New infinite families of congruences modulo 8 for partitions with even parts distinct

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Abstract

Let $\text{ped}(n)$ denote the number of partitions of an integer $n$ wherein even parts are distinct. Recently, Andrews, Hirschhorn and Sellers, Chen, and Cui and Gu have derived a number of interesting congruences modulo 2, 3 and 4 for $\text{ped}(n)$. In this paper we prove several new infinite families of congruences modulo 8 for $\text{ped}(n)$. For example, we prove that for $\alpha \geq 0$ and $n \geq 0$,

$$\text{ped}
\left(3^{4\alpha+4}n + \frac{11 \times 3^{4\alpha+3} - 1}{8}\right) \equiv 0 \pmod{8}.$$ 

Keywords: partition; congruence; regular partition

1 Introduction

Let $\text{ped}(n)$ denote the function which enumerates the number of partitions of $n$ wherein even parts are distinct (and odd parts are unrestricted). For a positive integer $t$ we say that a partition is $t$-regular if no part is divisible by $t$. Andrews, Hirschhorn and Sellers [1] found the generating function for $\text{ped}(n)$:

$$\sum_{n=0}^{\infty} \text{ped}(n) q^n = \prod_{n=1}^{\infty} \frac{1 + q^{2n}}{1 - q^{2n-1}} = \frac{f_4}{f_1}, \quad (1)$$

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where here and throughout this paper, and for any positive integer \( k \), \( f_k \) is defined by

\[
f_k := \prod_{n=1}^{\infty} \left(1 - q^{kn}\right). \tag{2}
\]

From (1) it is easy to see that \( \text{ped}(n) \) equals the number of 4-regular partitions of \( n \). In recent years many congruences for the number of regular partitions have been discovered (see for example, Cui and Gu [3, 4], Dandurand and Penniston [5], Furcy and Penniston [7], Gordon and Ono [8], Keith [10], Lin and Wang [11], Lovejoy and Penniston [12], Penniston [13, 14], Webb [15], Xia and Yao [16, 17], and Yao[18]).

Numerous congruence properties are known for the function \( \text{ped}(n) \). For example, Andrews, Hirschhorn and Sellers [1] proved that for \( \alpha \geq 1 \) and \( n \geq 0 \),

\[
\text{ped}(3n + 2) \equiv 0 \pmod{2}, \tag{3}
\]

\[
\text{ped}(9n + 4) \equiv 0 \pmod{4}, \tag{4}
\]

\[
\text{ped}(9n + 7) \equiv 0 \pmod{12}. \tag{5}
\]

\[
\text{ped} \left( 3^{2\alpha+2}n + \frac{11 \times 3^{2\alpha+1} - 1}{8} \right) \equiv 0 \pmod{2}, \tag{6}
\]

\[
\text{ped} \left( 3^{2\alpha+1}n + \frac{17 \times 3^{2\alpha} - 1}{8} \right) \equiv 0 \pmod{6}. \tag{7}
\]

\[
\text{ped} \left( 3^{2\alpha+2}n + \frac{19 \times 3^{2\alpha+1} - 1}{8} \right) \equiv 0 \pmod{6}. \tag{8}
\]

Recently, Chen [2] obtained many interesting congruences modulo 2 and 4 for \( \text{ped}(n) \) using the theory of Hecke eigenforms and Cui and Gu [3] found infinite families of wonderful congruences modulo 2 for the function \( \text{ped}(n) \).

The aim of this paper is to establish several new infinite families of congruences modulo 8 for \( \text{ped}(n) \) by employing some results of Andrews, Hirschhorn and Sellers [1], and Cui and Gu [3]. The main results of this paper can be stated as the following theorems.

**Theorem 1.** For \( \alpha \geq 0 \) and \( n \geq 0 \),

\[
\text{ped} \left( 3^{2\alpha}n + \frac{3^{2\alpha} - 1}{8} \right) \equiv \text{ped}(n) \pmod{4}, \tag{9}
\]

\[
\text{ped} \left( 3^{4\alpha}n + \frac{3^{4\alpha} - 1}{8} \right) \equiv 5^\alpha \text{ped}(n) \pmod{8}, \tag{10}
\]

\[
\text{ped} \left( 3^{4\alpha+4}n + \frac{11 \times 3^{4\alpha+3} - 1}{8} \right) \equiv 0 \pmod{8}, \tag{11}
\]

\[
\text{ped} \left( 3^{4\alpha+4}n + \frac{19 \times 3^{4\alpha+3} - 1}{8} \right) \equiv 0 \pmod{8}. \tag{12}
\]
In view of (9) and the facts \( \text{ped}(1) = 1, \text{ped}(2) = 2, \text{ped}(3) = 3, \text{ped}(4) = 4 \), we obtain the following corollary.

**Corollary 2.** For \( \alpha \geq 0 \) and \( i = 0, 1, 2, 3 \) we have that

\[
\text{ped} \left( \frac{t_i \times 3^{2\alpha} - 1}{8} \right) \equiv i \pmod{4},
\]

where \( t_0 = 33, t_1 = 9, t_2 = 17 \) and \( t_3 = 25 \).

Replacing \( \alpha \) by \( 2\alpha \) in (10), we find that for \( \alpha \geq 0 \),

\[
\text{ped} \left( \frac{3^{8\alpha} n + 3^{8\alpha} - 1}{8} \right) \equiv \text{ped}(n) \pmod{8}.
\]

Employing (14) and the facts \( \text{ped}(1) = 1, \text{ped}(2) = 2, \text{ped}(3) = 3, \text{ped}(4) = 4, \text{ped}(10) = 29, \text{ped}(5) = 6, \text{ped}(253) = 5178754681431 \) and \( \text{ped}(8) = 16 \), we obtain the following congruences modulo 8.

**Corollary 3.** For \( \alpha \geq 0 \) and \( 0 \leq j \leq 7 \) we have that

\[
\text{ped} \left( \frac{s_j \times 3^{8\alpha} - 1}{8} \right) \equiv j \pmod{8},
\]

where \( s_0 = 65, s_1 = 9, s_2 = 17, s_3 = 25, s_4 = 33, s_5 = 81, s_6 = 41 \) and \( s_7 = 2025 \).

Utilizing the generating functions of \( \text{ped}(9n + 4), \text{ped}(9n + 7) \) discovered by Andrews, Hirschhorn and Sellers [1] and the \( p \)-dissection identities of two Ramanujan’s theta functions due to Cui and Gu [3], we will prove the following theorem.

**Theorem 4.** Let \( p \) be a prime such that \( p \equiv 5, 7 \pmod{8} \) and \( 1 \leq i \leq p - 1 \). Then for \( n \geq 0 \) and \( \alpha \geq 1 \),

\[
\text{ped} \left( 9p^{2\alpha} n + \frac{(72i + 33p)p^{2\alpha} - 1}{8} \right) \equiv 0 \pmod{8}
\]

and

\[
\text{ped} \left( 9p^{2\alpha} n + \frac{(72i + 57p)p^{2\alpha} - 1}{8} \right) \equiv 0 \pmod{8}.
\]

### 2 Proof of Theorem 1

Andrews, Hirschhorn and Sellers [1] established the following results for \( \text{ped}(3n + 1) \):

\[
\sum_{n=0}^{\infty} \text{ped}(9n + 1)q^n = \frac{f_2^2 f_3 f_4}{f_5 f_6^2} + 24q\frac{f_2^3 f_3^2 f_4 f_5^3}{f_1^{10}},
\]

\[
\sum_{n=0}^{\infty} \text{ped}(9n + 4)q^n = 4\frac{f_2 f_3 f_4 f_6}{f_1^4} + 48q\frac{f_2^2 f_4 f_6^3}{f_1^9}.
\]
and
\[ \sum_{n=0}^{\infty} \text{ped}(9n + 7)q^n = 12 \frac{f_4^3 f_6 f_4}{f_1^{11}}. \]  
(20)

By the binomial theorem it is easy to see that for all positive integers \( m \) and \( k \),
\[ f_k^{2m} \equiv f_{2k}^m \pmod{2}. \]  
(21)

By (21) we see that
\[ \frac{f_2^7}{f_2} \equiv f_1^2 \equiv 1 \pmod{2}, \]  
(22)

which yields
\[ \frac{f_2^7}{f_1^2} \equiv \frac{f_4^4}{f_6^2} \equiv 1 \pmod{4}. \]  
(23)

It follows from (18) and (23) that
\[ \sum_{n=0}^{\infty} \text{ped}(9n + 1)q^n \equiv \frac{f_4^3}{f_1} \pmod{4}. \]  
(24)

In view of (1) and (24) we see that for \( n \geq 0 \),
\[ \text{ped}(9n + 1) \equiv \text{ped}(n) \pmod{4}. \]  
(25)

Congruence (9) follows from (25) and mathematical induction.

Andrews, Hirschhorn and Sellers [1] also established the following 3-dissection formula of the generating function of \( \text{ped}(n) \):
\[ \sum_{n=0}^{\infty} \text{ped}(n)q^n = \frac{f_{12} f_4^4}{f_3^3 f_2^2} + q \frac{f_6^2 f_9^3}{f_3^3 f_2^2} + 2q^2 \frac{f_6 f_{18} f_{36}}{f_3^3}. \]  
(26)

Fortin, Jacob and Mathieu [6], and Hirschhorn and Sellers [9] independently derived the following 3-dissection formula of the generating function of overpartitions:
\[ \frac{f_2}{f_2^2} = \frac{f_4 f_6}{f_3^3 f_1^2} + 2q \frac{f_6^3 f_3^2}{f_3^3} + 4q^2 \frac{f_6^2 f_3^2}{f_3^3}. \]  
(27)

Combining (1), (18), (26), (27) we deduced that
\[ \sum_{n=0}^{\infty} \text{ped}(9n + 1)q^n \equiv \frac{f_4^3 f_2^3}{f_6^3} \frac{f_4}{f_1} \]  
\[ \equiv \frac{f_4^3}{f_6^3} \left( \frac{f_6 f_9}{f_3 f_{18}} + 2q \frac{f_6^2 f_9}{f_3^2} + 4q^2 \frac{f_6^2 f_3}{f_3^3} \right)^2 \left( \frac{f_{12} f_4^4}{f_3^3 f_2^2} + q \frac{f_6^2 f_9^3}{f_3^3 f_2^2} + 2q^2 \frac{f_6 f_{18} f_{36}}{f_3^3} \right) \]  
\[ \equiv \frac{f_6^2 f_1^2}{f_3^3} \frac{f_2}{f_3 f_{18}} + q \frac{f_6^2 f_9}{f_3^3} \frac{f_4}{f_1} + 2q \frac{f_6 f_{18} f_{36}}{f_3^3} \]  
\[ + 4q^2 \frac{f_6^2 f_9}{f_3^3} \frac{f_2^2}{f_1} \frac{f_4}{f_3 f_{18}} + 4q^3 \frac{f_6 f_{18} f_{36}}{f_3^3} \frac{f_2^2}{f_1} \frac{f_4}{f_3 f_{18}} \pmod{8}. \]  
(28)
Extracting those terms associated with powers $q^{3n+1}$ on both sides of (28), then dividing by $q$ and replacing $q^3$ by $q$, we find that

$$\sum_{n=0}^{\infty} ped(27n + 10)q^n \equiv \frac{f_3^8 f_5^{15} f_{12}}{f_1^{16} f_6^8} + 4 \frac{f_2^3 f_3^9 f_4 f_6}{f_1^{14} f_{12}} \pmod{8}. \quad (29)$$

By the binomial theorem and (22) we have

$$\frac{f_8^2}{f_1^{16}} \equiv \frac{f_3^{16}}{f_6^8} \equiv 1 \pmod{8}, \quad (30)$$

which yields

$$\frac{f_2^8 f_3^{15} f_{12}}{f_1^{16} f_6^8} \equiv \frac{f_{12}}{f_3} \pmod{8}. \quad (31)$$

It follows from (21) that

$$\frac{f_2^5 f_3^9 f_4 f_6}{f_1^{14} f_{12}} \equiv \frac{f_{12}}{f_3} \pmod{2}. \quad (32)$$

Substituting (31) and (32) into (29), we see that

$$\sum_{n=0}^{\infty} ped(27n + 10)q^n \equiv \frac{5 f_{12}}{f_3} \pmod{8}, \quad (33)$$

which implies that

$$\sum_{n=0}^{\infty} ped(81n + 10)q^n \equiv \frac{5 f_4}{f_1} \pmod{8} \quad (34)$$

and for $n \geq 0$,

$$ped(81n + 37) \equiv 0 \pmod{8}, \quad (35)$$

$$ped(81n + 64) \equiv 0 \pmod{8}. \quad (36)$$

Thanks to (1) and (34), we see that for $n \geq 0$,

$$ped(81n + 10) \equiv 5 ped(n) \pmod{8}. \quad (37)$$

Congruence (10) follows from (37) and mathematical induction. Replacing $n$ by $81n + 37$ in (10) and employing (35), we obtain (11). Replacing $n$ by $81n + 64$ in (10) and using (36), we deduce (12). The proof is complete.
3 Proof of Theorem 4

Thanks to (19) and (21), we have

\[ \sum_{n=0}^{\infty} ped(9n + 4)q^n \equiv 4f_2\psi(q^3) \pmod{8}, \] (38)

where \( \psi(q) \) is defined by

\[ \psi(q) := \frac{f_2^2}{f_1}. \] (39)

In their nice paper [3], Cui and Gu established \( p \)-dissection formulas for \( f_1 \) and \( \psi(q) \). They proved that for any odd prime \( p \),

\[ \psi(q) = \sum_{k=0}^{p-1} (\frac{p-1}{2})^k f \left( \frac{q^{2+(2k+1)p}}{2}, \frac{q^{p^2-(2k+1)p}}{2} \right) + q^\frac{p^2-1}{8} \psi(q^{p^2}) \] (40)

and for any prime \( p \geq 5 \),

\[ f_1 = \sum_{k=1}^{\frac{p-1}{2}} (-1)^k q^{\frac{2k+1}{2}} f \left( -q^{2p^2-(6k+1)p}, -q^{2p^2-(6k+1)p} \right) + (-1)^{\frac{p-1}{6}} q^{\frac{p^2-1}{24}} f_p^2, \] (41)

where

\[ \frac{\pm p - 1}{6} := \begin{cases} \frac{p - 1}{6} & \text{if } p \equiv 1 \pmod{6}, \\ \frac{-p - 1}{6} & \text{if } p \equiv -1 \pmod{6} \end{cases} \] (42)

and the Ramanujan theta function \( f(a, b) \) is defined by

\[ f(a, b) := \sum_{n=\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \] (43)

where \( |ab| < 1 \).

Let \( a(n) \) be defined by

\[ \sum_{n=0}^{\infty} a(n)q^n := f_2\psi(q^3). \] (44)

It follows from (38) and (44) that for \( n \geq 0 \),

\[ ped(9n + 4) \equiv 4a(n) \pmod{8}. \] (45)
Substituting (40) and (41) into (44), we see that for any prime $p \equiv 5, 7 \pmod{8}$,

\[
\sum_{n=0}^{\infty} a(n)q^n = \left( \sum_{m=\frac{p-1}{2}}^{p-1} (-1)^m q^{3m^2+m} f\left(-q^{3p^2+(6m+1)p}, -q^{3p^2-(6m+1)p}\right) + (-1)^{\frac{p-1}{6}} q^{\frac{p^2-1}{12}} f_{2p^2} \right)
\]

\[
\times \left( \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{3k^2+k}{2}} f\left(q^{\frac{3(p^2+2k+1)p}{2}}, q^{\frac{3(p^2-(2k+1)p)}{2}}\right) + q^{\frac{3(p^2-1)}{8}} \psi(q^{3p^2}) \right).
\]

(47)

Now, we consider the congruence

\[
3m^2 + m + \frac{3(k^2 + k)}{2} \equiv \frac{11(p^2 - 1)}{24} \pmod{p},
\]

(48)

where $-\frac{p-1}{2} \leq m \leq \frac{p-1}{2}$ and $0 \leq k \leq \frac{p-1}{2}$. Congruence (48) can be rewritten as follows

\[
2(6m + 1)^2 + (6k + 3)^2 \equiv 0 \pmod{p}.
\]

(49)

Since $p \equiv 5, 7 \pmod{8}$, we have that $-2$ is a ratic nonresidue modulo $p$ and hence (49) is equivalent to

\[
6m + 1 \equiv 6k + 3 \equiv 0 \pmod{p}.
\]

(50)

Thus, $m = \frac{p-1}{6}$ and $k = \frac{p-1}{2}$. Extracting those terms associated with powers $q^{pn+\frac{11(p^2-1)}{24}}$ on both sides of (46) and employing the fact that Congruence (48) holds if and only if $m = \frac{p-1}{6}$ and $k = \frac{p-1}{2}$, we have

\[
\sum_{n=0}^{\infty} a \left( pn + \frac{11(p^2 - 1)}{24} \right) q^{pn+\frac{11(p^2-1)}{24}} = (-1)^{\frac{p-1}{6}} q^{\frac{11(p^2-1)}{24}} f_{2p^2} \psi(q^{3p^2}).
\]

(51)

Dividing $q^{\frac{11(p^2-1)}{24}}$ on both sides of (51) and then replacing $q^n$ by $q$, we get

\[
\sum_{n=0}^{\infty} a \left( pn + \frac{11(p^2 - 1)}{24} \right) q^n = (-1)^{\frac{p-1}{6}} f_{2p} \psi(q^{3p}),
\]

(52)

which implies that

\[
\sum_{n=0}^{\infty} a \left( p^2n + \frac{11(p^2 - 1)}{24} \right) q^n = (-1)^{\frac{p-1}{6}} f_{2} \psi(q^{3}).
\]

(53)
and
\[ a\left(p(pn + i) + \frac{11(p^2 - 1)}{24}\right) = 0 \]  \hspace{1cm} (54)

for \(n \geq 0\) and \(1 \leq i \leq p - 1\). Combining (44) and (53), we have
\[ a\left(p^2n + \frac{11(p^2 - 1)}{24}\right) \equiv a(n) \pmod{2}. \]  \hspace{1cm} (55)

By (55) and mathematical induction, we find that for \(n \geq 0\) and \(\alpha \geq 0\),
\[ a\left(p^{2\alpha}n + \frac{11(p^{2\alpha} - 1)}{24}\right) \equiv a(n) \pmod{2}. \]  \hspace{1cm} (56)

Replacing \(n\) by \(p(pn + i) + \frac{11(p^2 - 1)}{24} (1 \leq i \leq p - 1)\) in (56) and using (54), we deduce that for \(n \geq 0\) and \(\alpha \geq 1\),
\[ a\left(p^{2\alpha}n + \frac{(24i + 11p)p^{2\alpha - 1} - 11}{24}\right) \equiv 0 \pmod{2}. \]  \hspace{1cm} (57)

Finally, replacing \(n\) by \(p^{2\alpha}n + \frac{(24i + 11p)p^{2\alpha - 1} - 11}{24} (1 \leq i \leq p - 1)\) in (45) and using (57), we get (16).

We conclude the paper by proving (17). In view of (20) and (21), we find that
\[ \sum_{n=0}^{\infty} ped(9n + 7)q^n \equiv 4f_1\psi(q^6) \pmod{8}, \]  \hspace{1cm} (58)

where \(\psi(q)\) is defined by (39). Let \(b(n)\) be defined by
\[ \sum_{n=0}^{\infty} b(n)q^n := f_1\psi(q^6). \]  \hspace{1cm} (59)

By (58) and (59), we find that for \(n \geq 0\),
\[ ped(9n + 7) \equiv 4b(n) \pmod{8}. \]  \hspace{1cm} (60)

Substituting (40) and (41) into (59), we see that for any prime \(p \equiv 5, 7 \pmod{8}\),
\[ \sum_{n=0}^{\infty} b(n)q^n = \left( \sum_{m=1}^{p-1} (-1)^m q^{\frac{3m^2 + m}{2}} f \left( -q^{\frac{3p^2 + (6m+1)p}{2}}, -q^{\frac{3p^2 - (6m+1)p}{2}} \right) + (-1)^{\frac{p-1}{2}} q^{\frac{p^2 - 1}{24}} \right) \]
\[ \times \left( \sum_{k=0}^{p-3} q^{3(k^2 + k)} f \left( q^{3(p^2 + (2k+1)p)}, q^{3(p^2 - (2k+1)p)} \right) + q^{\frac{3(p^2 - 1)}{4}} \psi(q^6p^2) \right). \]  \hspace{1cm} (61)
As above, for any prime $p \equiv 5, 7 \pmod{8}$, $-\frac{p-1}{2} \leq m \leq \frac{p-1}{2}$ and $0 \leq k \leq \frac{p-1}{2}$, the congruence relation

$$\frac{3m^2 + m}{2} + 3(k^2 + k) \equiv \frac{19(p^2 - 1)}{24} \pmod{p}$$

holds if and only if $m = \pm \frac{p-1}{6}$ and $k = \frac{p-1}{2}$. This implies that

$$\sum_{n=0}^{\infty} b \left( pn + \frac{19(p^2 - 1)}{24} \right) q^n = (-1)^{\pm \frac{p-1}{6}} f_p \psi(q^6).$$

Thanks to (63), we find that

$$\sum_{n=0}^{\infty} b \left( p^2 n + \frac{19(p^2 - 1)}{24} \right) q^n = (-1)^{\pm \frac{p-1}{6}} f_1 \psi(q^6)$$

and

$$b \left( p(pn + i) + \frac{19(p^2 - 1)}{24} \right) = 0$$

for $n \geq 0$ and $1 \leq i \leq p - 1$. It follows from (59) and (64) that for $n \geq 0$,

$$b \left( p^2 n + \frac{19(p^2 - 1)}{24} \right) \equiv b(n) \pmod{2}.$$ 

By (66) and mathematical induction, we deduce that for $n \geq 0$ and $\alpha \geq 0$,

$$b \left( p^{2\alpha} n + \frac{19(p^{2\alpha} - 1)}{24} \right) \equiv b(n) \pmod{2}.$$ 

Replacing $n$ by $p(pn + i) + \frac{19(p^2 - 1)}{24}$ $(1 \leq i \leq p - 1)$ in (67) and employing (65), we find that

$$b \left( p^{2\alpha} n + \frac{(24i + 19p)p^{2\alpha - 1} - 19}{24} \right) \equiv 0 \pmod{2}$$

for $n \geq 0$, $\alpha \geq 1$ and $1 \leq i \leq p - 1$. Congruence (17) follows from (60) and (68). This completes the proof of Theorem 4.

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References


