min\{ \varphi(S, R) : \langle S | R \rangle \cong G, \#(R) \leq \#(S) < \infty \} = \min\{ \min\{ \psi(G; q) : \mu(G) \leq q < \infty \} : \mu(G) \leq q < \infty \} \leq \psi(G; \rho(G)) \leq \psi(G) \).
2.3 Lens Spaces

Consider the 3-sphere $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$. Let $p$ and $q$ be relatively prime integers. Then the action of $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ on $S^3$ generated by $e^{2\pi i/p} \cdot (z_1, z_2) = (e^{2\pi i/p} z_1, e^{2\pi i/q} z_2)$ is free and hence properly discontinuous. Therefore the quotient space $L(p, q) := S^3/\mathbb{Z}_p$ is a 3-manifold whose fundamental group is isomorphic to $\mathbb{Z}_p$. The 3-manifolds $L(p, q)$ are called the lens spaces. It is a classical theorem of Reidemeister that $L(p, q')$ is homeomorphic to $L(p, q)$ if and only if $q' \equiv \pm q^\pm 1 \pmod{p}$.

If $T_1, T_2$ are two solid tori (i.e., each cell in $X$ and $Y$ are homeomorphic. A regular CW-complex $X$ is said to be simplicial cell complex and is said to be the boundary cell complex of an $\alpha$-simplex then we say that $\alpha \in X$ is a vertex of $X$. For $\alpha \in X$, the set $\partial \alpha := \{\gamma \in X : \alpha \neq \gamma \leq \alpha\}$ is a subcomplex of $X$ with induced partial order and is said to be the boundary of $\alpha$. If $\partial \alpha$ is isomorphic to the boundary complex of an $i$-simplex then we say that $\alpha$ is an $i$-cell or a cell of dimension $i$. For $\beta \in X$, the set $\{\sigma \in X : \beta \lesssim \sigma\}$ is also simplicial cell complex and is said to be the link of $\alpha$ in $X$ and is denoted by $\text{lk}_X(\alpha)$.

Let $K$ be a simplicial cell complex with partial ordering $\leq$. If $\beta \leq \alpha \in K$ then we say $\beta$ is a face of $\alpha$. For $\alpha \in K$, the set $\partial \alpha := \{\gamma \in K : \alpha \neq \gamma \leq \alpha\}$ is a subcomplex of $K$ with induced partial order and is said to be the boundary of $\alpha$. If $\partial \alpha$ is isomorphic to the boundary complex of an $i$-simplex then we say that $\alpha$ is an $i$-cell or a cell of dimension $i$.

For $\beta \in K$, the set $\{\sigma \in K : \beta \lesssim \sigma\}$ is also simplicial cell complex and is said to be the link of $\alpha$ in $K$ and is denoted by $\text{lk}_K(\alpha)$.

If all the maximal cells of a $d$-dimensional simplicial cell complex $K$ are $d$-cells then it is called pure. Maximal cells in a pure simplicial cell complex $K$ are called the facets of $K$. Clearly, if $K$ is pure of dimension $d$ and $\alpha$ is an $i$-cell then $\text{lk}_K(\alpha)$ is $(d - i - 1)$-dimensional and pure. A pure $d$-dimensional simplicial cell complex $K$ is said to be a normal pseudomanifold if each $(d - 1)$-cell is a face of exactly two facets and the link of each cell of dimension $\leq d - 2$ is connected. Clearly, a pseudotriangulation of a connected manifold is a normal pseudomanifold.

The 0-cells in a simplicial cell complex $K$ are said to be the vertices of $K$. If $u$ is a face of $\alpha$ and $u$ is a vertex then we say $u$ is a vertex of $\alpha$. Clearly, a $d$-dimensional simplicial
cell complex $\mathcal{X}$ has at least $d + 1$ vertices. If a $d$-dimensional simplicial cell complex $\mathcal{X}$ has exactly $d + 1$ vertices then $\mathcal{X}$ is called **contracted**.

Let $\mathcal{X}$ be a pure $d$-dimensional simplicial cell complex. Consider the graph $\Lambda(\mathcal{X})$ whose vertices are the facets of $\mathcal{X}$ and edges are the ordered pairs $\{\{\sigma_1, \sigma_2\}, \gamma\}$, where $\sigma_1, \sigma_2$ are facets, $\gamma$ is a $(d - 1)$-cell and is a common face of $\sigma_1, \sigma_2$. The graph $\Lambda(\mathcal{X})$ is said to be the **dual graph** of $\mathcal{X}$. Observe that $\Lambda(\mathcal{X})$ is in general a multigraph without loops. On the other hand, for $d \geq 1$, if $(\Gamma, \gamma)$ is a $(d + 1)$-colored graph with color set $C = \{1, \ldots, d + 1\}$ then we define a $d$-dimensional simplicial cell complex $K(\Gamma)$ as follows. For each $v \in V(\Gamma)$ we take a $d$-simplex $\sigma_v$ and label its vertices by $1, \ldots, d + 1$. If $u, v \in V(\Gamma)$ are joined by an edge $e$ and $\gamma(e) = i$, then we identify the $(d - 1)$-faces of $\sigma_u$ and $\sigma_v$ opposite to the vertices labelled by $i$, so that equally labelled vertices are identified together. Since there is no identification within a $d$-simplex, this gives a simplicial CW-complex $W$ of dimension $d$. So, the face poset (denoted by $\mathcal{K}(\Gamma)$) of $W$ is a pure $d$-dimensional simplicial cell complex. We say that $(\Gamma, \gamma)$ **represents** the simplicial cell complex $\mathcal{K}(\Gamma)$. Clearly, the number of $i$-labelled vertices of $\mathcal{K}(\Gamma)$ is equal to the number of components of $\Gamma_{\{i\}}$ for each $i \in C$. Thus, the simplicial cell complex $\mathcal{K}(\Gamma)$ is contracted if and only if $\Gamma$ is contracted (cf. [8]).

A **crystallization** of a connected closed $d$-manifold $M$ is a $(d + 1)$-colored contracted graph $(\Gamma, \gamma)$ such that the simplicial cell complex $\mathcal{K}(\Gamma)$ is a pseudotriangulation of $M$. Thus, if $(\Gamma, \gamma)$ is a crystallization of a $d$-manifold $M$ then the number of vertices in $\mathcal{K}(\Gamma)$ is $d + 1$. On the other hand, if $K$ is a contracted pseudotriangulation of $M$ then the dual graph $\Lambda(K)$ gives a crystallization of $M$. Clearly, if $(\Gamma, \gamma)$ is a crystallization of a closed $d$-manifold $M$ then either $\Gamma$ has two vertices (in which case $M$ is $S^d$) or the number of edges between two vertices is at most $d - 1$. From [5], we know the following.

**Proposition 12** (Cavicchioli-Grasselli-Pezzana). Let $(\Gamma, \gamma)$ be a crystallization of an $n$-manifold $M$. Then $M$ is orientable if and only if $\Gamma$ is bipartite.

For $k \geq 2$, let $1, \ldots, k$ be the colors of a $k$-colored graph $(\Gamma, \gamma)$. For $1 \leq i \neq j \leq k$, $\Gamma_{ij}$ denote the graph $\Gamma_{\{i,j\}}$ and $g_{ij}$ denote the number of connected components of the graph $\Gamma_{ij}$. In [9], Gagliardi proved the following.

**Proposition 13** (Gagliardi). Let $(\Gamma, \gamma)$ be a contracted $4$-colored graph with $m$ vertices. Then $(\Gamma, \gamma)$ is a crystallization of a connected closed $3$-manifold if and only if

(i) $g_{ij} = g_{kl}$ for every permutation $ijkl$ of $1234$, and

(ii) $g_{12} + g_{13} + g_{14} = 2 + m/2$.

Let $(\Gamma, \gamma)$ be a crystallization (with the color set $C$) of a connected closed $n$-manifold $M$. So, $\Gamma$ is an $(n + 1)$-regular graph. Choose two colors, say, $i$ and $j$ from $C$. Let $\{G_1, \ldots, G_{s+1}\}$ be the set of all connected components of $\Gamma_{\{i,j\}}$ and $\{H_1, \ldots, H_{t+1}\}$ be the set of all connected components of $\Gamma_{ij}$. Since $\Gamma$ is regular, each $H_p$ is an even cycle. Note that, if $n = 2$, then $\Gamma_{ij}$ is connected and hence $H_1 = \Gamma_{ij}$. Take a set $\bar{S} = \{x_1, \ldots, x_s, x_{s+1}\}$ of $s + 1$ elements. For $1 \leq k \leq t + 1$, consider the word $\tilde{r}_k$ in $F(\bar{S})$.
as follows. Choose a vertex \( v_1 \) in \( H_k \). Let \( H_k = v_1e^i_1v_2e^j_2v_3e^i_3v_4 \cdots e^i_{2l-1}v_2e^j_2v_1 \), where \( e^i_p \) and \( e^j_q \) are edges with colors \( i \) and \( j \) respectively. Define
\[
\tilde{r}_k := x^{i_1}_{k_2}x^{i_1}_{k_3} \cdots x^{i_1}_{k_{2l-1}}x^{i_1}_{k_1},
\]
where \( G_{k_0} \) is the component of \( \Gamma \cap \{i,j\} \) containing \( v_h \). For \( 1 \leq k \leq t+1 \), let \( r_k \) be the word obtained from \( \tilde{r}_k \) by deleting \( x^{i_1}_{s+1} \)'s in \( \tilde{r}_k \). So, \( r_k \) is a word in \( F(S) \), where \( S = \widetilde{S} \setminus \{ x_{s+1} \} \).

In [10], Gagliardi proved the following.

**Proposition 14** (Gagliardi). For \( n \geq 2 \), let \((T, \gamma)\) be a crystallization of a connected closed \( n \)-manifold \( M \). For two colors \( i, j \), let \( s, t, x_p, r_q \) be as above. If \( \pi_1(M, x) \) is the fundamental group of \( M \) at a point \( x \), then
\[
\pi_1(M, x) \cong \left\{ \begin{array}{ll} \langle x_1, x_2, \ldots, x_s | r_1 \rangle & \text{if } n = 2, \\
\langle x_1, x_2, \ldots, x_s | r_1, \ldots, r_l \rangle & \text{if } n \geq 3. \end{array} \right.
\]

### 3 Proofs of Lemma 3, Theorem 4 and Corollary 5

Lemma 3 follows from the next lemma.

**Lemma 15.** (i) \( \psi(Z) = \psi(Z_2) = 8 \), (ii) \( \psi(Z_3) = 12 \), (iii) \( \psi(Z_5) = 16 \), (iv) \( \psi(Q_8) = 18 \) and (v) \( \psi(Z^3) = 24 \).

**Proof.** Any presentations of \( Z \) must have at least one generator and \( \langle x \rangle \) is a presentation of \( Z \). So, \( \mu(Z) = 1 \). If \( \langle S | R \rangle \cong Z \) with \( \#(S) = 1 \), then \( R = \emptyset \) and hence, by the definition (see (2.3)), \( \varphi(S, R) = \lambda(\emptyset) + 2(1 - 0) = 2 + 4 = 6 < 12 = 6(\mu(Z) + 1) \). Therefore, \( \psi(Z; q) \leq 4 \) for all \( q \geq 1 \). Thus, \( \psi(Z) = \max\{\psi(Z, \rho(Z)), 6\mu(Z) + 2\} = \max\{\psi(Z, \rho(Z)), 8\} = 8 \).

Let \( p \geq 2 \) be an integer. Since any presentations of \( Z_p \) must have at least one generator and \( \langle x | x^p \rangle \) is a presentation of \( Z_p \), it follows that \( \mu(Z_p) = 1 \). Clearly, if \( \langle S = \{ x \} | R = \{ r_1 \} \rangle \) is a presentation of \( Z_p \), then \( r_1 = x^{\pm p} \). Let \( r_2 \in R \) be of minimum weight. Since \( \langle x | r_2 \rangle \) is also a presentation of \( Z_p, r_2 = x^{\pm p} \). Therefore, by (2.3),
\[
\varphi(S, R) = \lambda(r_1) + \lambda(r_2) = (2p - \varepsilon(r_1)) + (2p - \varepsilon(r_2)) = 4p.
\]

First assume that \( p \leq 3 \). Since, \( \langle S | R \rangle \in P_1(Z_p) \) implies (up to renaming) \( (S, R) = (\{ x \}, \{ x^p \}) \) or \( (\{ x \}, \{ x^{-p} \}) \), it follows that \( \psi(Z_p; 1) = \varphi(\{ x \}, \{ x^{\pm p} \}) = 4p \leq 12 = 6(\mu(Z_p) + 1) \). This implies that \( \rho(Z_p) = \mu(Z_p) = 1 \). Thus, \( \psi(Z_p; \rho(Z_p)) = 4p \geq 8 = 6\mu(Z_p) + 2 \). Therefore, \( \psi(Z_p) = 4p \). This proves parts (i) and (ii).

Now, assume \( p = 5 \). By the similar arguments as for \( p \leq 3 \), \( \langle S | R \rangle \in P_1(Z_5) \) implies \( \varphi(S, R) = 4p = 20 \). Therefore, \( \psi(Z_5; 1) = 20 > 12 = 6(\mu(Z_5) + 1) \) and hence \( \rho(Z_5) > \mu(Z_5) = 1 \). If we take \( S = \{ x_1, x_2 \} \) and \( R = \{ r_1 = x_1x_2^{-1}, r_2 = x_1x_2{x_2}^{-1} \} \) then \( \varphi(S, R) \leq 16 \) (since \( r_3 = x_1x_2^2 \in R \) is of weight 6) and \( \langle S | R \rangle \in P_2(Z_5) \setminus P_1(Z_5) \). Thus, \( \psi(Z_5; 2) \leq 16 < 18 = 6(2 + 1) \). Therefore, \( \psi(Z_5) = 2 \) and hence \( \psi(Z_5) \leq 16 \).

Now, let \( \langle S | R \rangle \in P_2(Z_5) \setminus P_1(Z_5) \) with \( \varphi(S, R) \leq 16 \). Since there is no presentation \( \langle S | R \rangle \) of \( Z_5 \) with \( (\#(S), \#(R)) = (2, 1) \), it follows that \( \#(R) = \#(S) = 2 \). Let \( S = \)}
\{x_1, x_2 \} and \( R = \{r_1, r_2 \} \). If \( \lambda(r_1) = 2 \), then \( r_1 \) must be of the form \( x_i \pm 1 \) or \( x_i x_j^{-\varepsilon} \) for some \( j \neq i \in \{1, 2\} \) and \( \varepsilon = \pm 1 \). Since \( \langle S | R \rangle \cong \mathbb{Z}_5 \), it follows that \( r_2 \equiv x_j^{\pm 5} \pmod{r_1} \). This implies that \( \lambda(r_2) \geq \lambda(x_j^{\pm 5}) = 10 \). Let \( r_3 \in \overline{R} \) be of minimum weight. Then \( \langle x_1, x_2 | r_1, r_3 \rangle \) is also a presentation of \( \mathbb{Z}_5 \) and hence (by the same arguments) \( \lambda(r_3) \geq 10 \). Thus, \( \varphi(S, R) = \lambda(r_1) + \lambda(r_2) + \lambda(r_3) \geq 2 + 10 + 10 = 22 \), a contradiction. So, \( \lambda(r_1) \geq 4 \) for \( 1 \leq i \leq 2 \). Let \( A = \{x_1, x_2, x_1^2, x_2^2, x_1^2 x_2, x_2 x_1^2, x_1 x_2, x_1 x_2 - x_1 \} \) and let \( A^{-1} = \{w^{-1} : w \in A \} \). Then \( A \) is a set of pairwise independent elements of weight 4 in \( F(S) \) and \( w \in F(S) \) is an element of weight 4 imply that \( w \) is dependent with an element of \( A \). Note that \( \mathbb{Z}_5 \) has no presentation \( \langle S | R \rangle \in \mathcal{P}_2(\mathbb{Z}_5) \setminus \mathcal{P}_1(\mathbb{Z}_5) \) with \( R \subseteq A \cup A^{-1} \). So, at most one of \( r_1, r_2, r_3 \) has weight 4 and the weights of the other two are at least 6. Therefore, \( \varphi(S, R) \geq 16 \). This proves part (iii).

Clearly, \( \mu(Q_8) = 2 \). If we take \( S = \{x_1, x_2 \} \) and \( R = \{x_2 x_1 x_2 x_1^{-1}, x_1 x_2 x_1 x_2^{-1} \} \) then \( \langle S | R \rangle \in \mathcal{P}_2(Q_8) \) and \( \varphi(S, R) \leq 18 \) (since \( x_2 x_1^{-2} \in \overline{R} \) is of weight 6). Thus \( \psi(Q_8) \leq 18 = 6(2 + 1) \). Therefore, \( \rho(Q_8) = 2 \) and hence \( \psi(Q_8) \leq 18 \).

Now, let \( \varphi(S, R) \leq 18 \), where \( S = \{x_1, x_2 \} \) and \( \langle S | R \rangle \in \mathcal{P}_2(Q_8) \). Note that \( B = \{x_1 x_2, x_1^2, x_1^2 x_2^{-1}, x_2^2 x_1, x_1 x_2 x_1^{-2}, x_1 x_2 x_1^{-1}, x_2 x_1 x_2 x_1^{-1}, x_1 x_2 x_1^{-1}, x_2 x_1 x_2 x_1^{-1}, x_1 x_2 x_1^{-1}, x_2 x_1 x_2 x_1^{-1}, x_1 x_2 x_1^{-1}, x_2 x_1 x_2 x_1^{-1}, x_1 x_2 x_1^{-1}, x_2 x_1 x_2 x_1^{-1} \} \) is a set of pairwise independent elements of weight 4 or 6 in \( F(S) \). It is not difficult to see that \( w \in F(S) \) and \( 4 \leq \lambda(w) \leq 6 \) imply \( w \) is dependent with an element of \( B \). Let \( B^{-1} = \{w^{-1} : w \in B \} \). Then \( R \subseteq B \cup B^{-1} \). Clearly, the only possible choices of \( \{r_1, r_2 \} \) are \( \{x_2 x_1^{-2}, x_2 x_1 x_2^{-1} \} \), \( \{x_2 x_1^{-2}, x_2 x_1 x_2^{-1} \} \) and \( \{x_2 x_1 x_2^{-1}, x_1 x_2 x_1^{-1} \} \). Then \( \lambda(r) \geq 6 \) for \( r \in R \cup \overline{R} \). Thus, \( \varphi(S, R) \geq 18 \). Therefore, \( \psi(Q_8) = 18 \). This proves parts (iv).

Clearly, \( \mu(\mathbb{Z}^3) = 3 \). If \( S_0 = \{x_1, x_2, x_3 \} \) and \( R_0 = \{x_i x_j x_k^{-1}, x_i x_j^{-1} x_k : 1 \leq i < j < k \leq 3 \} \) then \( \langle S_0 | R_0 \rangle \in \mathcal{P}_3(\mathbb{Z}^3) \) and \( \varphi(S_0, R_0) \leq 24 \) (since \( x_1 x_2^{-1} x_3 x_1^{-1} x_2 x_3^{-1} \in \overline{R}_0 \) is of weight 6). Thus \( \psi(\mathbb{Z}^3) \leq 24 = 6(3 + 1) \). Therefore, \( \rho(\mathbb{Z}^3) = 3 \) and hence \( \psi(\mathbb{Z}^3) \leq 24 \).

Claim. If \( w \in N(R_0) \) is not the identity then \( \lambda(w) \geq 6 \).

If \( w \in N(R_0) \) is not the identity then clearly \( \lambda(w) \neq 2 \). Observe that, if \( w \in F(S_0) \) with \( \lambda(w) = 4 \), then \( w \) is dependent with an element of the set \( C = \{x_1^2 x_2^{-1}, x_2 x_1^{-1} x_3 x_2^{-1}, x_1 x_2, x_1 x_3, x_1 x_2 x_1^{-1} x_3 x_2^{-1} : i j k \text{ is a permutation of } 123 \} \). Since none of the element in \( C \) is in \( N(R_0) \), it follows that \( N(R_0) \) has no element of weight 4. This proves the claim.

Now, let \( \varphi(S, R) \leq 24 \), where \( S = \{x_1, x_2, x_3 \} \) and \( \langle S | R \rangle \in \mathcal{P}_3(\mathbb{Z}^3) \). Then \( N(R) = N(R_0) \) and hence, by the claim, weight of each element of \( R \) is at least 6. This implies \( \varphi(S, R) \geq 24 \) and hence \( \varphi(S, R) = 24 \). Therefore, \( \psi(\mathbb{Z}^3) = 24 \). This completes the proof.

**Proof of Theorem 4.** Let \( G = \pi(M, x) \) for some \( x \in M \). To prove the theorem, it is sufficient to show that any crystallization of \( M \) needs at least \( \psi(M) = \psi(G) \) vertices.

Let \( (\Gamma, \gamma) \) be a crystallization of \( M \) with \( m \) vertices and let \( \{1, 2, 3, 4\} \) be the color set. Then, by Proposition 14, we know that \( G \) has a presentation with \( g_{ij} - 1 \) generators and \( \leq g_{ij} - 1 \) relations. Therefore, by the definition of \( \mu(G) \) (in (2.4a)), \( \mu(G) \leq g_{ij} - 1 \). Then, by part (ii) of Proposition 13,

\[
m = 2(g_{12} + g_{13} + g_{14}) - 4 \geq 6(\mu(G) + 1) - 4 = 6\mu(G) + 2.
\]
From the definition of $\rho(G)$ (in (2.5)), $6(\rho(G) + 1) \geq \psi(G; \rho(G))$. Therefore, $m > 6(\rho(G) + 1)$ implies $m > \psi(G; \rho(G))$. Thus, if $m > 6(\rho(G) + 1)$ then the result follows from this and Eq. (3.2).

Now, assume that $m \leq 6(\rho(G) + 1)$. Then, by part (ii) of Proposition 13, $g_{12} + g_{13} + g_{14} \leq 2 + 3(\rho(G) + 1)$. This implies, $g_{1j} \leq \rho(G) + 1$ for some $j \in \{2, 3, 4\}$. Assume, without loss, that $g_{12} \leq \rho(G) + 1$.

As in Subsection 2.4, let $G_1, \ldots, G_{q+1}$ be the components of $\Gamma_{12}$ and $H_1, \ldots, H_{q+1}$ be the components of $\Gamma_{34}$, where $q + 1 = g_{34} = g_{12} \leq \rho(G) + 1$. By Proposition 14, $G$ has a presentation of the form $\langle x_1, x_2, \ldots, x_q \mid r_1, r_2, \ldots, r_q \rangle$, where $x_k$ corresponds to $G_k$ and $r_k$ corresponds to $H_k$ as in Subsection 2.4. Let $S = \{x_1, x_2, \ldots, x_q\}$ and $R = \{r_1, \ldots, r_q\}$.

For $1 \leq i \leq q$, let $r_i = x_{i_1}^{\varepsilon_1} \cdots x_{i_n}^{\varepsilon_n}$, where $x_{i_1}, \ldots, x_{i_n} \in \{x_1, \ldots, x_q\}$ and $\varepsilon_j = \pm 1$ for $1 \leq j \leq n - 1$. Then, $0 = (x_{ij}, x_{ij+1}) \neq (x_{ij}, -\varepsilon_j)$ for $1 \leq j \leq n - 1$ and $(x_{in}, \varepsilon_n) \neq (x_{in}, -\varepsilon_1)$.

Claim. For $1 \leq i < q$, the length of the cycle $H_i$ is at least $\lambda(r_i)$.

Consider the word $\tilde{r}_i$ in $F(\{x_1, \ldots, x_q, x_{q+1}\})$ which is obtained from $r_i$ by the following rules: if $\varepsilon_j = \varepsilon_{j+1}$ for $1 \leq j \leq n - 1$, then replace $x_{i_j}^{\varepsilon_j}$ by $x_{i_j}^{\varepsilon_j}x_{q+1}^{-\varepsilon_j}$ in $r_i$ and if $\varepsilon_n = \varepsilon_1$, then replace $x_{i_n}^{\varepsilon_n}$ by $x_{i_n}^{\varepsilon_n}x_{q+1}^{-\varepsilon_n}$ in $r_i$. Observe that $\tilde{r}_i$ is non empty (since $r_i$ is non empty) and the number of letters in $\tilde{r}_i$ is same as $\lambda(r_i)$ (see (2.6) and (2.2)). The claim follows from this.

Let $r_{q+1}$ be a word corresponding to $H_{q+1}$ in $\Gamma_{34}$. Then, any $q$ of the relations from the set $\{r_1, r_2, \ldots, r_q, r_{q+1}\}$ together with the generators $x_1, x_2, \ldots, x_q$ give a presentation of $G$. This implies, $r_{q+1} \in R$. Thus, $m \geq \lambda(r_1) + \lambda(r_2) + \cdots + \lambda(r_{q+1}) \geq \varphi(S, R) \geq \psi(G; \rho(G))$. Therefore, $m \geq \max\{\psi(G; \rho(G)), 6(a(G) + 2\} \geq \psi(G)$. This proves the theorem. \hfill \square

Proof of Corollary 5. Let $f_i$ be the number of $i$-cells in $X$. So, $f_0 = 4$. Therefore, $g_2(X) = f_1 - 16 + 10 = f_1 - 6 = f_1 - 12 + 6 = h_2(X)$. Since $|X|$ is a closed 3-manifold, each 2-cell is a face of two 3-cells and each 3-cell has four 2-dimensional faces. This implies that $2f_2 = 4f_3$. Then, $0 = f_0 - f_1 + f_2 - f_3 = 4 - f_1 + 2f_3 - f_3$. Thus, $f_1 = f_3 + 4$ and hence $g_2(X) = h_2(X) = f_3 - 2$. Therefore, by Theorem 4, $g_2(X) = h_2(X) = f_3 - 2 \geq \Psi(M) - 2 \geq \psi(M) - 2$.

From the definition of $\psi(G)$, $\psi(G) \geq 6\mu(G) + 2 \geq 6m(G) + 2$. So, $\psi(M) = \psi(\pi(M, *)) \geq 6m(\pi(M, *)) + 2$. Since any presentation of $\pi(M, *)$ has at least $\beta_1(M, \mathbb{F})$ generators, it follows that $m(M) = m(\pi(M, *)) \geq \beta_1(M, \mathbb{F})$. The corollary now follows. \hfill \square

Remark 16. If a crystallization $(\Gamma, \gamma)$ yields a presentation $\langle S \mid R \rangle$ then, from the proof of Theorem 4, we get $\varphi(S, R) \leq \varphi(S, R) \leq \beta_1$.

Remark 17. We found that $\rho(\mathbb{Z}^3) = 3$ and $\varphi(S, R) = 24$, where $\langle S \mid R \rangle = \mathcal{P}_3(Z^3)$. On the other hand, if $S' = \{x_1, \ldots, x_5\}$ and $R' = \{x_1x_4^{-1}x_5x_3^{-1}, x_1x_5x_3^{-1}, x_3x_4x_2^{-1}, x_4x_3^{-1}x_5x_4^{-1}, x_5x_1x_2^{-1}\}$ then $\langle S' \mid R' \rangle \in \mathcal{P}_3(Z^3) \setminus \mathcal{P}_4(Z^3)$ with $\varphi(S', R') = 24$. So, the minimum weight presentation of $\mathbb{Z}^3$ is not unique. This is true for most of the groups.
4 Uniqueness of some crystallizations

Here, we are interested in crystallizations of 3-manifolds $M$ with $\psi(M)$ vertices. For seven 3-manifolds, we show that there exists a unique such crystallization for each of them.

Throughout this section and behind, $1, 2, 3, 4$ are the colors of a 4-colored graph $(\Gamma, \gamma)$ and $g_{ij}$ is the number of components of $\Gamma_{ij} = \Gamma_{\{i,j\}}$ for $i \neq j$.

Let $\mathcal{X}$ be the pseudotriangulation of a connected closed 3-manifold $M$ determined by a crystallization $(\Gamma, \gamma)$. So, $(\Gamma, \gamma)$ is contracted, i.e., $\Gamma_{\{i,j,k\}}$ is connected for $i, j, k$ distinct. For $1 \leq i \leq 4$, we denote the vertex of $\mathcal{X}$ corresponding to the color $i$ by $v_i$. We identify a vertex $u$ of $\Gamma$ with the corresponding facet $\sigma_u$ of $\mathcal{X}$. For a facet $u \equiv \sigma_u$ of $\mathcal{X}$, the 2-face of $u$ not containing the vertex $v_i$ will be denoted by $u_i$. Similarly, the edge of $u$ not containing the vertices $v_i, v_j$ will be denoted by $u_{ij}$. Clearly, if $C_{2k}(u_1, u_2, \ldots, u_{2k})$ is a $2k$-cycle in $\Gamma$ with colors $i$ and $j$ alternately, then $u^1_{ij} = u^2_{ij} = \cdots = u^k_{ij}$ in $\mathcal{X}$.

Lemma 18. Let $\Gamma$ be a crystallization of a connected closed 3-manifold $M$ with $m$ vertices. If $\Gamma$ has a 2-cycle, then either $M$ has a crystallization with $m-2$ vertices or $\pi_1(M, x)$ (for $x \in M$) is isomorphic to the free product $\mathbb{Z} \ast H$ for some group $H$.

Proof. Without loss of generality, assume that $\Gamma$ has a 2-cycle with color 1 and 2, i.e., $\Gamma_{12}$ has a component of length 2. If this 2-cycle touches two different components of $\Gamma_{34}$ (say, at vertices $v$ and $w$, respectively), then $\Gamma[\{v, w\}]$ is a 3-dimensional dipole of type 2. Therefore, the crystallization $\Gamma$ can be reduced to a crystallization $\Gamma^1$ of $M$ with vertex set $V(\Gamma) \setminus \{v, w\}$ so that $\Gamma^1_{12}$ (resp., $\Gamma^1_{34}$) has one less components than $\Gamma_{12}$ (resp., $\Gamma_{34}$) as in Fig. 1 (see [7]). Thus, $M$ has a crystallization (namely, $\Gamma^1$) with $m-2$ vertices.

So, assume that the 2-cycle (say $G_1$) touches only one component (say, $H_1$) of $\Gamma_{34}$. Let $G_1, \ldots, G_{q+1}$ be the components of $\Gamma_{12}$ and $H_1, \ldots, H_{q+1}$ be the components of $\Gamma_{34}$, where $q + 1 = g_{12} = g_{34}$. Let $x_1, \ldots, x_{q+1}$ and $r_1, \ldots, r_{q+1}$ be as in Proposition 14. Then, by Proposition 14, $\pi_1(M, x)$ has a presentation of the form $\langle x_1, x_2, \ldots, x_q \vert r_2, r_3, \ldots, r_{q+1} \rangle$. Since $G_1$ touches only $H_1$, from the definition of $\tilde{r}_k$ in Eq. (2.6), $\tilde{r}_k$ does not contain $x_{\pm 1}$ for $k \neq 1$. Therefore, $\langle x_1, x_2, \ldots, x_q \vert r_2, \ldots, r_{q+1} \rangle = \langle x_1 \rangle \ast \langle x_2, \ldots, x_q \vert r_2, \ldots, r_{q+1} \rangle$. This proves the lemma. ☐

Figure 1: Cancellation of a dipole of type 2

THE ELECTRONIC JOURNAL OF COMBINATORICS 21(1) (2014), #P1.61

11
Lemma 19. There exist exactly three 8-vertex crystallizations of non-simply connected, connected, closed 3-manifolds. Moreover, these three are crystallizations of $S^2 \times S^1$, $S^2 \times S^1$ and $\mathbb{RP}^3$ respectively.

Proof. Let $(\Gamma, \gamma)$ be an 8-vertex crystallization of a non simply connected, connected, closed 3-manifold $M$. By Proposition 13, $g_{12} + g_{13} + g_{14} = \frac{8}{2} + 2 = 6$ and $g_{ij} = g_{kl}$ for $i, j, k, l$ distinct. Since $\pi_1(M, \ast)$ has at least one generator, $g_{ij} \geq 2$ for $1 \leq i \neq j \leq 4$. This implies that $g_{ij} = 2$ and hence $\Gamma_{ij}$ is of the form $C_2 \sqcup C_6$ or $C_4 \sqcup C_4$ for $1 \leq i \neq j \leq 4$.

Case 1: Suppose $(\Gamma, \gamma)$ has a 2-cycle. Since $M$ is not simply connected, $M$ has no crystallization with less than 8 vertices. Therefore, by Lemma 18, $\pi_1(M, \ast)$ must have a torsion free element. Again, $g_{ij} = 2$ implies $\pi_1(M, \ast)$ is generated by one element and hence isomorphic to $\mathbb{Z}$. Therefore, $M \cong S^2 \times S^1$ or $S^2 \times S^1$. Assume, without loss, $\Gamma_{12} = G_1 \sqcup G_2$, where $G_1 = C_2(v_3, v_4)$, $G_2 = C_6(v_1, v_2, v_5, v_6, v_7, v_8)$. Then there is no edge between $v_3$ and $v_4$ of color 3 or 4 and (see the proof of Lemma 18), $G_1$ touches only one component of $\Gamma_{34}$. Let $\Gamma_{34} = H_1 \sqcup H_2$, where $G_1 \cap H_1 = \emptyset$. Let $x$ and $y$ be the generators corresponding to the components $G_1$ and $G_2$ respectively. If $H_2$ is a 4-cycle then $H_2$ represents $xy^{-1}xy^{-1}$ by choosing some $v_1, i, j$ as in Eq. (2.6). But $xy^{-1}xy^{-1}$ does not give identity relation by deleting $x$ or $y$. Therefore, $H_2$ is a 6-cycle and hence $H_1$ is a 2-cycle. Similarly, $G_2 \cap H_2 = \emptyset$. Since the number of edges between any pair of vertices is at most 2, we can assume that $H_1 = C_2(v_1, v_6)$. Assume, without loss, that there is an edge of color 4 between $v_2$ and $v_3$. Since $\Gamma_{24}$ has two components, this implies $\Gamma_{24} = C_4(v_4, v_3, v_2, v_5) \sqcup C_4(v_8, v_1, v_6, v_7)$. So, there exists an edge of color 4 between $v_4$ and $v_5$ (resp. $v_7$ and $v_6$). Since $H_2$ is a 6-cycle on the vertex set $\{v_1, \ldots, v_8\} \setminus \{v_1, v_6\}$, this implies that $H_2 = C_6(v_2, v_3, v_8, v_7, v_4, v_5)$ or $C_6(v_2, v_3, v_7, v_8, v_4, v_5)$. In the first case, $(\Gamma, \gamma) = J_1$ and in the second case, $(\Gamma, \gamma) = J_2$ given in Fig. 2 (a) and (b) respectively.

Case 2: Suppose $(\Gamma, \gamma)$ has no 2-cycle. So, $\Gamma$ is a simple graph. Then, $\Gamma_{ij} = C_4 \sqcup C_4$ for $1 \leq i \neq j \leq 4$. Let $G_1 = C_4(a_1, b_1, a_2, b_2)$ and $G_2 = C_4(c_1, d_1, c_2, d_2)$ be the components of $\Gamma_{12}$. If $a_1a_2$ is an edge of color 3 then (since $\Gamma_{13} = C_4 \sqcup C_4$) $b_1b_2$ must be an edge of color 3. Then $\Gamma_{123}$ is disconnected. This is not possible. So, $a_1a_2$ cannot be an edge of color 3.
Similarly, $a_1a_2$ cannot be an edge of color 4. These imply, $a_1a_2$ cannot be an edge of $\Gamma$. Assume, without loss, $a_1c_1$ is an edge of color 4. Then $a_2c_2$, $b_1d_1$, $b_2d_2$ are edges of color 4 (since $\Gamma_{14} = C_4 \sqcup C_4$ for $1 \leq i \leq 2$). If $a_1d_1$ is an edge of color 3, then $C_4(a_1, b_1, c_1, d_1)$ would be a component of $\Gamma_{34}$. This implies $\Gamma \sim \{a_1, b_1, c_1, d_1\}$ would be proper component of $\Gamma_{\{2,3,4\}}$. This is not possible since $(\Gamma, \gamma)$ is a contracted graph. Thus, $a_1d_1$ is not an edge of color 3. Similarly, $a_1d_2$ is not an edge of color 3. These imply $a_1c_2$ is an edge of color 3. Similarly, $b_1d_2$, $a_2c_1$ and $b_2d_1$ are edges of color 3. Then, $(\Gamma, \gamma) = K_{2,1}$ given in Fig. 2 (c). Since $G_1 = C_4(a_1, b_1, a_2, b_2)$ and $H_1 = C_4(d_1, b_2, d_2, b_1)$ is a component of $\Gamma_{34}$, $\pi(M, *) = \langle x \mid x^2 \rangle \cong \mathbb{Z}_2$. This implies that $M = \mathbb{RP}^3$. This completes the proof. □

**Lemma 20.** There exists a unique 12-vertex crystallization of $L(3,1)$.

**Proof.** By Lemma 15 and Theorem 4, $L(3,1)$ has no crystallization with less than 12 vertices. Let $(\Gamma, \gamma)$ be a 12-vertex crystallization of $L(3,1)$. Since $\pi_1(L(3,1), *) (\cong \mathbb{Z}_3)$ has no torsion free element, by Lemma 18, $(\Gamma, \gamma)$ has no 2-cycle. So, $\Gamma$ is a simple graph. This implies that $g_{ij} \leq 3$ for $i \neq j$. Also (since $\mathbb{Z}_3$ has at least one generator) $g_{ij} \geq 2$. By Proposition 13, $g_{i2} + g_{i3} + g_{14} = 12/2 + 2 = 8$ and $g_{ij} = g_{kl}$ for $i, j, k, l$ distinct. So, without loss, we can assume that $g_{i2} = g_{i4} = 2$, $g_{i3} = g_{14} = 3$. Then $\Gamma_{i2} = C_4 \sqcup C_4 \sqcup C_4$ for $1 \leq i \leq 2, 3 \leq j \leq 4$. Let $G_1$, $G_2$ be the components of $\Gamma_{12}$ and $H_1, H_2$ be the components of $\Gamma_{34}$ such that $x_1, x_2$ represent the generators corresponding to $G_1$, $G_2$ respectively. Since $\langle x_j \mid x_j^3 \rangle$ is the only presentation in $P_1(\mathbb{Z}_3)$, $H_1$ must yield the relations $x_j^3$, for $1 \leq i, j \leq 2$. Therefore, $G_i$ and $H_i$ are 6-cycles. Let $G_1 = C_6(a_1, b_1, \ldots, a_3, b_3)$ and $G_2 = C_6(c_1, d_1, \ldots, c_3, d_3)$. Assume, without loss, $a_1c_1 \in \gamma^{-1}(4)$. Then $C_4(b_3, a_1, c_1, d_3) \subseteq \Gamma_{14}$ and hence $b_3d_3 \in \gamma^{-1}(4)$. Similarly, $a_3c_3, b_2d_2, a_2c_2, b_1d_1 \in \gamma^{-1}(4)$. Now, $a_1d_1 \in \gamma^{-1}(3)$ $\Rightarrow C_4(a_1, d_1, c_1, b_1) \subseteq \Gamma_{34} \Rightarrow \Gamma \{a_1, b_1, c_1, d_1\}$ is a component of $\Gamma_{\{2,3,4\}}$. This is not possible since $\Gamma$ is a contracted graph. So, $a_1d_1 \notin \gamma^{-1}(3)$. Similarly, $a_1d_3 \notin \gamma^{-1}(3)$. Again, $a_1d_2 \in \gamma^{-1}(3)$ $\Rightarrow C_4(a_1, d_2, c_2, b_1) \subseteq \Gamma_{23} \Rightarrow c_2b_1 \in \gamma^{-1}(3)$ $\Rightarrow C_4(a_2, b_1, c_2, d_1) \subseteq \Gamma_{13} \Rightarrow \Gamma \{a_2, b_1, c_2, d_1\}$ is a component of $\Gamma_{\{1,3,4\}}$, a contradiction. So, $a_1d_2 \notin \gamma^{-1}(3)$. Therefore, up to an isomorphism, $a_1c_2 \in \gamma^{-1}(3)$. Then $b_1d_2, a_2c_3, b_2d_3, a_3c_1, b_2d_1 \in \gamma^{-1}(3)$ and hence $(\Gamma, \gamma) = K_{3,1}$ given in Fig. 3 (a). Since $H_1 = C_6(d_1, b_1, d_2, b_2, d_3, b_3)$ is one of the two components of $\Gamma_{34}$, $(\Gamma, \gamma)$ yields $\langle x_1 \mid x_1^3 \rangle \cong \mathbb{Z}_3$. So, $(\Gamma, \gamma)$ is a crystallization of $L(3,1)$. This completes the proof. □

**Lemma 21.** There exists a unique 16-vertex 4-colored graph $(\Gamma, \gamma)$ which is a crystallization of a closed connected 3-manifold whose fundamental group is $\mathbb{Z}_5$.

**Proof.** Let $(\Gamma, \gamma)$ be a 16-vertex crystallization of a connected closed 3-manifold $M$ and $\pi(M, *) = \mathbb{Z}_5$. Then $M$ can not have a non-trivial 2-fold cover and hence $M$ is orientable. Also, by Lemma 15, $\psi(M) = 16$ and hence, by Theorem 4, $(\Gamma, \gamma)$ is the crystallization of $M$ with minimum number of vertices. Then, by Lemma 18, $(\Gamma, \gamma)$ has no 2-cycle. So, $\Gamma$ is a simple graph. Since $M$ is orientable, $\Gamma$ is bipartite. By Proposition 14 and Remark 16, $(\Gamma, \gamma)$ yields a presentation $\langle S \mid R \rangle$ of $\mathbb{Z}_5$ with $\varphi(S, R) = 16$.

**Claim 1.** If $\langle S = \{x_1, x_2\} \mid R = \{r_1, r_2\} \rangle \in P_2(\mathbb{Z}_5)$, $\varphi(S, R) = 16$ and $r_3 \in R$ is of minimum weight then $\{r_1, r_2, r_3\} = \{(x_1^3x_2^2)^{\pm 1}, (x_2^3x_1^{2})^{\pm 1}, (x_1^3x_2^{-1})^{\pm 1}, (x_1^{-1}x_2^{2})^{\pm 1}, (x_2^{-1}x_1x_2^{-1})^{\pm 1}\}$ or $\{(x_1^3x_2^{-1}x_1x_2^{-1})^{\pm 1}, (x_1^{-1}x_2^{2})^{\pm 1}, (x_2^{-1}x_1x_2^{-1})^{\pm 1}\}$. (So, the set $\{r_1, r_2, r_3\}$ has 16 choices.)
Let $B$ be the set as in the proof of Lemma 15. Then $w \in F(S)$ and $4 \leq \lambda(w) \leq 6$ imply $w$ is dependent with an element of $B$. Since $\Gamma$ has no 2-cycle, $R$ has no element of weight less than 4. Since $\varphi(S, R) = 16$, we can assume that $4 \leq \lambda(r_1), \lambda(r_2) \leq 6$. Since $\langle S \mid R \rangle \in \mathcal{P}_2(\mathbb{Z}_5) \setminus \mathcal{P}_1(\mathbb{Z}_5)$, the only possible choices of $\{r_1^\pm, r_2^\pm\}$ are $\{x_1^3 x_2^{-1}, x_2^3 x_1^{-1}\}$, $\{x_1^2 x_2^{-1}, x_2^2 x_1^{-1}\}$, $\{x_1^3 x_2^{-1}, x_1 x_2, x_2 x_1\}$ or $\{x_1^3 x_2^{-1}, x_1 x_2^{-1}, x_2 x_1^{-1}\}$. So, if $\langle S \mid R \rangle \in \mathcal{P}_2(\mathbb{Z}_5)$ and $\varphi(S, R) = 16$, then $(r_1^\pm, r_2^\pm, r_3^\pm) = (x_1^3 x_2^{-1}, x_2^3 x_1^{-1}, x_1 x_2)$ or $(x_1^3 x_2^{-1}, x_1 x_2^{-1}, x_2 x_1^{-1})$. This proves Claim 1.

If $g_{ij} = 2$ for some $i \neq j$ then $(\Gamma, \gamma)$ yields a presentation $\langle S \mid R \rangle \in \mathcal{P}_1(\mathbb{Z}_5)$ such that $\varphi(S, R) = 16$ (see Remark 16), which is not possible by Eq. (3.1). Thus, $g_{ij} \geq 3$. Since (by Proposition 13) $g_{12} + g_{13} + g_{14} = 16/2 + 2 = 10$, we can assume that $g_{12} = 3 = g_{13}, g_{14} = 4$. In particular, if we choose generators (resp., relations) corresponding to the components of $\Gamma_{12}$ (resp., $\Gamma_{34}$) then $(\Gamma, \gamma)$ yields a presentation $\langle S \mid R \rangle \in \mathcal{P}_2(\mathbb{Z}_5) \setminus \mathcal{P}_1(\mathbb{Z}_5)$ with $\varphi(S, R) = 16$.

Claim 2. If $x_1, x_2$ are generators corresponding to two components of $\Gamma_{12}$ then the relations corresponding to the components of $\Gamma_{34}$ are $(x_1^3 x_2^{-1})^{\varepsilon_1}, x_2^3 x_1^{-1}, (x_1 x_2)^{\varepsilon_2}$ for some $\varepsilon_1, \varepsilon_2 \in \{1, -1\}$.

Let $S, R, r_1, r_2, r_3$ be as in Claim 1. Then by choosing $(i, j) = (3, 4)$ or $(4, 3)$ as in Eq. (2.6), by Claim 1, we can assume $(r_1, r_2, r_3) = ((x_1^3 x_2^{-1})^{\pm 1}, x_2^3 x_1^{-1}, (x_1 x_2)^{\pm 1})$ or $((x_1^3 x_2^{-1}, x_1 x_2)^{-1}, (x_1 x_2)^{-1}, (x_1^3 x_2^{-1}, x_2 x_1^{-1})^{\pm 1})$. In the first case, Claim 2 trivially holds. In the second case, $r_2 = (x_1 x_3^{-1}, x_2 x_3^{-1})^{-1}$, where $x_3$ corresponds to the third component of $\Gamma_{12}$ (see Eq. (2.6)). By deleting $x_2$ and renaming $x_3$ by $x_2$ in $r_2^{-1}$, we get the new relation $x_2^3 x_1^{-1}$. Claim 2 now follows from Claim 1.

To construct $\tilde{r}_i$ as in Eq. (2.6), we can choose, without loss, $(i, j) = (4, 3)$. Since $g_{23} = g_{14} = 4, \Gamma_{14}$ and $\Gamma_{23}$ are of the form $C_4 \sqcup C_4 \sqcup C_4 \sqcup C_4$. Again, $g_{12} = g_{34} = g_{24} = g_{13} = 3$ implies $\Gamma_{13}, \Gamma_{24}, \Gamma_{12}$ and $\Gamma_{34}$ are of the form $C_4 \sqcup C_6 \sqcup C_6$. Let $G_1, G_2, G_3$ be the components of $\Gamma_{12}$ and $H_1, H_2, H_3$ be the components of $\Gamma_{34}$ such that $x_1, x_2, x_3$ represent the generators corresponding to $G_1, G_2, G_3$ respectively and $(x_1^3 x_2^{-1})^{\varepsilon_1}, x_2^3 x_1^{-1}, (x_1 x_2)^{\varepsilon_2}$ represent the relations corresponding to $H_1, H_2, H_3$ respectively.
Let $G_1 = C_6(x^1, \ldots, x^6)$, $G_2 = C_4(y^1, \ldots, y^4)$ and $G_3 = C_6(z^1, \ldots, z^6)$. Then to form the relations $(x_1^3 x_2^{-1})^{\varepsilon_1}, (x_2^2 x_1^{-1})^{\varepsilon_2}, (x_1^2 x_2)^{\varepsilon_3}$, we need to add the following: (i) two edges of color 4 between $G_1$ and $G_2$, (ii) four edges of colors 4 between $G_1$ and $G_3$, (iii) two edges of color 4 between $G_2$ and $G_3$. These give all the 8 edges in $\gamma^{-1}(4)$. Therefore, we must have the following: (a) two 4-cycles between $G_1$ and $G_3$ in $\Gamma_{14}$, (b) one 4-cycle between $G_1$ and $G_2$ in $\Gamma_{14}$, (c) one 4-cycle between $G_2$ and $G_3$ in $\Gamma_{14}$. So, the 4-cycle in $\Gamma_{24}$ is in between $G_1$ and $G_3$. Thus, up to an isomorphism, $\gamma^{-1}(4)$ is unique. In particular, we can assume that $\Gamma_{124}$ is as in Fig. 3 (b). Now, $y^2 x^5$ is an edge of color 4 between $G_1$ and $G_2$. Thus, $y^1$ (resp., $y^2$) is in $H_1$ or $H_2$. Assume, without loss, $y^1 \in H_1$. Then $y^2 \in H_2$. Since $\Gamma$ is bipartite and $H_2$ represents $x_2 x_3^{-1} x_2 x_1^{-1}$, taking $v_1 = x^5$ as in Eq. (2.6), $H_2 = C_4(x^5, y^2, z^6, y^4)$. Since $\Gamma_{23} = C_4 \sqcup C_4 \sqcup C_4 \sqcup C_4$ and $\Gamma_{13} = C_4 \sqcup C_6 \sqcup C_6$, we have $x^4 y^1, x^3 z^5, x^2 z^4, x^1 z^3, y^3 z^1, x^6 z^2 \in \gamma^{-1}(3)$. Then $(\Gamma, \gamma) = \mathcal{M}_{2,3}$ of Fig. 3 (b). This completes the proof. 

Lemma 22. There exists a unique 18-vertex crystallization of $S^3/Q_8$.

Proof. Let $(\Gamma, \gamma)$ be an 18-vertex crystallization of $S^3/Q_8$. By Lemma 15 and Theorem 4, $(\Gamma, \gamma)$ is the crystallization of $S^3/Q_8$ with minimum number of vertices. So, by Lemma 18, $(\Gamma, \gamma)$ has no 2-cycle. Thus, $\Gamma$ is a simple graph. Since $S^3/Q_8$ is orientable, $\Gamma$ is bipartite. By Proposition 14 and Remark 16, $(\Gamma, \gamma)$ yields a presentation $(S \mid R)$ of $Q_8$ with $\varphi(S, R) = 18$. Again, $(\Gamma, \gamma)$ has no 2-cycle implies $g_{ij} \leq 4$ for $i \neq j$. By Proposition 13, $g_{12} + g_{13} + g_{14} = 18/2 + 2 = 11$. Assume, without loss, that $g_{12} = 3$ and $g_{13} = g_{14} = 4$. Therefore, if we choose generators (resp., relations) correspond to the components of $\Gamma_{12}$ (resp., $\Gamma_{34}$) then $(\Gamma, \gamma)$ yields a presentation $\langle S \mid R \rangle \in \mathcal{P}_2(Q_8) \setminus \mathcal{P}_1(Q_8)$ with $\varphi(S, R) = 18$. Then by the proof of part (v) in Lemma 15, $R = \{(x_1^2 x_2^{-2})^{\varepsilon_1}, (x_1 x_2 x_1 x_2^{-1})^{\varepsilon_2}, (x_2 x_1 x_2 x_1^{-1})^{\varepsilon_3}\}$ for some $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{1, -1\}$. Then, by choosing $(i, j) = (3, 4)$ or $(4, 3)$ as in Eq. (2.6), we can assume that the three relations correspond to components of $\Gamma_{34}$ are $(x_1^2 x_2^{-1})^{\varepsilon_1}, (x_1 x_2 x_1 x_2^{-1})^{\varepsilon_2}, (x_2 x_1 x_2 x_1^{-1})^{\varepsilon_3}$ for some $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}$. Since $\Gamma$ has no 2-cycle, $\Gamma_{ij} = C_4 \sqcup C_4 \sqcup C_4 \sqcup C_6$ for $1 \leq i \leq 2$ and $3 \leq j \leq 4$. Let $G_1, G_2, G_3$ be the components of $\Gamma_{12}$ and $H_1, H_2, H_3$ be the components of $\Gamma_{34}$ such that $x_1, x_2, x_3$ represent the generators corresponding to $G_1, G_2, G_3$ and $(x_1^2 x_2^{-1})^{\varepsilon_1}, (x_1 x_2 x_1 x_2^{-1})^{\varepsilon_2}, (x_2 x_1 x_2 x_1^{-1})^{\varepsilon_3}$ represent the relations corresponding to $H_1, H_2, H_3$ respectively. Then $G_i, H_i$ are 6-cycles for $1 \leq i \leq 3$. Let $G_1 = C_6(a_1, \ldots, a_6), G_2 = C_6(b_1, \ldots, b_6), G_3 = C_6(c_1, \ldots, c_6)$. Again, to form these relations, there are exactly three edges with color 4 between $G_i$ and $G_j$ for $i \neq j$. Since each of $\Gamma_{14}$ and $\Gamma_{24}$ has three 4-cycles, the three edges with color 4 between $G_i$ and $G_j$ for $i \neq j$, yield two 4-cycles. Then, up to an isomorphism, $\Gamma_{124}$ is as in Fig. 4. Same arguments hold for color 3.

To construct $\tilde{r}_k$ as in Eq. (2.6), choose $(i, j) = (4, 3)$. Since $H_2$ presents the relation $x_2 x_1 x_2 x_1^{-1}$, up to isomorphism, the starting vertex $v_1$ (as in Eq. (2.6)) is $a_2$ or $a_3$. If $v_1 = a_2$ then $H_2 = C_6(a_2, b_3, c_4, a_5, c_2, b_5)$ or $C_6(a_2, b_3, c_4, a_5, c_6, b_1)$. In the first case, if $b_4 c_3 \in \gamma^{-1}(3)$, then $b_4 c_3$ lies in a cycle of size at least 8 in $\Gamma_{23}$, which is not possible. Then the 4-cycle in $\Gamma_{13}$ between $G_2$ and $G_3$ must be $C_4(b_1, b_2, c_5, c_6)$. But this is not possible since $b_1 c_6 \in \gamma^{-1}(4)$. In the second case, if $b_2 c_5 \in \gamma^{-1}(3)$, then $b_2 c_5$ lies in a cycle of size at least 8 in $\Gamma_{13}$, which is not possible. Then the 4-cycle in $\Gamma_{23}$ between $G_2$ and $G_3$ must be
Figure 4: Crystallization $\mathcal{J}_3$ of $S^3/\mathbb{Q}_8$

$C_4(b_4, b_5, c_2, c_3)$. Again, this is not possible since $b_5c_2 \in \gamma^{-1}(4)$. Thus, $v_1 = a_3$. Now, if $b_2c_5$ is an edge of color 3 then $a_4c_1$ and $a_5b_6$ must be edges of color 3. Then $b_5c_2$ must be an edge of color 3 to make a 6-cycle in $\Gamma_{13}$, which is a contradiction (since $b_5c_2$ is already an edge of color 4). Thus, $H_2 = C_6(a_3, b_2, c_3, a_6, c_1, b_6)$. Since the three edges with color 3 between $G_2$ and $G_3$ yield two 4-cycles (in $\Gamma_{13}$ and $\Gamma_{23}$), $b_1c_4, b_3c_2$ must be edges of color 3 between $G_2$ and $G_3$. To make a 6-cycle in $\Gamma_{13}$, $a_5b_4$ must be an edge of color 3. By similar arguments, $a_1c_6, a_2c_5, a_4b_5 \in \gamma^{-1}(3)$. Then, $(\Gamma, \gamma) = \mathcal{J}_3$ of Fig. 4.

Now, the components $H_1 = C_6(a_2, b_3, c_2, a_4, c_3)$ and $H_3 = C_6(b_4, a_1, c_6, b_1, c_4, a_5)$ of $\Gamma_{34}$ yield the relations $x_2^2x_1^{-2}$ and $x_1x_2x_1x_2^{-1}$ respectively. Thus $(\Gamma, \gamma)$ yields the presentation $\langle x_1, x_2, x_3 | x_2^2x_1^{-2}, x_1x_2x_1x_2^{-1} \rangle \cong \mathbb{Q}_8$. This completes the proof.

**Lemma 23.** There exists a unique 24-vertex crystallization of $S^1 \times S^1 \times S^1$.

**Proof.** Let $(\Gamma, \gamma)$ be a 24-vertex crystallization of $(S^1)^3$. By Lemma 15 and Theorem 4, $(\Gamma, \gamma)$ is the crystallization of $(S^1)^3$ with minimum number of vertices. So, by Lemma 18, $(\Gamma, \gamma)$ has no 2-cycle. Thus, $\Gamma$ is a simple graph. Since $(S^1)^3$ is orientable, $\Gamma$ is bipartite. By Proposition 14 and Remark 16, $(\Gamma, \gamma)$ yields a presentation $(S \mid R)$ of $Z^3$ with $\varphi(S, R) = 24$. Since any presentation of $Z^3$ has at least three generators, $g_{ij} \geq 4$ for $i \neq j$. By Proposition 13, $g_{12} + g_{13} + g_{14} = 14$ and $g_{ij} = g_{kl}$ for $i, j, k, l$ distinct.

**Claim.** $(\Gamma, \gamma)$ does not yield a presentation $(S \mid R) \in \mathcal{P}_4(Z^3) \setminus \mathcal{P}_3(Z^3)$ with $\varphi(S, R) = 24$.

Assume $(S \mid R) \in \mathcal{P}_4(Z^3) \setminus \mathcal{P}_3(Z^3)$, where $S = \{x_1, x_2, x_3, x_4\}$. Then $A := \{(x_k^2x_l^{-1})^{\pm 1}, (x_jx_k^{-1}x_jx_l)^{\pm 2}, (x_jx_kx_l^{-1})^{\pm 1}, (x_kx_l^{-1})^{2}, x_i^2 : i, j, k, l \text{ are distinct}\}$ is the set of all relations of weight four. Since $\Gamma$ has no 2-cycle, $R$ has no element of weight two. This implies that $R$ has at least three elements of weights four. Since $Z^3$ has no torsion element, $x_i^2, (x_kx_l^{-1})^2 \notin R$. Consider an element $w \in R \cap A$. Assume, without loss, $w = (x_1x_4)^{\pm 1}$. Then $(S_1 \mid R_1) \in \mathcal{P}_3(Z^3)$, where $S_1 = \{x_1, x_2, x_3\}$ and $R_1$ consists of the elements $\tilde{r}$, where $\tilde{r}$ can be obtained from a relation $r \in R$ by replacing $x_4$ by $w_1$. Let $A(w) := \{r \in A \setminus \{w\}\}$. Observe that the weights of the elements in $A(w)$ are 6 or 8.

Since $(S_1 \mid R_1) \in \mathcal{P}_3(Z^3) \setminus \mathcal{P}_2(Z^3)$, we have $N(R_1) = N(R_0)$, where $R_0 = \{x_1x_2x_1^{-1}x_2^{-1}, x_1x_3x_1^{-1}x_3^{-1}, x_2x_3x_2^{-1}x_3^{-1}\}$ and hence $N(R_1)$ has no element of weight less than 6 (see
the proof of part (v) of Lemma 15). Again, since \( \#(R_1 \cap A(w)) \geq 2 \), \( R_1 \) has at least two elements of weights 6 or 8. Observe that \( D := \{ x_i x_j x_k^{-1} x_j^{-1} x_k^{-1} x_j^{-1} x_k^{-1} x_i x_j x_k^{-1} : i, j, k \in \{1, 2, 3\} \} \) is the set of all relations of weights at most 8 in \( N(R_0) \). So, \( R_1 \) has at least two independent relations from \( D \cap A(w) \). But \( D \cap A(w) \) does not contain two such elements, a contradiction. This proves the claim.

![Diagram](image-url)

Figure 5: Crystallization \( J_4 \) of \( S^1 \times S^1 \times S^1 \)

By the claim, \( g_{ij} \neq 5 \) for all \( 1 \leq i \neq j \leq 4 \). So, we can assume that \( g_{12} = g_{13} = 4 \) and \( g_{14} = 6 \). Then all the components of \( \Gamma_{14} \) and \( \Gamma_{23} \) are 4-cycles. Let \( \Gamma_{12} = G_1 \sqcup \cdots \sqcup G_4 \) and \( \Gamma_{34} = H_1 \sqcup \cdots \sqcup H_4 \) such that \( x_1, \ldots, x_4 \) represent the generators corresponding to \( G_1, \ldots, G_4 \) respectively and \( r_1, \ldots, r_1 \) represent the relations corresponding to \( H_1, \ldots, H_4 \) respectively. To construct \( \tilde{r}_k \) as in Eq. (2.6), choose \( (i, j) = (4, 3) \). Thus \( (\Gamma, \gamma) \) yields a presentation \( \langle S = \{ x_1, x_2, x_3 \} \mid R = \{ r_1, r_2, r_3 \} \rangle \in \mathcal{P}_{\mathbb{Z}}(\mathbb{Z}^3) \setminus \mathcal{P}_2(\mathbb{Z}^3) \) with \( \varphi(S, \Gamma) = 24 \). Then \( R \) contains three independent relations of weight 6 from the set \( \{ x_i x_j x_k^{-1} x_j^{-1} x_k^{-1} x_j^{-1} x_k^{-1} : i, j, k \in \{1, 2, 3\} \} \) (see the proof of part (v) of Lemma 15). Without loss of generality, we can assume that \( R = \{ x_1 x_2 x_1^{-1} x_2^{-1}, x_2 x_3 x_2^{-1} x_3^{-1}, (x_1 x_3 x_1^{-1} x_3^{-1})^{\varepsilon_1}, (x_1 x_3 x_1^{-1} x_3^{-1})^{\varepsilon_2} \} \) for some \( \varepsilon_1, \varepsilon_2 \in \{1, -1\} \). Then, all the components of \( \Gamma_{12} \) and \( \Gamma_{34} \) are 6-cycles. Similarly, all the components of \( \Gamma_{13} \) and \( \Gamma_{24} \) are 6-cycles. Let \( G_1 = C_6(a_1, \ldots, a_6) \), \( G_2 = C_6(b_1, \ldots, b_6) \), \( G_3 = C_6(c_1, \ldots, c_6) \) and \( G_4 = C_6(d_1, \ldots, d_6) \). To form the relations, there are exactly two edges of color 3 (resp., 4) between \( G_i \) and \( G_j \) for \( 1 \leq i \neq j \leq 4 \). Then, up to an isomorphism, \( \Gamma_{124} \) as in Fig. 5. Now for the relation \( x_1 x_2 x_1^{-1} x_2^{-1} \), we can choose \( v_1 = b_0 \) as in Eq. (2.6). Then the cycle for \( x_1 x_2 x_1^{-1} x_2^{-1} \) is \( H_1 = C_6(b_0, a_1, d_4, b_3, a_4, d_1) \). Since \( \Gamma_{23} \) consists of 4-cycles, it follows that \( a_0 d_5, a_5 b_2, b_1 d_6 \in \gamma^{-1}(3) \). Then the cycle for the relation \( x_2 x_3 x_2^{-1} x_3^{-1} \) is \( H_2 = C_6(c_6, b_1, d_6, c_3, b_4, d_3) \). Again (since \( \Gamma_{23} \) is union of 4-cycles and \( \Gamma_{13} \) is union of 6-cycles), \( d_2 c_1, b_5 c_2, a_3 c_4, a_2 c_5 \in \gamma^{-1}(3) \). Then \( (\Gamma, \gamma) = J_4 \) of Fig. 5.
Now, the components $H_1, H_2$ and $H_3 = C_6(c_1, a_6, d_5, c_4, a_3, d_2)$ yield the relations $x_1 x_2 x_1^{-1} x_2^{-1}, x_2 x_3 x_2^{-1} x_3^{-1}$ and $x_1 x_3 x_1^{-1} x_3^{-1}$ respectively. Thus $(\Gamma, \gamma)$ yields the presentation $\langle x_1, x_2, x_3 \mid x_1 x_2 x_1^{-1} x_2^{-1}, x_2 x_3 x_2^{-1} x_3^{-1}, x_1 x_3 x_1^{-1} x_3^{-1} \rangle \cong \mathbb{Z}^3$. This completes the proof. \hfill \qed

**Remark 24.** The crystallizations $\mathcal{K}_{2,1}, \mathcal{K}_{3,1}$ and $\mathcal{M}_{3,2}$ (in Figures 2 and 3) were originally found by Gagliardi et al. ([8], [10]). The first two have the following natural generalization: Consider the bipartite graph $\Gamma$ consists of two disjoint $2p$-cycles $G_1 = C_{2p}(a_1, b_1, \ldots, a_p, b_p)$, $G_2 = C_{2p}(c_1, d_1, \ldots, c_p, d_p)$ together with $4p$ edges $a_i c_i, b_i d_i, a_i c_{i+q}, b_i d_{i+q}$ for $1 \leq i \leq p$. Consider the edge-coloring $\gamma$ with colors $1, 2, 3, 4$ of $\Gamma$ as: $\gamma(b_i d_{i+1}) = \gamma(d_i c_{i+1}) = 1$, $\gamma(a_i b_i) = \gamma(c_i d_i) = 2$, $\gamma(a_i c_{i+q}) = \gamma(b_i d_{i+q}) = 3$ and $\gamma(a_i c_i) = \gamma(b_i d_i) = 4$, $1 \leq i \leq p$. (Summations in the subscripts are modulo $p$.) Then, $\mathcal{K}_{p,q} = (\Gamma, \gamma)$ is a $4p$-vertex crystallization of $L(p, q)$, for $p \geq 2$ and $q \geq 1$. This series is more or less known in the literature. In the next section, we present some generalizations of $\mathcal{M}_{3,2}$.

## 5 Two series of crystallizations of lens spaces

Generalizing the construction of $\mathcal{M}_{3,2}$ (Fig. 3 (b)) we have constructed the following series of crystallizations.

### 5.1 A $4(k + q - 1)$-vertex crystallization of $L(kq - 1, q)$

Let $q \geq 3$. For each $k \geq 2$, we construct a $4(k + q - 1)$-vertex 4-colored simple graph $\mathcal{M}_{k,q} = (\Gamma^k, \gamma^k)$ with the color set $\{1, 2, 3, 4\}$ inductively which yields the presentation $\langle x, y \mid x^q y^{-1}, y^k x^{-1} \rangle$. For this, we want $g_{12}^k = g_{34}^k = 3$. Then, without loss, $g_{13}^k = g_{24}^k = k + q - 2$ and $g_{14}^k = g_{23}^k = k + q - 1$, where $g_{ij}^k$ is the number of components of $\Gamma_{ij}^k$ for $i \neq j$. These imply, $\Gamma_{14}^k$ and $\Gamma_{23}^k$ must be union of $4$-cycles and $\Gamma_{13}^k$ (resp., $\Gamma_{24}^k$) has two 6-cycles and $(k + q - 4)$ 4-cycles. Then, by Proposition 13, $\mathcal{M}_{k,q}$ would be a crystallization of some connected closed 3-manifold $M_k$.

**k = 2 case:** The crystallization $\mathcal{M}_{2,q}$ is given in Fig. 6. Then, the components of $\Gamma_{12}^k$ are $G_1 = C_2(y^1, \ldots, y^{2q})$, $G_2 = C_{4}(y^1, \ldots, y^4)$, $G_3 = C_{2q}(z^1, \ldots, z^{2q})$ and the components of $\Gamma_{14}^k$ are $H_1 = C_{2q}(y^1, x^{2q}, z^1, x^2, x^{2q-2}, x^{2q-2})$, $H_2 = C_{4}(y^{2q-1}, z, z^{2q}, y^1)$, $H_3 = C_{2q}(x^{2q-1}, z, x^1, \ldots, z^{2q-3}, x^{2q-3})$. Let $x$, $y$ be the generators corresponding to $G_1$ and $G_2$ respectively. To construct $\tilde{r}_1$ (resp., $\tilde{r}_2$) as in Eq. (2.6), choose $(i,j) = (4,3)$ and $v_1 = y^1$ (resp., $v_1 = x^{2q-1}$). Then $H_1$ and $H_2$ represent the relations $x^q y^{-1}$ and $y^2 x^{-1}$ respectively. Therefore, by Proposition 14, $\pi(\mathcal{M}_{2,q}) \cong \langle x, y \mid x^q y^{-1}, y^2 x^{-1} \rangle \cong \mathbb{Z}_{2q-1}$.

Let $T$ and $T_2$ be the 3-dimensional simplicial cell complexes represented by the color graphs $T^3|\{z^1, z^2, z^3, z^4\}$ and $\Gamma^2|V(T^2)\setminus\{x^1, y^1, x^3, y^3\}$ respectively. Then $|T|$ and $|T_2|$ are solid tori and the facets (2-cells) of $T \cap T_2$ are $x_2 x_4^1 x_3^1, x_2 x_4^2 x_3^2, x_1 x_3^3, z_3^1, z_3^2$. Thus, $|T \cap T_2|$ is a torus (see Fig. 7 (b)) with $\pi_1(|T \cap T_2|, v_1) = \langle \alpha = [a], \beta = [b] | \alpha \beta \alpha^{-1} \beta^{-1} \rangle$, where $a = x_3^2 x_4^1$ and $b = x_3^3 x_4^1 x_3^2$. Then $b = x_3^3 x_4^1 x_3^2 = 0(x_3^1) \sim 1$ in $|T|$. Therefore, $\pi_1(|T|, v_1) = \langle \alpha, \beta | \beta \rangle$.

Since $\alpha \beta = \beta \alpha$ in $|T| \cap |T_2|$, $ab \sim ba$ in $|T_2|$. Now $ab = (x_3^2 x_4^1)(x_3^2 x_4^1 x_3^2) \sim x_3^2 x_4^1 x_3^2 x_3^1$ $=$ $x_4^1 x_3^2 x_3^1 x_3^2 x_3^1$ $\sim x_3^2 x_4^1 x_3^2$. Therefore, $a^2 b \sim ab$ $\sim x_3^1 x_3^1 x_3^2 x_3^1 x_3^1 = x_3^2 x_3^1 x_3^2 x_3^1$ $\sim x_3^1 x_3^2 x_3^1 x_3^2 x_3^1$ $\sim x_3^2 x_3^1 x_3^2 x_3^1 x_3^2$. Thus, $a^3 b \sim x_3^2 x_3^1 x_3^2 x_3^1 x_3^2$ $\sim x_3^2 x_3^1 x_3^2 x_3^1 x_3^2$ $\sim x_3^2 x_3^1 x_3^2 x_3^1$. Then $a^4 b \sim x_3^2 x_3^1 x_3^2 x_3^1 x_3^2$ $\sim x_3^2 x_3^1 x_3^2 x_3^1 x_3^2$ $\sim x_3^2 x_3^1 x_3^2 x_3^1 x_3^2$.
Figure 6: Crystallization $\mathcal{M}_{2,q}$ of $L(2q-1, q)$

$x_{23}^4 x_{34}^4 z_{23}^1 \sim x_{23}^4 z_{23}^1$. Similarly, $a^{q-1}b \sim x_{23}^2 z_{23}^1$. Therefore, $a^2 b \sim x_{34}^4 x_{34}^1 x_{23}^{2q-4} z_{23}^1 = x_{34}^2 z_{23}^1 - z_{23}^1$, $a^2 b \sim x_{34}^4 x_{34}^1 z_{23}^1 - z_{23}^1$, $a^2 b \sim x_{34}^4 z_{23}^1 - z_{23}^1$, $a^2 b \sim x_{34}^4 z_{23}^1 - z_{23}^1$. Since $k = 2$, we have $z_{23}^1 = z_{23}^1$. This implies, $a^{q-1}p_2 \sim x_{34}^2 z_{23}^1 - z_{23}^1$, $a^{q-1}p_2 \sim x_{34}^2 z_{23}^1 - z_{23}^1$. Since $k = 2$, we have $z_{23}^1 = z_{23}^1$. This implies, $a^{q-1}p_2 \sim x_{34}^2 z_{23}^1 - z_{23}^1$. Similarly, $\pi_1([T], v_1) = (\alpha, \beta | \alpha^{q-1} \beta, \alpha \beta \alpha^{-1} \beta^{-1})$. This implies that (see the second paragraph of Subsection 2.3) $|T| \cup |T_2| = L(2q-1, 2)$. Therefore, $\mathcal{M}_{2,q}$ is a crystallization of $L(2q-1, 2) \cong L(2q-1, q)$.

$k = 3$ case: Here $z_{13}^{2q-1} \neq z_{13}^{2q-1}$. Let

$\Gamma^3 = (V(\Gamma^2) \cup \{y^5, y^6, z_{2q+1}^1, z_{2q+2}^1\}, E(\Gamma^2) \cup \{y^2 z_{2q}, y^3 z_{2q}^1, y^4 z_{2q-1}^1, z_{2q}^2 q, y^5 z_{2q} + y^6 z_{2q}, y^5 z_{2q} + y^6 z_{2q}, y^5 z_{2q}, y^6 z_{2q}^1, y^3 z_{2q}, y^5 z_{2q}, y^6 z_{2q}^1\}$. Consider the following coloring $\gamma^3$ on the edges of $\Gamma^3$: same colors on the old edges as in $\mathcal{M}_{2,q}$, color 1 on the edges $y^3 y^5, y^6 y^4, z_{2q-1}^1 z_{2q+1}^1, z_{2q+2} z_{2q}^2$, color 2 on the edges $y^5 y^6, z_{2q+1}^1 z_{2q+2}^1$, color 3 on the edges $y^2 z_{2q+1}^1, y^3 z_{2q+2}^1, y^5 z_{2q}, y^6 z_{2q}^1$ and color 4 on the edges $y^5 z_{2q+1}^1, y^6 z_{2q+2}^1$ (see Fig. 7 (a)). Let $T$ be as in the case $k = 2$ and $T_3$ be the cell complex represented by the colored graph $\Gamma^3|_{V(\Gamma^3) \{x_{13}^1, x_{23}^1, x_{33}^1\}}$.

Then, $a^{q-1}p_2 \sim z_{34}^2 z_{13}^{2q-1} z_{23}^1 = z_{34}^2 z_{13}^{2q-1} z_{23}^1 \sim z_{34}^2 z_{13}^{2q-1} z_{23}^1 = z_{34}^2 z_{13}^{2q-1} z_{23}^1$. This implies, $a^{q-1}p_2 \sim z_{34}^2 z_{13}^{2q-1} z_{23}^1 = \partial(z_{23}^1) \sim 1$ in $|T_3|$. Thus, $\pi_1([T_3], v_1) = (\alpha, \beta | \alpha^{q-1} \beta, \alpha \beta \alpha^{-1} \beta^{-1})$ and hence $|T| \cup |T_3| = L(3q-1, 3)$. Therefore, $\mathcal{M}_{3,q}$ is a crystallization of $L(3q-1, 3) \cong L(3q-1, q)$. 

THE ELECTRONIC JOURNAL OF COMBINATORICS 21(1) (2014), #P1.61

19
**Claim.**

$k \geq 4$ case: Consider the graph

$\Gamma^k = (V(\Gamma^{k-1}) \cup \{y^{2k-1}, y^{2k}, z^{2q+2k-5}, z^{2q+2k-4}\}, E(\Gamma^{k-1}) \setminus \{y^{2k-3}z^q, y^{2k-2}z^1, y^{2k-2}y^4, z^{2q+2k-6}z^{2q}, (y^{2k-2}y^{-1}y^2, y^{2k-1}y^2, y^{2k}y^4, z^{2q+2k-6}z^{2q+2k-5}, z^{2q+2k-5}z^{2q+2k-4}, z^{2q+2k-4}z^{2q+2q+2k-5}, y^{2k}z^{2q+2k-4}, y^{2k-3}z^{2q+2k-5}, y^{2k-2}z^{2q+2k-4}, y^{2k-1}z^{2q}, y^{2k}z^1\}).$

Also, consider the following coloring $\gamma^k$ on the edges of $\Gamma^k$: same colors on the old edges as in $\mathcal{M}_{k-1,q}$, color 1 on the edges $y^{2k-2}y^{2k-1}, y^{2k}y^4, z^{2q+2k-6}z^{2q+2k-5}, z^{2q+2k-4}z^{2q}$, color 2 on the edges $y^{2k-1}y^{2k}, z^{2q+2q+2k-5}z^{2q+2k-3}$, color 3 on the edges $y^{2k-3}z^{2q+2k-5}, y^{2k-2}z^{2q+2k-4}, y^{2k-1}z^{2q}, y^{2k}z^1$ and color 4 on the edges $y^{2k-1}z^{2q+2k-5}, y^{2k}z^{2q+2k-4}$. Let $T$ be as in the case $k = 2$ and $T_k$ be the cell complex represented by the colored graph $\Gamma^k|_{V(\Gamma^k)\setminus \{x^1, x^2, x^3, z^3\}}$.

Claim. $a^{kq-1}b^k \sim z_{\frac{2q}{34}}z_{\frac{2q}{13}}z_{\frac{2q}{23}}$ in $|T_k|$.

We prove the claim by induction. It is true for $k = 3$. Assume that $a^{(k-1)q-1}b^{k-1} \sim z_{\frac{2q}{34}}z_{\frac{2q}{13}}z_{\frac{2q}{23}}$ in $|T_{k-1}|$. Now, $a^q b \sim z_{\frac{2q}{34}}z_{\frac{2q}{13}}z_{\frac{2q}{23}}$ in $|T_{k-1}|$. Thus, $a^{kq-1}b^k \sim (a^q b)(a^{(k-1)q-1}b^{k-1}) \sim z_{\frac{2q}{34}}z_{\frac{2q}{13}}z_{\frac{2q}{23}}$ in $|T_k|$. The claim now follows by induction.

Since $z_{\frac{2q}{13}}z_{\frac{2q}{13}}z_{\frac{2q}{23}}$ in $T_k$, by the claim we get $a^{kq-1}b^k \sim 1$ in $|T_k|$. Thus, $\pi_1(|T_k|, v_1) = \langle\alpha, \beta \mid \alpha^{kq-1}\beta^k, \alpha\beta\alpha^{-1}\beta^{-1}\rangle$ and hence $|T_k| = L(kq - 1, k) \cong L(kq - 1, q)$. Therefore, $\mathcal{M}_{k,q}$ is a crystallization of $L(kq - 1, q)$.

### 5.2 A 4($k + q$)-vertex crystallization of $L(kq + 1, q)$

Let $q \geq 4$. For each $k \geq 1$, we construct a 4($k + q$)-vertex 4-colored simple graph $\mathcal{N}_{k,q} = (\Gamma^k, \gamma^k)$ with the color set $\{1, 2, 3, 4\}$ inductively which yields the presentation $\langle x, y \mid x^qy^{-1}, xy^k \rangle$. For this, we want $g_{12}^k = g_{34}^k = 3$. Then, without loss, $g_{13}^k = g_{24}^k = k + q - 1$ and $g_{14}^k = g_{23}^k = k + q$, where $g_{ij}^k$ is the number of components of $\Gamma^k_{ij}$ for $i \neq j$. These imply, $\Gamma_{14}^k$ and $\Gamma_{23}^k$ must be union of 4-cycles and $\Gamma_{13}^k$ (resp., $\Gamma_{24}^k$) has two 6-cycles and ($k + q - 4$) 4-cycles. Then, by Proposition 13, $\mathcal{N}_{k,q}$ would be a crystallization of some connected closed 3-manifold $M_k$. 

---

**Figure 7:** (a) Crystallization $\mathcal{M}_{3,q}$ of $L(3q - 1, q)$  
(b) $|T_1 \cap T_k|$
**k = 1 case:** The crystallization $N_{1,q}$ is given in Fig. 8. Then, the components of $\Gamma_{12}$ are $G_1 = C_{2q}(x^1, \ldots, x^{2q})$, $G_2 = C_4(y^1, \ldots, y^4)$, $G_3 = C_{2q}(z^1, \ldots, z^{2q})$ and the components of $\Gamma_{34}$ are $H_1 = C_{2q}(y^3, x^2, z^2, x^4, \ldots, z^{2q-2}, x^{2q})$, $H_2 = C_4(z^{2q-1}, x^1, z^1, y^1)$, $H_3 = C_{2q}(x^3, y^2, z^{2q}, y^4, z^{2q-1}, z^{2q-3}, x^{2q-3}, \ldots, z^1, x^4, z^3)$. Let $x, y$ be the generators corresponding to $G_1$ and $G_2$ respectively. To construct $\tilde{r}_1$ (resp., $\tilde{r}_2$) as in Eq. (2.6), choose $(i, j) = (4, 3)$ and $v_1 = y^3$ (resp., $v_1 = z^{2q-1}$). Then $H_1$ and $H_2$ represent the relations $x^qy^{-1}$ and $xy$ respectively. Therefore, by Proposition 14, $\pi(M_1, *) \cong \langle x, y \mid x^qy^{-1}, xy \rangle \cong \mathbb{Z}_{q+1}$.

![Figure 8: Crystallization $N_{1,q}$ of $L(q+1, q)$](image)

Let $T$ and $T_1$ be the 3-dimensional simplicial cell complexes represented by the color graphs $\Gamma^1|_{\{x^5, y^4, x^3, z^3\}}$ and $\Gamma^1|_{\{y^4, x^3, z^3\}}$ respectively. Then $|T|$ and $|T_1|$ are solid tori and the facets (2-cells) of $T \cap T_1$ are $x_5^5, x_5^6, x_5^3, x_5^4, x_5^1, x_5^2, z_3^1, z_3^3$. Thus, $|T \cap T_1|$ is a torus (see Fig. 9 (b)) with $\pi_1(|T \cap T_1|, v_1) = \langle \alpha = [a], \beta = [b] \mid \alpha \beta \alpha^{-1} \beta^{-1} \rangle$, where $a = x_3^4x_3^4$ and $b = x_3^3x_3^3$. Then $b = x_3^3x_3^3 \sim \partial(x_3^3) \sim 1$ in $|T|$. Therefore, $\pi_1(|T_1|, v_1) = \langle \alpha, \beta \mid \beta \rangle$.

Since $\alpha \beta = \beta \alpha$ in $|T \cap T_1|$, it follows that $ab \sim ba$ in $|T_1|$. Now, $ab = (x_3^3x_3^4)(x_3^3x_3^4x_3^4x_3^4) \sim x_3^4x_3^6x_3^4x_3^3 = z_3^4z_3^2z_3^2z_3^3 \sim \hat{z}_3^2z_3^3 = z_3^1x_3^2$. Thus, $a^2b \sim aba \sim (x_3^2x_3^2)(x_3^4x_3^4) = z_3^1x_3^4x_3^4 = z_3^3x_3^3x_3^3 = z_3^3z_3^3x_3^3 = z_3^3z_3^2z_3^2z_3^3 \sim z_3^1x_3^2z_3^3 \sim z_3^1x_3^2z_3^3 \sim z_3^1x_3^2z_3^3$. Similarly we get, $a^{q-2}b \sim z_3^{q-2}z_3^4$. Thus, $a^{q-1}b \sim a^{q-2}ba \sim z_3^{q-2}z_3^4x_3^4z_3^4 = z_3^3z_3^2z_3^2z_3^2z_3^2z_3^2z_3^3z_3^2x_3^2z_3^3 \sim z_3^1x_3^2z_3^3 \sim z_3^1x_3^2z_3^3$. Therefore, $a^q b \sim \frac{1}{23}x_3^{23}x_3^{23}x_3^{23}$. 

THE ELECTRONIC JOURNAL OF COMBINATORICS 21(1) (2014), #P1.61 21
Again, \( a = x_4^3 x_{34}^2 = y_3^3 y_{34}^2 \sim z_{34}^1 y_3^2 y_{34}^2 \sim z_{34}^1 z_{13}^1 y_{34}^2 = z_{34}^1 y_3^2 y_{34}^2 \sim z_{34}^1 y_{34}^2 = z_{34}^1 z_{34}^2 \). Therefore, \( a^{q+1} b \sim a q^2 b \sim y_3^3 z_{34}^1 \sim z_{34}^2 y_{34}^2 = y_3^3 z_{34}^1 \sim z_{34}^2 z_{34}^1 \sim z_{34}^2 z_{34}^1 = \partial(z_4^2) \sim 1 \) in \( |T_1| \). Thus \( \pi_1(|T_1|, v_1) = (\alpha, \beta | \alpha^{q+2}, \alpha \beta \alpha^{-1} - 1) \). This implies that \( |T| \cup |T_1| = L(q + 1, 1) \). Therefore, \( N_{1,q} \) is a crystallization of \( L(q + 1, 1) \equiv L(q + 1, q) \).

**k = 2 case:** Here \( z_{14}^1 \neq z_{14}^q \). Let

\[
\Gamma^2 = (V(\Gamma)^1) \cup \{y_1^5, y_2^6, z_{2q}^1, z_{2q+2}^1\}, E(\Gamma)^1 \setminus \{y_1^5 z_{2q}^2, y_1^5 y_1^4, y_1^5 z_{2q}^2 \} \cup \{y_1^5 y_1^6, y_1^5 y_1^6, y_1^5 y_1^6, y_1^5 y_1^6, y_1^5 y_1^6, y_1^5 y_1^6\}.
\]

To construct \( N_{2,q} \), consider the following coloring \( \gamma^2 \) on the edges of \( \Gamma^2 \): same colors on the old edges as in \( N_{1,q} \), color 1 on the edges \( y_1^5 y_1^6, y_1^5 y_1^6, z_{2q}^1 z_{2q+2}^1, z_{2q}^1 z_{2q+2}^1 \) color 2 on the edges \( y_1^5 y_1^6, z_{2q}^1 z_{2q+2}^1, z_{2q}^1 z_{2q+2}^1 \), color 3 on the edges \( y_1^5 y_1^6, y_1^5 y_1^6, y_1^5 y_1^6, y_1^5 y_1^6, y_1^5 y_1^6 \), color 4 on the edges \( y_1^5 y_1^6, y_1^5 y_1^6, y_1^5 y_1^6, y_1^5 y_1^6 \) (see Fig. 9(a)). Let \( T \) be as in the case \( k = 2 \) and \( T_2 \) be the cell complex represented by the colored graph \( \Gamma^2 |_{V(\Gamma^2) \setminus \{x_3^i, x_4^i, x_3^i, x_3^i\}} \).

**k \geq 3 case:** Let

\[
\Gamma^k = (V(\Gamma^k)^1) \cup \{y_1^{2k+1}, y_1^{2k+1}, z_{2q}^1 z_{2q}^1, z_{2q}^1 z_{2q}^1, z_{2q}^1 z_{2q}^1\}, E(\Gamma^k)^1 \setminus \{y_1^{2k+1} z_{2q}^2, y_1^{2k+1} z_{2q}^2, y_1^{2k+1} z_{2q}^2, y_1^{2k+1} z_{2q}^2, y_1^{2k+1} z_{2q}^2, y_1^{2k+1} z_{2q}^2\}.
\]

To construct \( N_{k,q} \), consider the following coloring \( \gamma^k \) on the edges of \( \Gamma^k \): same colors on the old edges as in \( N_{k-1,q} \), color 1 on the edges \( y_1^{2k-1} y_1^{2k+1}, y_1^{2k-1} y_1^{2k+1}, z_{2q}^1 z_{2q}^1 z_{2q}^1 \) color 2 on the edges \( y_1^{2k-1} y_1^{2k+1}, y_1^{2k-1} y_1^{2k+1}, z_{2q}^1 z_{2q}^1 z_{2q}^1 \), color 3 on the edges \( y_1^{2k-1} y_1^{2k+1}, y_1^{2k-1} y_1^{2k+1}, y_1^{2k-1} y_1^{2k+1}, y_1^{2k-1} y_1^{2k+1}, y_1^{2k-1} y_1^{2k+1} \) and color 4 on the edges \( y_1^{2k-1} z_{2q}^1 z_{2q}^1 z_{2q}^1 \), \( y_1^{2k-1} z_{2q}^1 z_{2q}^1 z_{2q}^1 \), \( y_1^{2k-1} z_{2q}^1 z_{2q}^1 z_{2q}^1 \), \( y_1^{2k-1} z_{2q}^1 z_{2q}^1 z_{2q}^1 \), \( y_1^{2k-1} z_{2q}^1 z_{2q}^1 z_{2q}^1 \) (see Fig. 9(a)). Let \( T \) be as in the case \( k = 1 \) and \( T_k \) be the cell complex represented by the colored graph \( \Gamma^k |_{V(\Gamma^k) \setminus \{x_3^i, x_4^i, x_3^i, x_3^i\}} \).
Claim. \(a^{kq+1}b^k \sim z_{2q}^2 z_{14}^2 z_3^2 z_3^2\) in \(|T_k|\).

We prove the claim by induction. It is true for \(k = 2\). Assume that \(a^{(k-1)q+1}b^{k-1} \sim z_{2q}^{2q} z_{14}^{2q+2k-3} z_3^2 z_3^2 \) in \(|T_{k-1}|\). Now \(a^q b \sim z_{34}^{2q+2k-3} z_3^2 z_3^2\) and \(a^{q(k-1)+1}b^{(k-1)} \sim z_{24}^{2q} z_{24}^{2q+2k-4} = z_{24}^{2q} z_{24}^{2q+2k-3}\). So, \(a^{kq+1}b^k \sim (a^{q(k-1)+1}b^{(k-1)}(a^q b) \sim z_{24}^{2q} z_{24}^{2q+2k-3} z_3^2 z_3^2 = z_{24}^{2q} z_{24}^{2q+2k-3}\) in \(|T_k|\). The claim now follows by induction.

Since \(z_{2q}^2 = z_{14}^{2q+2k-3}\) in \(T_k\), by the claim we get \(a^{kq+1}b^k \sim 1\) in \(|T_k|\). Thus, \(\pi_1(|T_k|, v_1) = (|, \beta | \alpha^{kq+1} \beta^k, \alpha \beta \alpha^{-1} \beta^{-1})\) and hence \(|T| \cup |T_k| = L(kq + 1, k) \cong L(kq + 1, q)\). Therefore, \(N_{k,q}\) is a crystallization of \(L(kq + 1, q)\).

A few days after we posted the first version of this article (arXiv:1308.6137) in the arXiv, Casali and Cristofori posted an article on complexity of lens spaces [4] in the arXiv (arXiv:1309.5728). In that paper, the authors constructed crystallizations of \(L(p, q)\) with \(4S(p, q)\) vertices, where \(S(p, q)\) denotes the sum of all partial quotients in the expansion of \(q/p\) as a regular continued fraction. In particular, they have constructed \(L(kq - 1, q)\) with \(4(k + q - 1)\) vertices for \(k, q \geq 2\) and \(L(kq + 1, q)\) with \(4(k + q)\) vertices for \(k, q \geq 1\). Their constructions are different from ours.

Remark 25. From the enumeration of crystallizations of prime 3-manifolds with at most 30 vertices (see [3, 12]), we know that \(\Psi(L(9, 4)) = 24\) and \(\Psi(L(13, 4)) = 28\). From our constructions in Subsections 5.1 and 5.2, we know \(M_{2,5}\) and \(N_{2,4}\) are 24-vertex crystallizations of \(L(9, 4)\). The induced subgraphs of \(M_{2,5}\) on 2-colored edges are of the form \(2C_{10} \cup C_4, 2C_6 \cup 3C_4\) or \(6C_4\) and such subgraphs of \(N_{2,4}\) are of the form \(C_{10} \cup C_6 \cup C_6, 2C_6 \cup 3C_4\) or \(6C_4\). So, \(M_{2,5}\) and \(N_{2,4}\) are non-isomorphic. Thus, \(L(9, 4)\) has more than one (non-isomorphic) crystallizations with minimum number of vertices. The constructions in [4] give a 28-vertex of crystallization of \(L(13, 4)\) with \(\{g_{12}, g_{13}, g_{14}\} = \{4, 5, 7\}\). Observe that \(N_{3,4}\) is also a 28-vertex of crystallization of \(L(13, 4)\) with \(\{g_{12}, g_{13}, g_{14}\} = \{3, 6, 7\}\). Thus, these two crystallizations of \(L(13, 4)\) are non-isomorphic. So, the minimal crystallization \(N_{3,4}\) of \(L(13, 4)\) is not unique. Also, from the list of crystallizations in [12], we know that there are several 3-manifolds having more than one crystallizations with minimum number of vertices.

6 Proofs of Theorems 6, 8 and Corollary 7

Proof of Theorem 6. Let \(M_{2,3}\) be as in Subsection 5.1. Then, \(M_{2,3}\) is a 16-vertex crystallization of \(L(5, 3) = L(5, 2)\). Part (i) now follows from Lemmas 15, 19, \ldots, 23.

If \(f_2(X) < 8\) then, by Theorem 4, \(\psi(M) < 8\) and hence \(\psi(M) = 2\). Therefore \(\pi(M, *) = \{0\}\) and hence, by Perelman’s theorem (Poincaré conjecture), \(M = S^3\). Part (ii) now follows from Lemma 19. \(\Box\)

Proof of Corollary 7. From the proof of Lemma 22, \(m(Q_8) = 2\). Therefore, if \(X\) is a pseudotriangulation of \(S^3/Q_8\) then, by Corollary 5 and Lemma 3, \(h_2(X) \geq \psi(S^3/Q_8) - 2 = 18 - 2 > 12 = 6m(S^3/Q_8)\).

Again, if \(X\) is a pseudotriangulation of \(S^1 \times S^1 \times S^1\) then, by Corollary 5 and Lemma 3, \(h_2(X) \geq \psi(S^1 \times S^1 \times S^1) - 2 = 24 - 2 > 6 \times 3 = 6m(S^1 \times S^1 \times S^1)\).
For \(p, q\) relatively prime and \(p \geq 3\), let \(X\) be a pseudotriangulation of \(L(p, q)\). Then, by Theorem 6 (ii) and Corollary 5, \(h_2(X) \geq \psi(L(p, q)) - 2 > 8 - 2 = 6 \times 1 = 6m(L(p, q))\) for \(p \geq 3\). This completes the proof.

**Proof of Theorem 8.** Let \(K_{p,q}\) be as in Remark 24. Then \(K_{3,1}\) is a 12-vertex crystallization of \(L(3, 2)\). Part (a) now follows by the constructions in Subsection 5.1.

Again, \(K_{q+1,q}\) is a \(4(q + 1)\)-vertex crystallization of \(L(q + 1, q)\) for \(1 \leq q \leq 3\). Part (b) now follows by the constructions in Subsection 5.2.

**Acknowledgement**

This work is supported in part by UGC Centre for Advanced Studies. The first author thanks CSIR, India for SPM Fellowship. The authors thank M. R. Casali and P. Cristofori for pointing out an error in an earlier version of this paper. The authors also thank the anonymous referee for many useful comments.

**References**


