Abstract

For every $h \in \mathbb{N}$, a graph $G$ with the vertex set $V(G)$ and the edge set $E(G)$ is said to be $h$-magic if there exists a labeling $l : E(G) \to \mathbb{Z}_h \setminus \{0\}$ such that the induced vertex labeling $s : V(G) \to \mathbb{Z}_h$, defined by $s(v) = \sum_{uv \in E(G)} l(uv)$ is a constant map. When this constant is zero, we say that $G$ admits a zero-sum $h$-magic labeling. The null set of a graph $G$, denoted by $N(G)$, is the set of all natural numbers $h \in \mathbb{N}$ such that $G$ admits a zero-sum $h$-magic labeling. In 2012, the null sets of 3-regular graphs were determined. In this paper we show that if $G$ is an $r$-regular graph, then for even $r$ ($r > 2$), $N(G) = \mathbb{N}$ and for odd $r$ ($r \neq 5$), $\mathbb{N} \setminus \{2,4\} \subseteq N(G)$. Moreover, we prove that if $r$ is odd and $G$ is a 2-edge connected $r$-regular graph ($r \not= 5$), then $N(G) = \mathbb{N} \setminus \{2\}$. Also, we show that if $G$ is a 2-edge connected bipartite graph, then $\mathbb{N} \setminus \{2,3,4,5\} \subseteq N(G)$.

1 Introduction

Let $G$ be a finite and undirected graph with vertex set $V(G)$ and edge set $E(G)$. A graph in which multiple edges are admissible is called a multigraph. An $r$-regular graph is a graph each of whose vertex has degree $r$. The degree of a vertex $u$ in $G$ is denoted by $d_G(u)$. A cut-edge of $G$ is an edge in $E(G)$ such that its deletion results in a graph with one more connected component than $G$ has. A graph $G$ is $n$-edge connected if the minimum number of edges whose removal would disconnect $G$ is at least $n$. We denote the complete graph and the cycle of order $n$ by $K_n$ and $C_n$, respectively. A wheel is a
A subgraph $F$ of $G$ is a factor of $G$ if $F$ is a spanning subgraph of $G$. If a factor $F$ is $k$-regular for some integer $k \geq 0$, then $F$ is a $k$-factor. Thus a 2-factor is a disjoint union of cycles that cover all vertices of $G$. A $k$-factorization of $G$ is a partition of the edges of $G$ into disjoint $k$-factors. For integers $a$ and $b$ with $1 \leq a \leq b$, an $[a,b]$-multigraph is defined to be a multigraph $G$ such that for every $v \in V(G)$, $a \leq d_G(v) \leq b$. For a set $\{a_1, \ldots, a_r\}$ of non-negative integers an $\{a_1, \ldots, a_r\}$-multigraph is a multigraph each of whose vertices has degree from the set $\{a_1, \ldots, a_r\}$. Analogously, an $[a,b]$-factor and an $\{a_1, \ldots, a_r\}$-factor can be defined.

Let $G$ be a graph. A zero-sum flow for $G$ is an assignment of non-zero real numbers to the edges of $G$ such that the sum of values of all edges incident with each vertex is zero. Let $k$ be a natural number. A zero-sum $k$-flow is a zero-sum flow with values from the set $\{\pm 1, \ldots, \pm (k-1)\}$.

For an abelian group $A$, written additively, any mapping $l : E(G) \to A$ is called a labeling of a graph $G$. Given a labeling on the edge set of $G$, one can introduce a vertex labeling $s : V(G) \to A$, defined by $s(v) = \sum_{uv \in E(G)} l(uv)$, for $v \in V(G)$. A graph $G$ is said to be $A$-magic if there is a labeling $l : E(G) \to A \setminus \{0\}$ such that for each vertex $v$, the sum of the labels of edges incident with $v$ is all equal to the same constant, that is there exists constant $c$ such that for all vertices $v$, $s(v) = c \in A$. We call this labeling an $A$-magic labeling of $G$. In general, an $A$-magic graph may admit more than one $A$-magic labeling. For every positive integer $h \geq 2$, a graph $G$ is called an $h$-magic graph if there is a $\mathbb{Z}_h$-magic labeling of $G$. A graph $G$ is said to be zero-sum $h$-magic if there is an edge labeling from $E(G)$ into $\mathbb{Z}_h \setminus \{0\}$ such that the sum of values of all edges incident with each vertex is zero. If $s(v) = 0$ for a fixed vertex $v \in V(G)$, then we say that zero-sum $h$-magic rule holds in $v$. The null set of a graph $G$, denoted by $N(G)$, is the set of all natural numbers $h \in \mathbb{N}$ such that $G$ admits a zero-sum $h$-magic labeling.

Recently, Choi, Georges and Mauro [6] proved that if $G$ is 3-regular graph, then $N(G)$ is $\mathbb{N} \setminus \{2\}$ or $\mathbb{N} \setminus \{2,4\}$. In this article, we extend this result by showing that if $G$ is an $r$-regular graph, then for even $r$ ($r > 2$), $N(G) = \mathbb{N}$ and for odd $r$ ($r \neq 5$), $\mathbb{N} \setminus \{2,4\} \subseteq N(G)$. Moreover, we prove that if $r$ ($r \neq 5$) is odd and $G$ is a 2-edge connected $r$-regular graph, then $N(G) = \mathbb{N} \setminus \{2\}$.

The original concept of $A$-magic graph is due to Sedlacke [14], who defined it to be a graph with a real-valued edge labeling such that have distinct non-negative labels, and, in the manner described above, the sum of the labels of the edges incident to vertex $v$ is constant over $V(G)$. Stanley considered $Z$-magic graphs and showed that the theory of magic labeling can be put into the more general context of linear homogeneous diophantine equations, [16, 17]. Recently, there have been considerable research articles in graph labeling. Interested readers are referred to [7, 11, 12, 13, 18].

In [11], the null set of some classes of regular graphs are determined.

**Theorem 1.** If $n \geq 4$, then $N(K_n) = \begin{cases} \mathbb{N}, & \text{if } n \text{ is odd;} \\ \mathbb{N} \setminus \{2\}, & \text{if } n \text{ is even.} \end{cases}$
Theorem 2. \( N(C_n) = \begin{cases} \mathbb{N}, & \text{if } n \text{ is even;} \\ 2\mathbb{N}, & \text{if } n \text{ is odd.} \end{cases} \)

Recently, the following theorem was proved, in [2] and [3].

Theorem 3. Let \( r \geq 3 \) be a positive integer. Then every \( r \)-regular graph admits a zero-sum 5-flow.

This theorem implies that if \( G \) is an \( r \)-regular graph \( (r \geq 3) \), then \( \mathbb{N} \setminus \{2, 3, 4\} \subseteq N(G) \). Before establishing our results we need some theorems.

Theorem 4.[9] Let \( r \geq 3 \) be an odd integer and let \( k \) be an integer such that \( 1 \leq k \leq \frac{2r}{3} \). Then every \( r \)-regular graph has a \([k-1,k]\)-factor each component of which is regular.

Also, the following theorems were proved.

Theorem 5.[4, p.179] Let \( r \geq 3 \) be an odd integer, and \( G \) be a 2-edge connected \([r-1,r]\)-multigraph having exactly one vertex \( w \) of degree \( r-1 \). Then for every even integer \( k \), \( 2 \leq k \leq \frac{2r}{3} \), \( G \) has a \( k \)-factor.

Theorem 6.[5] Every 2-edge connected \((2r+1)\)-regular multigraph contains a 2-factor.

Theorem 7.[10] Every \( 2r \)-regular multigraph admits a 2-factorization.

2 Regular Graphs

Let \( G \) be an \( r \)-regular graph. In this section we prove that for every even natural number \( r \) \((r > 2)\), \( N(G) = \mathbb{N} \) and for every odd natural number \( r \) \((r \neq 5)\), \( \mathbb{N} \setminus \{2, 4\} \subseteq N(G) \).

We start this section with the following theorem.

Theorem 8. Let \( r \) be an odd integer and \( r \geq 3 \). Then every \( r \)-regular multigraph with at most one cut-edge admits a zero-sum 4-magic labeling.

Proof. Obviously, we may suppose that \( G \) is connected. First assume that \( G \) is a 2-edge connected \( r \)-regular multigraph. By Theorem 6, \( G \) has a 2-factor, say \( H \). Now, assign 1 and 2 to the edges of \( H \) and the edges of \( G \setminus E(H) \), respectively. It is not hard to see that \( G \) admits a zero-sum 4-magic labeling.

Now, suppose that \( G \) has a cut-edge, say \( e \). Let \( G' = G \setminus \{e\} \). Clearly, \( G' \) has two components, say \( G_1 \) and \( G_2 \). Since both \( G_1 \) and \( G_2 \) are 2-edge connected \([r-1,r]\)-multigraphs, by Theorem 5, \( G_1 \) and \( G_2 \) have 2-factors and so \( G \) has a 2-factor. Hence by the same argument as we did before, \( G \) has a zero-sum 4-magic labeling. \( \square \)

Remark 9. If \( G \) is a \( 2r \)-regular multigraph, then by assigning 2 to all edges of \( G \), one can obtain a zero-sum 4-magic labeling.
The following remark shows that there are some regular graphs with no zero-sum 4-magic labeling.

**Remark 10.** Let \( r \) be an odd integer (\( r \geq 3 \)) and \( G \) be an \( r \)-regular multigraph. If there is a vertex \( u \) such that every edge adjacent to \( u \) is a cut-edge, then \( G \) does not admit a zero-sum 4-magic labeling.

**Proof.** For contradiction assume that \( G \) admits a zero-sum 4-magic labeling, say \( l \). Since \( G \) admits a zero-sum 4-magic labeling it is not hard to see that there exists at least one edge adjacent to \( u \), say \( uv \), with label 1 or 3. Assume that \( G' \) is the connected component of \( G \setminus \{ u \} \) containing \( v \). Clearly, we have \( \sum_{x \in V(G')} s(x) = 2 \sum_{e \in E(G')} l(e) + l(uv) \). But \( \sum_{x \in V(G')} s(x) = 0 \pmod{2} \). On the other hand, \( 2 \sum_{e \in E(G')} l(e) + l(uv) = 1 \pmod{2} \), a contradiction.

**Lemma 11.** Let \( G \) be a \( \{1, 7\} \)-multigraph with no component that is isomorphic to \( K_2 \). Suppose that the subgraph induced by the set of vertices of degree 7 has no cut-edge. Fix \( a \in \{1, 2\} \). Then if \( h \) is a fixed pendant edge of \( G \), then there exists a function \( l \) from \( E(G) \) into \( \{1, 2\} \) such that \( l(h) = a \) and for every vertex \( v \) of degree 7 in \( V(G) \), the zero-sum 3-magic rule holds in \( v \) under \( l \).

**Proof.** First assume that \( a = 1 \) and \( G \) is a multigraph with exactly one pendant edge \( h = uv \). Assume that \( d_G(v) = 1 \). Let \( G' = G \setminus \{v\} \). Note that \( G' \) is a 2-edge connected \( [6, 7] \)-multigraph in which \( u \) is the only vertex of degree 6. By Theorem 5, \( G' \) has a 2-factor \( H \). Define \( l(e) = 2 \), for every \( e \in E(H) \) and define \( l(e) = 1 \), for every \( e \in E(G') \setminus E(H) \). Hence we obtain the desired labeling for \( G \).

Now, for \( a = 2 \), we define \( l^* \) to be the labeling defined as above, and let \( l = 2l^* \pmod{3} \).

Next, suppose that the number of pendant edges of \( G \) is at least two and \( a = 1 \). Consider two copies of \( G \), say \( G_1 \) and \( G_2 \). Assume that \( u_i, v_i, 1 \leq i \leq k \) (\( k \geq 2 \)) are all edges of \( G_1 \), such that \( u_i, v_i \in V(G_1) \) and \( d_{G_1}(v_i) = 1 \). Also, suppose that \( u_i' \) and \( v_i' \) are the vertices corresponding to \( u_i \) and \( v_i \) (\( i = 1, \ldots, k \)) in \( G_2 \). Let \( G^* \) be the multigraph obtained by removing the vertices \( v_1, \ldots, v_k \) and \( u_1', \ldots, u_k' \) and joining \( u_i \) and \( u_i' \) in \( G_1 \cup G_2 \), for \( i = 1, \ldots, k \). Since none of the connected components of \( G \) is \( K_2 \), \( G^* \) is a 2-edge connected \( 7 \)-regular multigraph. Thus by Theorem 6, \( G^* \) has a 2-factor, say \( H \). If the edge in \( G^* \) corresponding to \( h \) belongs to \( E(H) \), then let \( l(e) = 2 \) for every \( e \in E(G^*) \setminus E(H) \) and \( l(e) = 1 \) for every \( e \in E(H) \). Otherwise, define \( l(e) = 1 \) for every \( e \in E(G^*) \setminus E(H) \) and \( l(e) = 2 \) for every \( e \in E(H) \). Hence we obtain the desired labeling.

Now, for \( a = 2 \), we define \( l^* \) to be the labeling defined as above, and let \( l = 2l^* \pmod{3} \), we obtain the desired labeling and the proof is complete.

In the following theorem, we prove that for every \( r \)-regular graph \( G \) (\( r \geq 3 \), \( r \neq 5 \)), \( 3 \in N(G) \).

**Theorem 12.** Let \( r \) be an integer such that \( r \geq 3 \) and \( r \neq 5 \). Then every \( r \)-regular graph admits a zero-sum 3-magic labeling.
Proof. First assume that $r$ is an even positive integer and $r \neq 2$. The proof is by induction on $r$. If $r = 4$, then by Theorem 7, $G$ is decomposed into 2-factors $G_1$ and $G_2$. Now, assign 1 and 2 to all edges of $G_1$ and $G_2$, respectively. Thus $G$ admits a zero-sum 3-magic labeling. If $r = 6$, then assign 1 to the edges of $G$ to obtain a zero-sum 3-magic labeling. Now, suppose that $r \geq 8$. So, by Theorem 7, $G$ is decomposed into 2-factors. Choose two 2-factors $G_1$ and $G_2$. Now, by induction hypothesis $G \setminus (E(G_1) \cup E(G_2))$ admits a zero-sum 3-magic labeling. On the other hand, by the case $r = 4$, $G_1 \cup G_2$ admits a zero-sum 3-magic labeling and the proof is complete.

Now, assume that $r$ is an odd positive integer. If $r$ is divisible by 3, then assign 1 to all edges of $G$ to obtain a zero-sum 3-magic labeling.

If $r$ is not divisible by 3, then $r \equiv 1, 5, 7, 11 \pmod{12}$.

First, suppose that $r = 7$. For finding a zero-sum 3-magic labeling we construct a rooted tree $T$ from $G$, where every maximal 2-edge connected subgraph of $G$ is considered as a vertex of $T$ and every edge of $T$ is corresponding to a cut-edge of $G$. Now, by traversing $T$, level by level, we find a zero-sum 3-magic labeling for $G$. We start from the root of $T$ say $H$ (The root can be taken to be any vertex). Let $h$ be an arbitrary cut-edge incident with $H$. Assign the label 1 to $h$. By Lemma 11, one can assign 1 or 2 to each edge of $H$ and cut-edges of $G$ which are incident with $H$ such that every cut-edge of $G$ incident with $H$ has value 1 or 2 and moreover the zero-sum 3-magic rule holds in every vertex of $H$. Now, we move to the next vertex level of $T$. Let $H'$ be a vertex adjacent to $H$ in $T$. At this stage there exists just one cut-edge of $G$ incident with $H'$ which has been labeled by 1 or 2. Now, by Lemma 11, we can label each edge of $H'$ and each cut-edge of $G$ that is incident to $H'$ (except $h$ which is already labeled 1 or 2) with 1 or 2 such that the zero-sum 3-magic rule holds in every vertex of $H'$. By continuing this procedure we obtain a zero-sum 3-magic labeling for $G$, as desired.

Now, assume that $r = 11$. Then by Theorem 4, $G$ has a $[6, 7]$-factor, say $H$ whose components are regular. Let $H_1$ and $H_2$ be the union of 6-regular components and 7-regular components of $H$, respectively. Also, by Theorem 7, $H_1$ is decomposed into 2-factors $G_1$, $G_2$ and $G_3$. Now, assign 1 to all edges of $G_1$ and assign 1 to the edges of $G \setminus (E(H_1) \cup E(H_2))$ and $G_3$. Then it is not hard to see that $G$ admits a zero-sum 3-magic labeling.

Now, suppose that $r = 12k + 1$ or $r = 12k + 7$, and $k \geq 1$. By Theorem 4, $G$ has a $[6k-2, 6k-1]$-factor, say $H$, whose components are regular. Let $H_1$ and $H_2$ be the union of $(6k-2)$-regular components and $(6k-1)$-regular components of $H$, respectively. Since $6k-2$ is even, $H_1$ admits a zero-sum 3-magic labeling. Now, assign 2 to the edges of $H_2$ and assign 1 to all edges of $G \setminus (E(H_1) \cup E(H_2))$. Then $G$ admits a zero-sum 3-magic labeling.

Now, assume that $r = 12k + 5$ or $r = 12k + 11$, and $k \geq 1$. By Theorem 4, $G$ has a $[6k+1, 6k+2]$-factor, say $H$, whose components are regular. Let $H_1$ and $H_2$ be the union of $(6k+1)$-regular components and $(6k+2)$-regular components of $H$, respectively. Since $6k+2$ is even, $H_2$ admits a zero-sum 3-magic labeling. Now, assign 2 to all edges of $H_1$ and assign 1 to all edges of $G \setminus (E(H_1) \cup E(H_2))$. Therefore, $G$ admits a zero-sum 3-magic labeling, as desired. $\Box$
Now, we are in a position to prove our main theorem for regular graphs.

**Theorem 13.** Let $G$ be an $r$-regular graph ($r \geq 3$, $r \neq 5$). If $r$ is even, then $N(G) = \mathbb{N}$, otherwise $\mathbb{N} \setminus \{2, 4\} \subseteq N(G)$.

**Proof.** First, assume that $r$ is even. Clearly, by assigning 1 to all edges of $G$, it is seen that $2 \in N(G)$. Moreover, Theorem 3 immediately follows, $k \in N(G)$ for $k \geq 5$ and $k = 1$. By Theorem 12 and Remark 9, $N(G)$ contains 3 and 4 as well, giving the result. Next, assume that $r$ is an odd integer. Then by Theorems 3 and 12 we are done. 

**Lemma 14.** If $r$ ($r \neq 5$) is odd and $G$ is a 2-edge connected $r$-regular graph, then $N(G) = \mathbb{N} \setminus \{2\}$.

**Proof.** Since the degree of each vertex is odd, $2 \notin N(G)$. Now, the result follows from Theorems 3, 8 and 12. 

We close this section with the following conjecture.

**Conjecture 15.** Every 5-regular graph admits a zero-sum 3-magic labeling.

It is easily seen that a 5-regular graph $G$ admitting a zero-sum 3-magic labeling is equivalent to $G$ having a factor with the degree sequence 1 or 4.

### 3 Bipartite Graphs

In this section we show that $\mathbb{N} \setminus \{2, 3, 4, 5\} \subseteq N(G)$ if $G$ is a 2-edge connected bipartite graph. Before establishing this result we need some definitions and theorems.

Let $G$ be a directed graph. A $k$-flow on $G$ is an assignment of integers with maximum absolute value at most $k-1$ to each edge of $G$ such that for each vertex of $G$, the sum of the labels on incoming edges is equal to that of the labels on outgoing edges. A nowhere-zero $k$-flow is a $k$-flow with no zeros.

A $\mathbb{Z}_k$-flow on $G$ is an assignment of element of $\mathbb{Z}_k$ to each edge of $G$ such that for any vertex of $G$, the sum of the labels on incoming edges is equal to that of the labels on outgoing edges (mod $k$). A nowhere-zero $\mathbb{Z}_k$-flow is a $\mathbb{Z}_k$-flow with no zero, for every $k \in \mathbb{N}$.

The following theorem was proved in [15].

**Theorem 16.** Every 2-edge connected directed graph admits a nowhere-zero 6-flow.

The following well-known theorem is due to Tutte.

**Theorem 17.**[8, p.294] If $G$ is a directed graph and $k \geq 1$ is an integer, then $G$ admits a nowhere-zero $k$-flow if and only if $G$ admits a nowhere-zero $\mathbb{Z}_k$-flow.

In [11], the null set of a complete bipartite graph was determined.
Theorem 18. If \( m, n \geq 2 \), then \( N(K_{m,n}) = \begin{cases} \mathbb{N}, & \text{if } m+n \text{ is even;} \\ \mathbb{N} \setminus \{2\}, & \text{if } m+n \text{ is odd.} \end{cases} \)

In the following theorem we determine a necessary condition for the existence of a zero-sum \( h \)-magic labeling in bipartite graphs.

Theorem 19. Let \( G \) be bipartite in which \( G \) admits a zero-sum \( h \)-magic labeling, for some \( h \in \mathbb{N} \). Then \( G \) is 2-edge connected.

Proof. Assume that \( G \) admits a zero-sum \( h \)-magic labeling, say \( l \). To the contrary, let \( e = uv \) be a cut-edge of \( G \). Note that \( G \setminus \{e\} \) is bipartite graph. Let \( H \) be one of the connected components of \( G \setminus \{e\} \) with two parts \( X \) and \( Y \) such that \( Y \cap \{u, v\} \neq \emptyset \). It is not hard to see that in \( G \), \( \sum_{x \in X} s(x) = \sum_{y \in Y} s(y) - l(uv) \). On the other hand, by assumption
\[
\sum_{x \in X} s(x) = \sum_{y \in Y} s(y) \equiv 0 \pmod{h}.
\]
This implies that \( l(uv) \equiv 0 \pmod{h} \), which is a contradiction. \( \square \)

Next, we determine the null set of a 2-edge connected bipartite graph.

Theorem 20. Let \( G \) be a 2-edge connected bipartite graph. Then \( G \) admits a zero-sum \( k \)-magic labeling, for \( k \in \mathbb{N} \setminus \{2, 3, 4, 5\} \).

Proof. First, orient all edges from one part of \( G \) to the other part and call the resultant directed graph by \( G' \). By Theorem 16, \( G' \) admits a nowhere-zero 6-flow. Thus \( G' \) admits a nowhere-zero \( k \)-flow, for every \( k \in \mathbb{N} \setminus \{2, 3, 4, 5\} \) and so by Theorem 17, \( G' \) admits a nowhere-zero \( \mathbb{Z}_k \)-flow, for \( k \in \mathbb{N} \setminus \{2, 3, 4, 5\} \). Now, by removing the direction of all edges we conclude that \( G \) admits a zero-sum \( k \)-magic labeling, for every \( k \in \mathbb{N} \setminus \{2, 3, 4, 5\} \) and the proof is complete. \( \square \)

In the following remark, we show that there are some 2-edge connected bipartite graphs with no zero-sum \( k \)-magic labeling, for \( k = 2, 3, 4 \).

Remark 21. In a bipartite graph the existence of a zero-sum \( k \)-flow is equivalent to the existence of a zero-sum \( k \)-magic labeling. To see this first orient all edges from one part to the other part and call the directed graph by \( G' \). Therefore, \( G' \) admits a nowhere-zero \( k \)-flow. Now, by removing the direction of all edges we conclude that \( G \) admits a zero-sum \( k \)-flow. So, \( G \) admits a zero-sum \( k \)-flow if and only if \( G' \) admits a nowhere-zero \( k \)-flow. Thus by Theorem 17, \( G' \) admits a nowhere-zero \( \mathbb{Z}_k \)-flow. But the later condition implies that \( G \) admits a zero-sum \( k \)-magic labeling.
Let $G$ be the following graph. By a computer search one can see that $G$ does not admit a zero-sum 4-flow, see [1]. So $G$ does not admit a zero-sum 4-magic labeling.

Since $G$ does not admit a zero-sum 4-flow, $G$ does not admit a zero-sum $k$-flow, for $k \leq 4$. Hence $G$ does not admit a zero-sum $k$-magic labeling, for $k = 2, 3, 4$.

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