A Slight Improvement to the Colored Bárány’s Theorem

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Abstract

Suppose $d+1$ absolutely continuous probability measures $m_0, \ldots, m_d$ on $\mathbb{R}^d$ are given. In this paper, we prove that there exists a point of $\mathbb{R}^d$ that belongs to the convex hull of $d+1$ points $v_0, \ldots, v_d$ with probability at least $\frac{2^d}{(d+1)!^2(d+1)}$, where each point $v_i$ is sampled independently according to probability measure $m_i$.

1 Introduction

Let $P \subset \mathbb{R}^d$ be a set of $n$ points. Every $d+1$ of them span a simplex, for a total of $\binom{n}{d+1}$ simplices. The point selection problem asks for a point contained in as many simplices as possible. Boros and Füredi [BF84] showed for $d = 2$ that there always exists a point in $\mathbb{R}^2$ contained in at least $\frac{2}{9}\binom{n}{3} - O(n^2)$ simplices. A short and clever proof of this result was given by Bukh [Buk06]. Bárány [Bár82] generalized this result to higher dimensions:

**Theorem 1** (Bárány [Bár82]). There exists a point in $\mathbb{R}^d$ that is contained in at least $c_d\binom{n}{d+1} - O(n^2)$ simplices, where $c_d > 0$ is a constant depending only on the dimension $d$.

This general result, the Bárány’s theorem, is also known as the first selection lemma. We will henceforth denote by $c_d$ the largest possible constant for which the Bárány’s theorem holds true. Bukh, Matoušek and Nivasch [BMN10] used a specific construction called the stretched grid to prove that the constant $c_2 = \frac{2}{9}$ in the planar case found by Boros and Füredi [BF84] is the best possible. In fact, they proved that $c_d \leq \frac{d}{(d+1)^2}$. On the other hand, Bárány’s proof in [Bár82] implies that $c_d \geq (d+1)^{-d}$, and Wagner [Wag03] improved it to $c_d \geq \frac{d^2+1}{(d+1)^{d+1}}$.

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Figure 1: 3 red points, 3 green points and 3 blue points are placed in the plane. The point marked by a square is contained in 6 (= $\frac{2}{3} \cdot 3^3$) colorful triangles.

Gromov [Gro10] further improved the lower bound on $c_d$ by topological means. His method gives $c_d \geq \frac{2^d}{(d+1)(d+1)!}$. Matoušek and Wagner [MW11] provided an exposition of the combinatorial component of Gromov’s approach in a combinatorial language, while Karasev [Kar12] found a very elegant proof of Gromov’s bound, which he described as a “decoded and refined” version of Gromov’s proof.

The exact value of $c_d$ has been the subject of ongoing research and is unknown, except for the planar case. Basit, Mustafa, Ray and Raza [BMRR10] and successively Matoušek and Wagner [MW11] improved the Bárány’s theorem in $\mathbb{R}^3$. Král’, Mach and Sereni [KMS12] used flag algebras from extremal combinatorics and managed to further improve the lower bound on $c_3$ to more than 0.07480, whereas the best upper bound known is 0.09375.

However, in this paper, we are concerned with a colored variant of the point selection problem. Let $P_0, \ldots, P_d$ be $d + 1$ disjoint finite sets in $\mathbb{R}^d$. A colorful simplex is the convex hull of $d + 1$ points each of which comes from a distinct $P_i$. For the colored point selection problem, we are concerned with the point(s) contained in many colorful simplices. Karasev proved:

**Theorem 2** (Karasev [Kar12]). Given a family of $d + 1$ absolutely continuous probability measures $m = (m_0, \ldots, m_d)$ on $\mathbb{R}^d$, an $m$-simplex$^1$ is the convex hull of $d + 1$ points $v_0, \ldots, v_d$ with each point $v_i$ sampled independently according to probability measure $m_i$. There exists a point of $\mathbb{R}^d$ that is contained in an $m$-simplex with probability $p_d \geq \frac{1}{(d+1)!}$. In addition, if two probability measures coincide, then the probability can be improved to $p_d \geq \frac{2^d}{(d+1)(d+1)!}$.

By a standard argument which we will provide immediately, a result on the colored point selection problem follows:

**Corollary 3.** If $P_0, \ldots, P_d$ each contains $n$ points, then there exists a point that is contained in at least $\frac{1}{(d+1)!} \cdot n^{d+1}$ colorful simplices.

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An $m$-simplex is actually a simplex-valued random variable.
Our result drops the additional assumption in theorem 2, hence improves corollary 3:

**Main Theorem.** There is a point in $\mathbb{R}^d$ that belongs to an $m$-simplex with probability $p_d \geq \frac{2d}{(d+1)(d+1)!}$.

**Corollary 4.** There exists a point that is contained in at least $\frac{2d}{(d+1)(d+1)!} \cdot n^{d+1}$ colorful simplices.

**Proof of corollary 4 from the main theorem.** Given $d+1$ sets $P_0, \ldots, P_d$ in $\mathbb{R}^d$ each of which contains $n$ points. Let $\Psi : \mathbb{R}^d \to \mathbb{R}$ be the bump function defined by $\Psi(x_1, \ldots, x_d) = \prod_{i=1}^d \psi(x_i)$, where $\psi(x) = e^{-1/(1-x^2)}1_{|x| < 1}$, and set $\Psi_n(x_1, \ldots, x_d) = n^d\Psi(nx_1, \ldots, nx_d)$ for $n \in \mathbb{N}$. It is a standard fact that $\Psi$ and $\Psi_n$ are absolutely continuous probability measures supported on $[-1, 1]^d$ and $[-1/n, 1/n]^d$ respectively.

For each $n \in \mathbb{N}$ and $0 \leq k \leq d$, define $m_k^{(n)}(x) := \frac{1}{n} \sum_{p \in P_k} \Psi_n(x - p)$ for $x \in \mathbb{R}^d$. Note that $m_k^{(n)}$ is an absolutely continuous probability measure supported on the Minkowski sum of $P_k$ and $[-1/n, 1/n]^d$. Let $m^{(n)}$ be the family of $d+1$ probability measures $m_0^{(n)}, \ldots, m_d^{(n)}$. By the main theorem, there is a point $p^{(n)}$ of $\mathbb{R}^d$ that belongs to an $m^{(n)}$-simplex with probability at least $\frac{2d}{(d+1)(d+1)!}$.

Because no point in a certain neighborhood of infinity is contained in any $m^{(n)}$-simplex, the set $\{p^{(n)} : n \in \mathbb{N}\}$ is bounded, and consequently the set has a limit point $p$. Suppose $p$ is contained in $N$ colorful simplices. Let $\epsilon > 0$ be the distance from $p$ to all the colorful simplices that do not contain $p$. Choose $n$ large enough such that $1/n \ll \epsilon$ and $|p^{(n)} - p| \ll \epsilon$. By the choice of $n$, if $p$ is not contained in a colorful simplex spanned by $v_0, \ldots, v_d$, then $p^{(n)}$ is not contained the convex hull of $v_0', \ldots, v_d'$ for all $v_i' \in v_i + [-1/n, 1/n]^d$. This implies that the probability that $p^{(n)}$ is contained in an $m^{(n)}$-simplex is at most $\frac{N}{n^{d+1}}$. Hence $p$ is the desired point contained in $N \geq \frac{2d}{(d+1)(d+1)!} \cdot n^{d+1}$ colorful simplices. \hfill \Box

Readers who are familiar with Karasev’s work [Kar12] would notice that our proof of the main theorem heavily relies on his arguments. The author is deeply in debt to him.

# 2 Proof of the Main Theorem

In this section, we provide the proof of the main theorem. The topological terms in the proof are standard, and can be found in [Mat03]. In addition to the notion of an $m$-simplex, in the proof, we will often refer to an $(m_k, \ldots, m_d)$-face which means the convex hull of $d-k+1$ points $v_k, \ldots, v_d$ with each point $v_i$ sampled independently according to probability measure $m_i$. An $m$-simplex and an $(m_k, \ldots, m_d)$-face are both set-valued random variables.

**Proof of the main theorem.** To obtain a contradiction, we suppose that for any point $v$ in $\mathbb{R}^d$, the probability that $v$ belongs to an $m$-simplex is less than $p_d := \frac{2d}{(d+1)(d+1)!}$. Since this probability, as a function of point $v$, is continuous and uniformly tends to 0 as $v$ goes to infinity, there is an $\epsilon > 0$ such that $v$ is contained in an $m$-simples with probability at most $p_d - \epsilon$ for all $v$ in $\mathbb{R}^d$. 
Let $S^d := \mathbb{R}^d \cup \{\infty\}$ be the one-point compactification of the Euclidean space $\mathbb{R}^d$. Take $\delta = \epsilon/d$. Choose a finite triangulation of $S^d$ with one of the $d$-simplices containing $\infty$ such that for $0 < k \leq d$, any $k$-face of $T$ intersects an $(m_k, \ldots, m_d)$-face with probability less than $\delta$ and that the measure of any $d$-face of $T$ under $(m_{d-1} + m_d)/2$ is less than $\delta$. This can be done by taking a sufficiently fine triangulation of $S^2$ with one $d$-simplex having $\infty$ in its relative interior.

We use cone($\cdot$) as the cone functor with apex $O$. A triangulation $T$ of $S^d$ naturally extends to a triangulation cone($T$) of cone($S^d$). We denote the $k$-skeleton of $T$ and cone($T$) by $T_{\leq k}$ and cone($T$)$_{\leq k}$ respectively.

We are going to define a continuous map $f : \text{cone}(T)_{\leq d} \to S^d$. Put $f(x) = x$ for all $x \in S^d = ||T|| \subset ||\text{cone}(T)_{\leq d}||$, and set $f(O) = \infty$. We proceed to define $f$ on cone($\sigma$) for all the $k$-faces $\sigma$ of $T$ inductively on dimension $k$ of $\sigma$ while we maintain the property that the image of the boundary of cone($\sigma$) under $f$, that is $f(\partial\text{cone}(\sigma))$, intersects an $(m_k, \ldots, m_d)$-face with probability at most $(k + 1)!(p_d - \epsilon + k\delta)$. We say $f$ is economical over a $k$-face $\sigma$ of $T_{\leq d-1}$ if $f$ and $\sigma$ satisfy the above property. Unlike Karasev [Kar12], our inductive construction of $f$ follows the same pattern until $k = d - 2$ instead of $d - 1$. The main innovation of this proof is a different construction for $k = d - 1$, which enables us to remove the additional assumption in theorem 2.

Note that for any 0-face $\sigma$ in $T$, $f(\partial\text{cone}(\sigma)) = f(\{\sigma, O\}) = \{\sigma, \infty\}$. According to the assumption at the beginning of the proof, $f(\partial\text{cone}(\sigma))$ intersects an $(m_0, \ldots, m_d)$-face,
that is, an $m$-simplex, with probability at most $p_d - \epsilon$. Therefore $f$ is economical over 0-faces of $T$. This finishes the first step.

Suppose $f$ is already defined on cone$(T)^\leq k$ and it is economical over $k$-faces of $T$. We are going to extend the domain of $f$ to cone$(T)^\leq k+1$. Indeed, we only need to define $f$ on cone$(\sigma)$ for every $k$-face $\sigma$ of $T$.

Take any $k$-face $\sigma$ of $T$. Suppose convex hull of $v_k, \ldots, v_d$, denoted by conv$(v_k, \ldots, v_d)$, is an $(m_k, \ldots, m_d)$-face. Notice that the following statements are equivalent:

- $f(\partial$ cone$(\sigma))$ intersects conv$(v_k, \ldots, v_d)$;
- for some $v \in f(\partial$ cone$(\sigma))$, the ray with initial point $v$ in the direction $\overrightarrow{v_k}v$ intersects conv$(v_{k+1}, \ldots, v_d)$.

We call the union of such rays the shadow of $f(\partial$ cone$(\sigma))$ centered at $v_k$. Since $f$ is economical over $\sigma$, the probability for an $(m_k, \ldots, m_d)$-face to meet $f(\partial$ cone$(\sigma))$ is at most $(k+1)! (p_d - \epsilon + k\delta)$, and so there exists $v_\sigma^k \in \mathbb{R}^d$ such that the shadow of $f(\partial$ cone$(\sigma))$ centered at $v_\sigma^k$ intersects conv$(v_{k+1}, \ldots, v_d)$ with probability at most $(k+1)! (p_d - \epsilon + k\delta)$.

Now, we define $f$ on cone$(\sigma)$. First, let $g$ be the homeomorphism from cone$(\sigma)$ onto the cone over $\partial$ cone$(\sigma)$ with apex $c$ such that $g$ is an identity on $\partial$ cone$(\sigma)$. This can be done because cone$(\sigma)$ is homeomorphic to a $(k+1)$-simplex $\Delta$ and it is easy to find a homeomorphism from $\Delta$ to cone$(\partial \Delta)$ that keeps $\partial \Delta$ fixed.

![Figure 3](image1.png)  
Figure 3: An illustration of an 1-simplex $\Delta$, $\partial \Delta$, cone$(\partial \Delta)$ and a homeomorphism from $\Delta$ to cone$(\partial \Delta)$.

![Figure 4](image2.png)  
Figure 4: The illustration shows a cone over part of $\partial$ cone$(\sigma)$ with apex $c$ and a point $v$ on the boundary, and how a point $w$ on the line segment $[v, c]$ are mapped under $h$.

Next, note that every point $w$ in cone$(\sigma)$ except $c$ is on a line segment $[v, c]$ for a unique point $v$ on $\partial$ cone$(\sigma)$. If $t = \overrightarrow{vw}/\overrightarrow{vc} \in [0, \infty)$, then put $h(w) = f(v) + t \cdot v_\sigma^k f(v)$. In
addition, set \( h(c) = \infty \). The function \( h \) maps \([v, c)\) onto \([f(v), v_k^\tau]\) linearly and then takes the inversion centered at \( v_k^\tau \) with radius \( v_k^\tau f(v) \) so that \([f(v), v_k^\tau]\) gets mapped onto the ray with the initial point \( f(v) \) in the direction \( v_k^\tau f(v) \). Evidently, \( h \) is a continuous map from \( \partial \text{cone}(\sigma) \) onto the shadow of \( f(\partial \text{cone}) \) centered at \( v_k^\tau \) that coincides with \( f \) on \( \partial \text{cone}(\sigma) \).

Define \( f \) on \( \text{cone}(\sigma) \) to be the composition of \( g \) and \( h \):

\[
\begin{align*}
\partial \text{cone}(\sigma) & \xrightarrow{=\ f} f(\partial \text{cone}(\sigma)) \\
\text{cone}(\sigma) & \xrightarrow{g} \text{cone}(\partial \text{cone}(\sigma)) \xrightarrow{h} \text{the shadow of } f(\partial \text{cone}(\sigma)) \text{ centered at } v_k^\tau.
\end{align*}
\]

According to the commutative diagram above, \( f \) is well-defined on \( \text{cone}(\sigma) \) in the sense that it is compatible with its definition on \( \text{cone}(T)^{<k} \). We use the phrase “fill in the boundary of \( \text{cone}(\sigma) \) against the center \( v_k^\tau \)” to represent the above process that extends the domain of \( f \) from \( \partial \text{cone}(\sigma) \) to \( \text{cone}(\sigma) \).

To complete the inductive step, we must demonstrate that \( f \) is economical over \((k+1)\)-faces of \( T \). Pick any \((k+1)\)-face \( \tau \) of \( T \). Let \( \sigma_0, \ldots, \sigma_{k+1} \) be the \( k \)-faces of \( \tau \). Observing that \( f(\partial \text{cone}(\tau)) = f(\tau \cup \text{cone}(\partial \tau)) = \tau \cup f(\text{cone}(\sigma_0)) \cup \ldots \cup f(\text{cone}(\sigma_{k+1})) \) and that \( f(\text{cone}(\sigma_i)) \) is the shadow of \( f(\partial \text{cone}(\sigma_i)) \) centered at \( v_k^\tau \) which intersects an \((m_k+1, \ldots, m_d)\)-face with probability at most \((k+1)!(p_d - \epsilon + k\delta) \leq (k+2)!(p_d - \epsilon + (k+1)\delta) \).

We have so far defined a continuous map \( f \) on \( \text{cone}(T)^{<d-1} \) such that for any \((d-1)\)-face \( \sigma \) of \( T \) the probability for an \((m_{d-1}+1)\)-face to intersect \( D := f(\text{cone}(\sigma)) \) is at most \( d!(p_d - \epsilon + (d-1)\delta) \). We write \( f(X) \bmod 2 := \{ y \in f(X) : |f^{-1}(y) \cap X| = 1 \ (\bmod 2) \} \) for the set of points in \( f(X) \) whose fibers in \( X \) have an odd number of points. Set \( \tilde{m} := (m_{d-1}+1)/2 \). We are going to define \( f \) on \( \text{cone}(\sigma) \) such that \( \tilde{m} (f(\text{cone}(\sigma)) \bmod 2) \) is less than \( \frac{1-\delta}{d+1} \).

Fix a point \( s \) in \( \mathbb{R}^d \setminus D \). For any point \( t \) in \( \mathbb{R}^d \setminus D \), if a generic piecewise linear path from \( s \) to \( t \) intersects with \( D \) an odd number of times, then put \( t \) in \( B \), otherwise put it in \( A \). Here the number of intersections of a piecewise linear path \( L \) and \( D \) might not be the cardinality of \( L \cap D \). Instead, the number of intersections is precisely \( \sum_{x \in L \cap D} |f^{-1}(x) \cap \partial \text{cone}(\sigma)| \), that is, it takes the multiplicity into account. Thus we have partitioned \( \mathbb{R}^d \setminus D \) into \( A \) and \( B \) such that any generic piecewise linear path from a point in \( A \) to a point in \( B \) meets \( D \) an odd number of times. Suppose \( a := m_{d-1}(A), b := m_d(A) \) and \( x := \tilde{m}(A) = (a+b)/2 \). The probability that an \((m_{d-1}+1)\)-face intersects with \( D \) is at least \( a(1-b) + (1-a)b \). Hence \( a(1-b) + (1-a)b < d!(p_d - \epsilon + (d-1)\delta) < 2 \frac{(\frac{1-\delta}{d+1})}{(1 - \frac{1-\delta}{d+1})} \). Because \( a(1-b) + (1-a)b = (a + b) - 2ab \geq (a + b) - (a + b)^2/2 = 2x(1-x) \), either \( x \) or \( 1-x \) is less than \( \frac{1-\delta}{d+1} \). In other words, one of \( \tilde{m}(A) \) and \( \tilde{m}(B) \) is less than \( \frac{1-\delta}{d+1} \). We may assume that \( \tilde{m}(B) < \frac{1-\delta}{d+1} \).

Fix a point \( c \in A \). Again, we fill in the boundary of \( \text{cone}(\sigma) \) against the center \( c \). For any generic point \( x \in A \), the line segment \([c, x]\) intersects with \( D \) an even number of times. For every \( v \) on \( \partial \text{cone}(\sigma) \), the ray with the initial point \( f(v) \) in the direction \( cf(v) \)
Figure 5: An illustration of the partition, the result of filling in against \( c \), and \( f(\text{cone}(\sigma)) \mod 2 \).

covers \( x \) once if and only if the line segment \([c, x]\) intersects with \( D \) at \( f(v) \). Because \( f(\text{cone}(\sigma)) \) is the union of such rays, the number of times that \( x \) is covered by \( f(\text{cone}(\sigma)) \) is exactly the number of intersections between \([c, x]\) and \( D \). This implies that \( x \) is not in \( f(\text{cone}(\sigma)) \mod 2 \). Therefore \( f(\text{cone}(\sigma)) \mod 2 \) is a subset of \( B \cup D \) almost surely. Noticing that \( \bar{m}(D) = 0 \), the extension of \( f \) has the desired property \( \bar{m}(f(\text{cone}(\sigma)) \mod 2) < \frac{1-\delta}{d+1} \).

Pick any \( d \)-face \( \tau \) of \( T \). Suppose the \((d-1)\)-faces of \( \tau \) are \( \sigma_0, \ldots, \sigma_d \). By a parity argument, we have

\[
\begin{align*}
f(\partial\text{cone}(\tau)) \mod 2 &= [\tau \cup f(\text{cone}(\sigma_0)) \cup \ldots \cup f(\text{cone}(\sigma_d))] \mod 2 \\
&\subset \tau \cup f(\text{cone}(\sigma_0)) \mod 2 \cup \ldots \cup f(\text{cone}(\sigma_d)) \mod 2.
\end{align*}
\]

Therefore \( \bar{m}(f(\partial\text{cone}(\tau)) \mod 2) \) is less than \( \delta + (d+1)\frac{1-\delta}{d+1} = 1 \), and so the degree of \( f \) on \( \partial\text{cone}(\tau) \), denoted by \( \deg(f, \partial\text{cone}(\tau)) \), is even. Because

\[
\sum_{\tau} \deg(f, \partial\text{cone}(\tau)) = 2 \sum_{\sigma} \deg(f, \text{cone}(\sigma)) + \deg(f, T) = \deg(f, T) \mod 2,
\]

where the first sum and the second sum are over all \( d \)-faces and all \((d-1)\)-faces of \( T \) respectively, we know that \( \deg(f, T) \) is even, which contradicts with the fact that \( f \) is identity on \( T \).

\[ \square \]

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