Summation of rational series twisted by strongly $B$-multiplicative coefficients

Jean-Paul Allouche∗      Jonathan Sondow
CNRS, Institut de Mathématiques de Jussieu-PRG          209 West 97th Street
Université Pierre et Marie Curie, Case 247          New York
F-75252 Paris Cedex 05, France          NY 10025, U.S.A
jean-paul.allouche@imj-prg.fr          jsondow@alumni.princeton.edu

Submitted: Aug 26, 2014; Accepted: Feb 17, 2015; Published: Mar 6, 2015
Mathematics Subject Classifications: 11A63, 11B83, 11B85, 68R15, 05A19

Abstract

We evaluate in closed form series of the type $\sum u(n)R(n)$, with $(u(n))_n$ a strongly $B$-multiplicative sequence and $R(n)$ a (well-chosen) rational function. A typical example is:

$$\sum_{n \geq 1} (-1)^{s_2(n)} \frac{4n + 1}{2n(2n+1)(2n+2)} = -\frac{1}{4}$$

where $s_2(n)$ is the sum of the binary digits of the integer $n$. Furthermore closed formulas for series involving automatic sequences that are not strongly $B$-multiplicative, such as the regular paperfolding and Golay-Shapiro-Rudin sequences, are obtained; for example, for integer $d \geq 0$:

$$\sum_{n \geq 0} \frac{v(n)}{(n+1)^{2d+1}} = \frac{\pi^{2d+1}|E_{2d}|}{(2^{2d+2} - 2)(2d)!}$$

where $(v(n))_n$ is the ±1 regular paperfolding sequence and $E_{2d}$ is an Euler number.

Keywords: summation of series; strongly $B$-multiplicative sequences; paperfolding sequence; Golay-Shapiro-Rudin sequence

1 Introduction

The problem of evaluating a series $\sum_n R(n)$ where $R$ is a rational function with integer coefficients is classical: think of the values of the Riemann $\zeta$ function at integers. Such

∗The author was partially supported by the ANR project “FAN” (Fractals et Numération), ANR-12-

IS01-0002.
sums can also be “twisted”, usually by a character (think of the $L$-functions), or by the usual arithmetic functions (e.g., the Möbius function $\mu$).

Another possibility is to twist such sums by sequences related to the digits of $n$ in some integer base. Examples can be found in [5] with, in particular, series $\sum \frac{a(n)}{n(n+1)}$, and in [7] with, in particular, series $\sum \frac{u(n)}{2n(2n+1)}$ (also see [9]): in both cases $u(n)$ counts the number of occurrences of a given block of digits in the $B$-ary expansion of the integer $n$, or is equal to $s_B(n)$, the sum of the $B$-ary digits of the integer $n$ ($B$ being an integer $\geq 2$).

Two emblematic examples are (see [10, Problem B5, p. 682] and [12, 5] for the first one, and [14, 7] for the second one):

$$\sum_{n\geq 1} \frac{s_B(n)}{n(n+1)} = \frac{B}{B-1} \quad \text{and} \quad \sum_{n\geq 1} \frac{s_2(n)}{2n(2n+1)} = \gamma + \log \frac{4}{\pi}$$

where $\gamma$ is the Euler-Mascheroni constant.

Similarly one can try to evaluate infinite products $\prod_n R(n)$, where $R(n)$ is a rational function, as well as twisted such products $\prod_n R(n)^{a(n)}$, where the sequence $(u(n))_{n\geq 0}$ is related to the digits of $n$ in some integer base. An example can be found in [2] (also see [11] for the original problem):

$$\prod_{n\geq 1} \left( \frac{(4n+2)(4n+2)}{(4n+1)(4n+3)} \right)^{2s(n)} = \frac{4}{\pi}$$

where $z(n)$ is the sum of the number of 0’s and the number of 1’s in the binary expansion of $n$, i.e., the length of this expansion. Other examples can be found in [4], e.g.,

$$\prod_{n\geq 0} \left( \frac{(4n+2)(8n+7)(8n+3)(16n+10)}{(4n+3)(8n+6)(8n+2)(16n+11)} \right)^{a(n)} = \frac{1}{\sqrt{2}}$$

where $u(n) = (-1)^{a(n)}$ and $a(n)$ is equal to the number of blocks 1010 occurring in the binary expansion of $n$. The products studied in [4] (also see references therein) are of the form $\prod_n R(n)^{-1}\cdot a(n)$ where $R(n)$ is a (well-chosen) rational function with integer coefficients, and $a(n)$ counts the number of occurrences of a given block of digits in the $B$-ary expansion of the integer $n$. The case where $a(n)$ counts the number of 1’s occurring in the binary expansion of $n$ is nothing but the case $a(n) = s_2(n)$. If $a(n) = s_B(n)$, the sequence $((-1)^{a(n)})_{n\geq 0}$ is strongly $B$-multiplicative: the more general evaluation of the product $\prod_n R(n)^{a(n)}$ where $(u(n))_{n\geq 0}$ is a strongly $B$-multiplicative sequence, is addressed in [8] (also see [13]). Recall that a strongly $B$-multiplicative sequence $(u(n))_{n\geq 0}$ satisfies $u(0) = 0$, and $u(Bn + j) = u(n)u(j)$ for all $j \in [0, B-1]$ and all $n \geq 0$. In particular, $(u(n))_{n\geq 0}$ is $B$-regular (or even $B$-automatic if it takes only finitely many values): recall that a sequence $(u(n))_{n\geq 0}$ is called $B$-automatic if its $B$-kernel, i.e., the set of subsequences $\{(u(B^n r + r))_{n\geq 0} \mid a \geq 0, \ 0 \leq r \leq B^a - 1\}$, is finite; a sequence $(u(n))_{n\geq 0}$ with values
in $\mathbb{Z}$ is called $B$-regular if the $\mathbb{Z}$-module spanned by its $B$-kernel has finite type (for more on these notions, see, e.g., [6]).

Since $\log \prod_n R(n)^{u(n)} = \sum_n u(n) \log R(n)$, it is natural to look at “simpler” series of the form $\sum_n u(n)R(n)$ with $R$ and $u$ as previously. All the examples above involve sequences $(u(n))_{n \geq 0}$ that are $B$-regular or even $B$-automatic. Unfortunately we were not able to address the general case where $(u(n))_{n \geq 0}$ is any $B$-regular or any $B$-automatic sequence.

The purpose of the present paper is to study the special case where, as in [8], the sequence $u(n)$ is strongly $B$-multiplicative and $R(n)$ is a well-chosen rational function. The paper can thus be seen as a companion paper to [8]. We will end with the evaluation of similar series where $(u(n))_{n \geq 0}$ is the regular paperfolding sequence or the Golay-Shapiro-Rudin sequence.

2 Preliminary definitions and results

This section quickly recalls definitions and results from [8].

**Definition 1.** Let $B \geq 2$ be an integer. A sequence of complex numbers $(u(n))_{n \geq 0}$ is strongly $B$-multiplicative if $u(0) = 1$ and, for all $n \geq 0$ and all $k \in \{0, 1, \ldots, B - 1\}$,

$$u(Bn + k) = u(n)u(k).$$

**Example 2.** Let $B \geq 2$ be an integer and $s_B(n)$ be the sum of the $B$-ary digits of $n$. Then for every complex number $a \neq 0$ the sequence $(a^{s_B(n)})_{n \geq 0}$ is strongly $B$-multiplicative. This sequence is $B$-regular (see the introduction); it is $B$-automatic if and only if $a$ is a root of unity.

The following lemma is a variation of Lemma 1 in [8].

**Lemma 3.** Let $B > 1$ be an integer. Let $(u(n))_{n \geq 0}$ be a strongly $B$-multiplicative sequence of complex numbers different from the sequence $(1, 0, 0, \ldots)$. We suppose that $|u(n)| \leq 1$ for all $n \geq 0$ and that $|\sum_{0 \leq k < B} u(k)| < B$. Let $f$ be a map from the set of nonnegative integers to the set of complex numbers such that $|f(n + 1) - f(n)| = O(n^{-2})$. Then the series $\sum_{n \geq 0} u(n)f(n)$ is convergent.

**Proof.** Use [8, Lemma 1] to get the upper bound $|\sum_{0 \leq n < N} u(n)| < C N^\alpha$ for some positive constant $C$ and some real number $\alpha$ in $(0, 1)$. Then use summation by parts. \qed

3 Main results

We state in this section some basic identities as well as first applications and examples. First we define $\delta_k$, a special case of the Kronecker delta:

$$\delta_k = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$
Theorem 4. Let $B > 1$ be an integer. Let $(u(n))_{n \geq 0}$ be a strongly $B$-multiplicative sequence, and let $f$ be a map from the nonnegative integers to the complex numbers, such that $(u(n))_{n \geq 0}$ and $f$ satisfy the conditions of Lemma 3. Define the series $S(k, B, u, f)$, for $k = 0, 1, \ldots, B - 1$, by

$$S(k, B, u, f) := \sum_{n \geq 0} u(n) f(Bn + k).$$

Then the following linear relations hold:

$$\sum_{n \geq 0} u(n) f(n) = \sum_{0 \leq k \leq B - 1} u(k) S(k, B, u, f)$$

and

$$\sum_{n \geq 0} u(n) \sum_{0 \leq k \leq B - 1} f(Bn + k) = \sum_{0 \leq k \leq B - 1} S(k, B, u, f).$$

In particular, define the series $S_1(k, B, u)$ and $S_2(k, B, u)$, for $k = 0, 1, \ldots, B - 1$, by

$$S_1(k, B, u) := \sum_{n \geq \delta_k} \frac{u(n)}{Bn + k} \quad \text{and} \quad S_2(k, B, u) := \sum_{n \geq \delta_k} \frac{u(n)}{(Bn + k)(Bn + k + 1)}.$$

Then the following linear relations hold:

$$(B - 1) S_1(0, B, u) - \sum_{0 \leq k \leq B - 1} u(k) S_1(k, B, u) = 0$$

and

$$\sum_{0 \leq k \leq B - 1} (B - u(k)) S_2(k, B, u) = B - 1.$$ 

Proof. It follows from Lemma 3 that all the series in the theorem converge. To prove the first relation, we split $\sum_{n \geq 0} u(n) f(n)$, obtaining

$$\sum_{n \geq 0} u(n) f(n) = \sum_{0 \leq k \leq B - 1} \sum_{n \geq 0} u(Bn + k) f(Bn + k) = \sum_{0 \leq k \leq B - 1} \sum_{n \geq 0} u(n) u(k) f(Bn + k)$$

$$= \sum_{0 \leq k \leq B - 1} u(k) \sum_{n \geq 0} u(n) f(Bn + k) = \sum_{0 \leq k \leq B - 1} u(k) S(k, B, u, f).$$

To prove the second relation, we write

$$\sum_{n \geq 0} u(n) \sum_{0 \leq k \leq B - 1} f(Bn + k) = \sum_{0 \leq k \leq B - 1} \sum_{n \geq 0} u(n) f(Bn + k) = \sum_{0 \leq k \leq B - 1} S(k, B, u, f).$$

To prove the last part of the theorem, we make two choices for $f$. First we take $f$ defined by $f(n) = 1/n$ for $n \neq 0$ and $f(0) = 0$. Then we take $f(n) = 1/(n + 1)$ if $n \neq 0$ and $f(0) = 0$. \qed
Remark The formula \( S_2(k, B, u) = S_1(k, B, u) - (S_1(k + 1, B, u) - \delta_k) \) (0 \( \leq k \leq B - 2 \)) holds. Nevertheless, the last two relations in Theorem 4 are independent, because \( S_2(B - 1, B, u) \) cannot be expressed in terms of the \( S_1(k, B, u) \) for \( k = 0, 1, \ldots, B - 1 \).

Corollary 5. If \((u(n))_{n \geq 0}\) is a strongly \( B \)-multiplicative sequence satisfying the conditions of Lemma 3, then

\[
\sum_{n \geq 1} u(n) \sum_{0 \leq k \leq B-1} \left( \frac{1}{Bn} - \frac{u(k)}{Bn + k} \right) = \sum_{1 \leq k \leq B-1} \frac{u(k)}{k}
\]

and

\[
\sum_{n \geq 1} u(n) \sum_{0 \leq k \leq B-1} \frac{B - u(k)}{(Bn + k)(Bn + k + 1)} = \sum_{1 \leq k \leq B-1} \frac{u(k)}{k(k + 1)}.
\]

Proof. This follows from the last part of Theorem 4 by substitution and manipulation. \( \square \)

Recall that the \( n \)-th harmonic number \( H_n \) and the \( n \)-th alternating harmonic number \( H_n^* \) are defined by

\[
H_n := \sum_{1 \leq k \leq n} \frac{1}{k} \quad \text{and} \quad H_n^* := \sum_{1 \leq k \leq n} \frac{(-1)^{k-1}}{k}.
\]

Corollary 6. If \( N_{j,B}(n) \) is the number of occurrences of the digit \( j \in \{0, 1, \ldots, B-1\} \) in the \( B \)-ary expansion of \( n \), then the following summations hold when \( j \neq 0 \):

\[
\sum_{n \geq 1} (-1)^{N_{j,B}(n)} \left( \frac{2}{Bn + j} + \frac{1}{Bn} \sum_{1 \leq k \leq B-1} \frac{k}{Bn + k} \right) = H_{B-1} - \frac{2}{j}
\]

and

\[
\sum_{n \geq 1} (-1)^{N_{j,B}(n)} \left( \frac{B - 1}{n(n + 1)} + \frac{2B}{(Bn + j)(Bn + j + 1)} \right) = B - 1 - \frac{2B}{j(j + 1)}.
\]

Proof. It is not hard to see that, if \( j \neq 0 \), we can apply the last part of Theorem 4 to the sequence \( u(n) := (-1)^{N_{j,B}(n)} \). Using Corollary 5 and the fact that \( N_{j,B}(k) = \delta_{k,j} \) when \( 0 \leq k < B \), the result follows. \( \square \)

Example 7. Taking \( B = 2 \) and \( j = 1 \), we get

\[
\sum_{n \geq 1} (-1)^{N_{1,2}(n)} \frac{4n + 1}{2n(2n + 1)} = -1
\]

and

\[
\sum_{n \geq 1} (-1)^{N_{1,2}(n)} \frac{4n + 1}{2n(2n + 1)(2n + 2)} = -\frac{1}{4}.
\]
Subtracting the second equation from the first, we multiply by 4 and obtain
\[
\sum_{n \geq 1} (-1)^{N_{1,2}(n)} \frac{4n + 1}{n(n + 1)} = -3.
\]
With \( B = 3 \) and \( j = 1 \) we get
\[
\sum_{n \geq 1} (-1)^{N_{1,3}(n)} \frac{18n^2 + 21n + 4}{3n(3n + 1)(3n + 2)} = -\frac{1}{2}
\]
and
\[
\sum_{n \geq 1} (-1)^{N_{1,3}(n)} \frac{6n^2 + 6n + 1}{3n(3n + 1)(3n + 2)(3n + 3)} = -\frac{1}{36}.
\]

**Corollary 8.** If \( s_B(n) \) is the sum of the \( B \)-ary digits of \( n \), then
\[
\sum_{n \geq 1} (-1)^{s_B(n)} \sum_{1 \leq k \leq B-1} \left( \frac{1}{Bn} - \frac{(-1)^k}{Bn+k} \right) = -H_{B-1}^*
\]
and
\[
\sum_{n \geq 1} (-1)^{s_B(n)} \sum_{0 \leq k \leq B-1} \frac{B - (-1)^k}{(Bn+k)(Bn+k+1)} = 1 + \frac{(-1)^B}{B} - 2H_{B-1}^*.
\]

**Proof.** Setting \( u(n) := (-1)^{s_B(n)} \), it is not hard to see that \( u(2n + 1) = -u(2n) \) for all \( n \geq 0 \). (Hint: look at the cases \( B \) even and \( B \) odd separately.) It follows that \( (u(n))_{n \geq 0} \) satisfies the conditions of Lemma 3. Noting that \( u(k) = (-1)^k \) when \( 0 \leq k < B \), the result follows from Corollary 5. \( \square \)

**Example 9.** Taking \( B = 2 \) or 3 gives the same pair of series as those with that value of \( B \) in Example 1, since \( s_2(n) = N_{1,2}(n) \) and \( s_3(n) = N_{1,3}(n) + 2N_{2,3}(n) \). (We can also replace \( s_3(n) \) with \( n \), as \( (-1)^{s_3(n)} = (-1)^n \) when \( B \) is odd.) With \( B = 4 \) we get
\[
\sum_{n \geq 1} (-1)^{s_4(n)} \frac{128n^3 + 176n^2 + 76n + 9}{4n(4n + 1)(4n + 2)(4n + 3)} = \frac{5}{12}
\]
and
\[
\sum_{n \geq 1} (-1)^{s_4(n)} \frac{128n^3 + 184n^2 + 80n + 9}{4n(4n + 1)(4n + 2)(4n + 3)(4n + 4)} = -\frac{5}{12}.
\]

**4 More examples**

Using Corollary 5 with sequences \( (u(n))_{n \geq 0} \) taking complex values yields other examples of sums of series.
Example 10. We may let \( u(n) := i^{s_2(n)} \) in Corollary 5. This gives the two summations
\[
\sum_{n \geq 1} \left( \frac{i^{s_2(n)}}{2n} - \frac{i^{s_2(n)+1}}{2n+1} \right) = i \sum_{n \geq 1} \frac{i^{s_2(n)}(3n + 1) - i^{s_2(n)+1}n}{n(n+1)(2n+1)},
\]
and by taking the imaginary and real parts we obtain the following result:

If \( \chi \) is the non-principal Dirichlet character modulo 4, defined by
\[
\chi(n) := \begin{cases} 
+1 & \text{if } n \equiv 1 \mod 4, \\
-1 & \text{if } n \equiv 3 \mod 4, \\
0 & \text{otherwise},
\end{cases}
\]

then
\[
\sum_{n \geq 1} \left( \frac{\chi(s_2(n))}{2n} - \frac{\chi(s_2(n)+1)}{2n+1} \right) = 1 = \sum_{n \geq 1} \frac{(3n + 1)\chi(s_2(n)) - n\chi(s_2(n)+1)}{n(n+1)(2n+1)}
\]
and
\[
\sum_{n \geq 1} \left( \frac{\chi(s_2(n)+1)}{2n} - \frac{\chi(s_2(n)+2)}{2n+1} \right) = 0 = \sum_{n \geq 1} \frac{(3n + 1)\chi(s_2(n)+1) - n\chi(s_2(n)+2)}{n(n+1)(2n+1)}.
\]

Example 11. Generalizing Example 10 by replacing \( i^{s_2(n)} \) with \( e^{2\pi i s_2(n)/d} \), for integer \( d \geq 2 \), is straightforward, yielding the following summations (Example 10 is another formulation for the case \( d = 4 \)):
\[
\sum_{n \geq 1} \left( \frac{\sin \frac{2\pi s_2(n)}{d}}{2n} - \frac{\sin \frac{2\pi (s_2(n)+1)}{d}}{2n+1} \right) = \sin \frac{2\pi}{d} = \sum_{n \geq 1} \frac{(3n + 1)\sin \frac{2\pi s_2(n)}{d} - n\sin \frac{2\pi (s_2(n)+1)}{d}}{n(n+1)(2n+1)}
\]
and
\[
\sum_{n \geq 1} \left( \frac{\cos \frac{2\pi s_2(n)}{d}}{2n} - \frac{\cos \frac{2\pi (s_2(n)+1)}{d}}{2n+1} \right) = \cos \frac{2\pi}{d} = \sum_{n \geq 1} \frac{(3n + 1)\cos \frac{2\pi s_2(n)}{d} - n\cos \frac{2\pi (s_2(n)+1)}{d}}{n(n+1)(2n+1)}.
\]

5 The paperfolding and Golay-Shapiro-Rudin sequences

The results above involve sums \( \sum u(n)R(n) \) where \( (u(n))_{n \geq 0} \) is a strongly \( B \)-multiplicative sequence, which, in all of our examples except Example 2 with alpha not a root of unity, happens to take only finitely many values. This implies that \( (u(n))_{n \geq 0} \) is \( B \)-automatic (see the introduction). One can then ask about more general sums \( \sum u(n)R(n) \) where the sequence \( (u(n))_{n \geq 0} \) is \( B \)-automatic. We give two cases where such series can be summed.
Theorem 12. Let \((v(n))_{n \geq 0}\) be the regular paperfolding sequence. Its first few terms are given by (replacing \(+1\) by \(+\) and \(-1\) by \(-\))

\[
(v(n))_{n \geq 0} = + + - + + - \cdots;
\]
it can be defined by: \(v(2n) = (-1)^n\) and \(v(2n + 1) = v(n)\) for all \(n \geq 0\). Then, for all integers \(d \geq 0\), we have the relation

\[
\sum_{n \geq 0} \frac{v(n)}{(n + 1)^{2d+1}} = \frac{\pi^{2d+1} |E_{2d}|}{(2^{2d+2} - 2)(2d)!}.
\]

where the \(E_{2d}\)'s are the Euler numbers defined by:

\[
\frac{1}{\cosh t} = \sum_{n \geq 0} \frac{E_{2n}}{(2n)!} t^{2n} \text{ for } |t| < \frac{\pi}{2}.
\]

Proof. First note that the series \(\sum_{n \geq 0} \frac{v(n)}{(n + 1)^{2d+1}}\) converges for \(\Re(s) > 0\): use the inequality \(|\sum_{n<N} v(n)| = O(\log N)\) (see, e.g., [6, Exercise 28, p. 206]) and summation by parts; note that the sequence \((R_n)_{n \geq 1}\) in [6, Exercise 28, p. 206] is equal to the sequence \((v(n))_{n \geq 0}\) here. Now, Exercise 27 in [6, p. 205–206] asks to prove, for all complex numbers \(s\) with \(\Re(s) > 0\), the equality (again with slightly different notation)

\[
\sum_{n \geq 0} \frac{v(n)}{(n + 1)^s} = \frac{2^s}{2^s - 1} \sum_{n \geq 0} \frac{(-1)^n}{(2n + 1)^s}.
\]

This can be easily done by splitting the sum on the left into even and odd indexes. Recalling that the Dirichlet beta function is defined by \(\beta(s) = \sum_{n \geq 0} \frac{(-1)^n}{(2n + 1)^s}\) for \(\Re(s) > 0\), we thus have, for any nonnegative integer \(d\),

\[
\sum_{n \geq 0} \frac{v(n)}{(n + 1)^{2d+1}} = \frac{2^{2d+1}}{2^{2d+1} - 1} \beta(2d + 1).
\]

But, when \(s\) is an odd integer, the value of \(\beta(s)\) can be expressed as a rational multiple of \(\pi\) (see, e.g., [1, 23.2.22, p. 807]):

\[
\beta(2d + 1) = \frac{(\pi/2)^{2d+1}}{2(2d)!} |E_{2d}|.
\]

Example 13. Taking \(d = 0\) in Theorem 12 yields a result due to F. von Haeseler (see [6, Exercise 27, p. 205–206])

\[
\sum_{n \geq 0} \frac{v(n)}{n + 1} = \frac{\pi}{2}.
\]

The second result we give in this section involves the Golay-Shapiro-Rudin sequence.
Theorem 14. Let \((r(n))_{n \geq 0}\) be the \pm 1 Golay-Shapiro-Rudin sequence. This sequence can be defined by \(r(n) = (-1)^{a(n)}\), where \(a(n)\) is the number of possibly overlapping occurrences of the block 11 in the binary expansion of \(n\), so that (replacing +1 by + and -1 by -)

\[
(r(n))_{n \geq 0} = + + + - + + - + \ldots ;
\]

alternatively it can be defined by

\[
r(0) = 1, \quad \text{and} \quad r(2n) = r(n), \quad r(2n + 1) = (-1)^n r(n) \quad \text{for} \quad n \geq 0.
\]

Let \(R(n)\) be a function from the nonnegative integers to the complex numbers, such that \(|R(n + 1) - R(n)| = \mathcal{O}(n^{-2})\). Then we have the relation

\[
\sum_{n \geq 1} r(n)(R(n) - R(2n) + R(2n + 1) - 2R(4n + 1)) = R(1).
\]

Proof. It is well known that \(|\sum_{n \leq N} r(n)| < K \sqrt{n}\) for some positive constant \(K\) (actually more is known; see, e.g., [6, Theorem 3.3.2, p. 79] and the historical comments given in [6, 3.3, p. 121]). Thus, by summation by parts, the series \(\sum_{n \geq 0} r(n)R(n)\) is convergent. Now we write

\[
\sum_{n \geq 0} r(n)R(n) = \sum_{n \geq 0} r(2n)R(2n) + \sum_{n \geq 0} r(2n + 1)R(2n + 1)
= \sum_{n \geq 0} r(n)R(2n) + \sum_{n \geq 0} (-1)^n r(n)R(2n + 1)
= \sum_{n \geq 0} r(n)R(2n) + \sum_{n \geq 0} r(2n)R(4n + 1) - \sum_{n \geq 0} r(2n + 1)R(4n + 3)
= \sum_{n \geq 0} r(n)(R(2n) + R(4n + 1)) - \sum_{n \geq 0} r(2n + 1)R(4n + 3).
\]

Hence

\[
\sum_{n \geq 0} r(n)(R(n) - R(2n) - R(4n + 1)) = -\sum_{n \geq 0} r(2n + 1)R(4n + 3)
= -(\sum_{n \geq 0} r(n)R(2n + 1) - \sum_{n \geq 0} r(2n)R(4n + 1))
= -\sum_{n \geq 0} r(n)R(2n + 1) + \sum_{n \geq 0} r(n)R(4n + 1)
\]

where the penultimate equality is obtained by splitting the sum \(\sum_{n \geq 0} r(n)R(2n + 1)\) into even and odd indices. Thus, finally

\[
\sum_{n \geq 0} r(n)(R(n) - R(2n) + R(2n + 1) - 2R(4n + 1)) = 0,
\]

hence

\[
\sum_{n \geq 1} r(n)(R(n) - R(2n) + R(2n + 1) - 2R(4n + 1)) = R(1).
\]

\[\square\]
Example 15. Taking $R(n) = 1/n$ if $n \neq 0$ and $R(0) = 1$ in Theorem 14 above yields

$$\sum_{n \geq 1} r(n) \frac{8n^2 + 4n + 1}{2n(2n+1)(4n+1)} = 1.$$  

Example 16. Taking $R$ defined by $R(n) = \log n - \log(n+1)$ for $n \neq 0$ and $R(0) = 0$ in Theorem 14 above yields

$$\sum_{n \geq 1} r(n) \log \frac{(2n+1)^4}{(n+1)^2(4n+1)^2} = -\log 2.$$  

Hence

$$\sum_{n \geq 0} r(n) \log \frac{(2n+1)^2}{(n+1)(4n+1)} = -\frac{1}{2} \log 2.$$  

After exponentiating we obtain:

$$\prod_{n \geq 0} \left( \frac{(2n+1)^2}{(n+1)(4n+1)} \right)^{r(n)} = \frac{1}{\sqrt{2}}$$  

thus recovering the value of an infinite product obtained in [3, Theorem 2, p. 148] (also see [4]).

References


