A Bijective Proof of the
Alladi-Andrews-Gordon Partition Theorem

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Submitted: Jul 28, 2014; Accepted: Mar 1, 2015; Published: Mar 13, 2015
Mathematics Subject Classifications: 05A17, 05A19

Abstract

Based on the combinatorial proof of Schur’s partition theorem given by Bressoud, and the combinatorial proof of Alladi’s partition theorem given by Padmavathamma, Raghavendra and Chandrashekara, we obtain a bijective proof of a partition theorem of Alladi, Andrews and Gordon.

Keywords: bijection; partition; Schur’s partition theorem; Göllnitz’s partition theorem; the Alladi-Andrews-Gordon partition theorem

1 Introduction

In 1926, Schur [15] proved one of the most profound results in the theory of partitions, which can be stated as follows.

**Theorem 1.1** (Schur). The number of partitions of $n$ into distinct parts $\equiv 1, 2 \pmod{3}$ is equal to the number of partitions of $n$ into distinct parts $\lambda_1 > \lambda_2 > \lambda_3 > \cdots$ where $\lambda_i - \lambda_{i+1} \geq 3$ with strict inequality if $\lambda_i \equiv 3 \pmod{3}$.

Throughout this paper $x \equiv y \pmod{M}$ means that $x = y + kM$ for a nonnegative integer $k$, where $x \geq y$ and $x > 0$. Theorem 1.1 is usually called Schur’s celebrated partition theorem of 1926. It was extended by Göllnitz [13] in 1967.

**Theorem 1.2** (Göllnitz). Let $B(n)$ be the number of partitions of $n$ into distinct parts $\equiv 2, 4, 5 \pmod{6}$. Let $C(n)$ be the number of partitions of $n$ into distinct parts $\lambda_1 > \lambda_2 > \lambda_3 > \cdots$ where no part equals 1 or 3, and $\lambda_i - \lambda_{i+1} \geq 6$ with strict inequality if $\lambda_i \equiv 6, 7$ or 9 (mod 6). Then $B(n) = C(n)$. 
Theorem 1.2 is one of the most striking extensions of Theorem 1.1. It is not a priori evident that $B(n) = C(n)$. Göllnitz’s proof is quite involved. Andrews gave two simpler proofs of Theorem 1.2, one by generating functions [8], and the other by computer algebra [9, §10]. Göllnitz [13] also gave the following refinement of Theorem 1.2:

\[ B(n; s) = C(n; s), \]  

where $B(n; s)$ and $C(n; s)$ denote, respectively, the number of partitions enumerated by $B(n)$ and $C(n)$ with exactly $s$ parts and the parts $\equiv 6, 7, 9$ (mod 6) are counted twice. Andrews [9] asked for a proof which would offer more insights into the refinement (1.1) of Göllnitz’s theorem.

There has been a lot of progress towards this direction, see [1, 5, 14]. The first combinatorial approach to Theorem 1.2 was provided by Alladi [1]. Precisely, Alladi constructed a bijection to prove a three-parameter $q$-identity [1, Eq. (1.2)], which first appeared in [5] and is a deep refinement of Theorem 1.2. However, as mentioned by Alladi [1], his construction can not be used to give a bijection between the sets of partitions of $n$ counted by $B(n)$ and $C(n)$. Padmavathamma, Raghavendra and Chandrashekara [14] presented a bijective proof of another partition theorem due to Alladi [2, Theorem 1], and remarked that their bijection also implies Theorem 1.2. They also noted that their method is very similar in spirit to Bressoud’s [11] combinatorial proof of Schur’s partition theorem.

By using weighted words introduced by Alladi and Gordon [6, 7], Alladi, Andrews and Gordon [5] obtained a more general partition theorem.

**Theorem 1.3** (Alladi-Andrews-Gordon). Let $M \geq 6$ and let $r_1, r_2, r_3$ be residues satisfying the following conditions:

\[0 < r_1 < r_2 < r_3 < M \leq r_1 + r_2 \quad \text{and} \quad r_1 + M < r_2 + r_3.\]  

Let $B(n; s)$ denote the number of partitions of $n$ into $s$ distinct parts congruent to $r_1, r_2, r_3$ (mod $M$). Let $C(n; s)$ denote the number of partitions of $n$ into $s$ distinct parts $\lambda_1 > \lambda_2 > \lambda_3 > \cdots$ such that

(i) each part $\lambda_i$ is $\equiv r_1, r_2, r_3, r_1 + r_2, r_1 + r_3$ or $r_2 + r_3$ (mod $M$),

(ii) $\lambda_i - \lambda_{i+1} \geq M$ with strict inequality if $\lambda_i \equiv r_1 + r_2, r_1 + r_3$ or $r_2 + r_3$ (mod $M$),

(iii) the parts $\equiv r_1 + r_2, r_1 + r_3$ or $r_2 + r_3$ (mod $M$) are counted twice.

Then $B(n; s) = C(n; s)$.

Clearly, Theorem 1.3 reduces to (1.1) by setting $M = 6, r_1 = 2, r_2 = 4,$ and $r_3 = 5$. As remarked by Alladi, Andrews and Gordon [5, §1], Theorem 1.3 also generalizes two extensions of (1.1) given by Göllnitz [13, Sätze 4.8 and 4.10]. In fact, Alladi, Andrews and Gordon established a three-parameter key identity [5, Eq. (1.4)] which implies Theorem 1.3. Alladi [1, §6] noticed that Jacobi’s triple product identity [12, p. 12] can be derived from a special case of this key identity.
Alladi, Andrews and Berkovich [4] found an interpretation of Theorem 1.3 in terms of partitions into six colored integers, and they obtained a more general theorem on partitions into eleven colored integers. Moreover, they showed that the partition theorem involving eleven colored integers is combinatorially equivalent to a four-parameter key-identity [4, Eq. (1.7)]. Further studies related to Theorem 1.2 and Theorem 1.3 can be found in Alladi and Andrews [3] and Andrews, Bringmann and K. Mahlburg [10].

The objective of this paper is to provide a bijective proof of Theorem 1.3. Our proof is in the spirit of the combinatorial proof of Alladi’s partition theorem [2, Theorem 1] given by Padmavathamma, Raghavendra and Chandrashekara [14].

2 A Bijective Proof of Theorem 1.3

In this section, we present a bijective proof of Theorem 1.3. Let $B(n; s)$ and $C(n; s)$ denote the sets of partitions counted by $B(n; s)$ and $C(n; s)$, respectively. We define a map from $B(n; s)$ to $C(n; s)$, then we show that it is a bijection. We need Lemma 2.1 to transform the congruence condition for integers congruent to $r_i + r_j$ modulo $M$ $(1 \leq i < j \leq 3)$ into difference conditions for consecutive integers congruent to $r_i$ and $r_j$ modulo $M$.

By the conditions in (1.2), we see that

$$0 \leq r_1 + r_2 - M < r_1 + r_3 - M < r_1 < r_2 + r_3 - M < r_2 < r_3 < M. \quad (2.1)$$

This implies that $r_1, r_2, r_3, r_1 + r_2, r_1 + r_3$ and $r_2 + r_3$ are distinct modulo $M$. For a partition $\mu$ in $C(n, s)$, if a part $\mu_k$ is congruent to $r_i + r_j$ modulo $M$, where $1 \leq i < j \leq 3$, we can represent $\mu_k$ as a sum of two positive integers congruent to $r_i$ and $r_j$ modulo $M$ subject to a difference condition. This property also holds for $\mu_k - tM$, where $t$ is an integer such that $\mu_k - tM \geq r_i + r_j$.

Lemma 2.1. Let $r_1, r_2$ and $r_3$ be integers satisfying the conditions in (1.2). Let $u$ be a positive integer congruent to $r_i + r_j$ modulo $M$ and $u \geq r_i + r_j$, where $1 \leq i < j \leq 3$. Let $w = (u - r_i - r_j)/M$. Then for integer $0 \leq t \leq w$, $u - tM$ can be uniquely expressed as

$$u - tM = a_t + b_t, \quad (2.2)$$

where $a_t$ and $b_t$ are positive integers such that

$$a_t, b_t \equiv r_i \text{ or } r_j \pmod{M} \quad \text{and} \quad a_t \not\equiv b_t \pmod{M}, \quad (2.3)$$

and

$$0 < a_t - b_t < M. \quad (2.4)$$

More precisely,

$$a_t = \ell M + r_j, \quad b_t = \ell M + r_i, \quad (2.5)$$

if $u - tM = 2\ell M + r_i + r_j$, and

$$a_t = (\ell + 1)M + r_i, \quad b_t = \ell M + r_j, \quad (2.6)$$

if $u - tM = (2\ell + 1)M + r_i + r_j$, where $\ell$ is a nonnegative integer.
Proof. Clearly, \( u - tM \equiv r_i + r_j \pmod{M} \) can be deduced from (2.2) and (2.3). To determine \( a_i \) and \( b_i \) from (2.2), (2.3) and (2.4), we may represent \( u - tM \) by \( 2tM + r_i + r_j \) or \((2\ell + 1)M + r_i + r_j\), where \( \ell \) is a nonnegative integer. First consider the case \( u - tM = 2\ell M + r_i + r_j \). There are two possibilities. Subcase 1: \( a_i = \ell' M + r_i \) and \( b_i = \ell'' M + r_j \), where \( \ell' \) and \( \ell'' \) are nonnegative integers such that \( \ell' + \ell'' = 2\ell \). Hence we have
\[
a_t - b_i = (\ell' - \ell'')M + r_i - r_j = 2(\ell' - \ell)M + r_i - r_j. \tag{2.7}
\]
Since \( 0 < r_1 < r_2 < r_3 < M \) as given in (1.2), we have
\[
-M < r_i - r_j < 0. \tag{2.8}
\]
Under the condition \( a_t - b_i > 0 \), it follows from (2.7) and (2.8) that \( 2(\ell' - \ell) \geq 1 \). Moreover, since \( a_t - b_i < M \), by (2.7) and (2.8) we get \( 2(\ell' - \ell) \leq 1 \). So we deduce that \( 2(\ell' - \ell) = 1 \). But this is impossible since \( \ell' \) and \( \ell \) are integers. This means that Subcase 1 cannot happen.

We now consider Subcase 2: \( a_t = \ell' M + r_j \) and \( b_i = \ell'' M + r_i \), where \( \ell' \) and \( \ell'' \) are nonnegative integers such that \( \ell' + \ell'' = 2\ell \). In this case, we have
\[
a_t - b_i = (\ell' - \ell'')M + r_j - r_i = 2(\ell' - \ell)M + r_j - r_i. \tag{2.9}
\]
Under the condition \( a_t - b_i > 0 \), it follows from (2.9) and (2.8) that \( 2(\ell' - \ell) \geq 0 \). Moreover, since \( a_t - b_i < M \), by (2.9) and (2.8) we get \( 2(\ell' - \ell) \leq 0 \). So we deduce that \( \ell' = \ell'' = \ell \), which yields (2.5).

For the case \( u - tM = (2\ell + 1)M + r_i + r_j \), we also consider two subcases. Subcase 1: \( a_t = \ell' M + r_j \) and \( b_i = \ell'' M + r_i \), where \( \ell' \) and \( \ell'' \) are nonnegative integers such that \( \ell' + \ell'' = 2\ell + 1 \). Subcase 2: \( a_t = \ell' M + r_i \) and \( b_i = \ell'' M + r_j \), where \( \ell' \) and \( \ell'' \) are nonnegative integers such that \( \ell' + \ell'' = 2\ell + 1 \). In Subcase 1, there is no solution for \( \ell' \). In Subcase 2, there is only one solution, that is, \( \ell' = \ell + 1 \) and \( \ell'' = \ell \). So we arrive at (2.6). The detailed proof is similar to the argument for the first case and hence it is omitted.

We are now ready to give a bijective proof of Theorem 1.3.

Proof of Theorem 1.3. Define a map \( \Phi : B(n; s) \rightarrow C(n; s) \) by the following procedure. Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) \) be a partition in \( B(n; s) \). We aim to construct a partition \( \mu \) such that \( \mu_k - \mu_{k+1} \geq M \) with strictly inequality if \( \mu_k \equiv r_i + r_j \pmod{M} \) \((1 \leq i < j \leq 3)\). Assume that \( \lambda \) has only positive parts. For notational convenience, set \( \lambda_0 = +\infty \). Consider the following two cases.

Case 1: Condition (ii) in Theorem 1.3 holds for all consecutive parts of \( \lambda \), that is, for any \( 1 \leq i \leq s - 1 \), we have \( \lambda_i - \lambda_{i+1} \geq M \) with strict inequality if \( \lambda_i \) is congruent to \( r_1 + r_2 \), \( r_1 + r_3 \) or \( r_2 + r_3 \) modulo \( M \). In this case, we see that \( \lambda \in C(n; s) \), and we set \( \mu = \lambda \).

Case 2: Condition (ii) in Theorem 1.3 does not hold, that is, there exists an integer \( i \) such that \( \lambda_i - \lambda_{i+1} < M \). We choose \( i_1 \) to be the minimum integer subject to this condition. We aim to construct a partition, denoted \( \alpha^{(1)} \), such that the condition (ii) holds for the
first $i_1$ parts of $\alpha^{(1)}$. If this can be achieved, then one can iterate this process to find a desired partition in $C(n, s)$. Here are two subcases.

Subcase 2.1: $\lambda_{i_1-1} - (\lambda_{i_1} + \lambda_{i_1+1}) \geq M$. Let

$$\alpha^{(1)} = (\lambda_1, \ldots, \lambda_{i_1-2}, \lambda_{i_1-1}, \lambda_{i_1} + \lambda_{i_1+1}, \lambda_{i_1+2}, \ldots).$$

It is easily checked that the condition (ii) holds for the first $i_1$ parts of $\alpha^{(1)}$, that is, for any $1 \leq j \leq i_1 - 1$ we have $\alpha_j^{(1)} - \alpha_j^{(1)} \geq M$ with strict inequality if $\alpha_j^{(1)} \equiv r_1 + r_2, r_1 + r_3$ or $r_2 + r_3$ (mod $M$).

Since $0 < \lambda_i - \lambda_{i+1} < M$, we get $\lambda_i \not\equiv \lambda_{i+1} \pmod{M}$. This means that $\lambda_i + \lambda_{i+1} \equiv r_i + r_{i+1} \pmod{M}$. So we need also show that $\lambda_i + \lambda_{i+1} - \lambda_{i+2} > M$ when $s \geq i_1 + 2$. This relation is obvious when $\lambda_i - \lambda_{i+2} \geq M$. We now assume that $\lambda_i - \lambda_{i+2} < M$. Note that $\lambda_i, \lambda_{i+1}$ and $\lambda_{i+2}$ are positive integers congruent to $r_1, r_2$ or $r_3$ modulo $M$. By the condition $0 < r_1 < r_2 < r_3 < M$ as given in (1.2) and the assumption $\lambda_i - \lambda_{i+2} < M$, we see that $(\lambda_i, \lambda_{i+1}, \lambda_{i+2})$ can be expressed in one of the three forms $(\ell M + r_1, \ell M + r_2, \ell M + r_1)$, $((\ell + 1) M + r_1, \ell M + r_2, \ell M + r_3)$ and $((\ell + 1) M + r_2, (\ell + 1) M + r_1, \ell M + r_3)$, where $\ell$ is a nonnegative integer. Using the conditions $0 < r_1 < r_2 < r_3 < M$, and the condition $\lambda_i - \lambda_{i+2} < M$, one can check that $\lambda_i + \lambda_{i+1} - \lambda_{i+2} > M$ holds in any of the above three cases. So we have shown that $\alpha^{(1)}$ is a desired partition in Subcase 2.1.

Subcase 2.2: $\lambda_{i_1-1} - (\lambda_{i_1} + \lambda_{i_1+1}) < M$. There is a unique integer $1 \leq k_1 \leq i_1 - 1$ such that

$$\lambda_{i_1-1-t} - (\lambda_{i_1} + \lambda_{i_1+1} + t M) < M$$

for $0 \leq t \leq k_1 - 1$, and

$$\lambda_{i_1-1-k_1} - (\lambda_{i_1} + \lambda_{i_1+1} + k_1 M) \geq M.$$  \hspace{1cm} (2.11)

Let

$$\alpha^{(1)} = (\lambda_1, \ldots, \lambda_{i_1-1-k_1}, \lambda_{i_1} + \lambda_{i_1+1} + k_1 M, \lambda_{i_1-1-k_1} - M, \ldots, \lambda_{i_1-1} - M, \lambda_{i_1+2}, \ldots).$$

As $i_1$ is chosen to be the minimum integer $i$ such that $\lambda_i - \lambda_{i+1} < M$, for any $1 \leq j \leq i_1 - 1$, we have $\lambda_j - \lambda_{j+1} \geq M$. This implies that for $i_1 - k_1 \leq j \leq i_1 - 2$, $(\lambda_j - M) - (\lambda_{j+1} - M) \geq M$. By (2.11), $\lambda_{i_1-1-k_1} - (\lambda_{i_1} + \lambda_{i_1+1} + k_1 M) \geq M$. To verify the condition (ii) for the first $i_1$ parts of $\alpha^{(1)}$, it remains to show that

$$(\lambda_{i_1} + \lambda_{i_1+1} + k_1 M) - (\lambda_{i_1-k_1} - M) > M,$$  \hspace{1cm} (2.12)

since the part $\lambda_{i_1} + \lambda_{i_1+1} + k_1 M$ is congruent to $r_i + r_j$ modulo $M$. Notice that (2.12) can be deduced from (2.10) by setting $t = k_1 - 1$. This completes the proof in Subcase 2.2.

For the partition $\alpha^{(1)}$, if condition (ii) holds for all consecutive parts, then we set $\mu = \alpha^{(1)}$. Otherwise, we can find a minimum integer $i_2$ such that $i_2 \geq i_1$ and $\alpha_{i_2}^{(1)} - \alpha_{i_2+1}^{(1)} < M$. Then we may repeat the above process in Case 2. Finally, we obtain a partition $\mu$ for which condition (ii) holds for all consecutive parts.
We observe that each part of $\mu$ is congruent to $r_1, r_2, r_3, r_1 + r_2, r_1 + r_3$ or $r_2 + r_3$ modulo $M$, and the number of parts of $\lambda$ is equal to the number of parts of $\mu$ if the number of parts congruent to $r_1 + r_2, r_1 + r_3$ or $r_2 + r_3$ modulo $M$ are counted twice. Hence conditions (i) and (iii) in Theorem 1.3 also hold for $\mu$. So we have $\mu \in \mathcal{C}(n; s)$.

To prove that $\Phi$ is a bijection, we now describe the inverse map $\Phi^{-1}$. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_v)$ be a partition in $\mathcal{C}(n; s)$. Assume that $\mu_1 > \mu_2 > \cdots > \mu_v > 0$. We aim to construct a partition $\lambda$ such that $\Phi(\lambda) = \mu$ by transforming the congruence condition for parts congruent to $r_1 + r_j$ modulo $M$ into difference conditions for consecutive parts congruent to $r_1$ and $r_j$ modulo $M$. For notational convenience, set $\mu_{v+1} = 0$ if $\mu_t$ is the last positive part of $\mu$. Consider the following two cases.

Case 1: There is no part of $\mu$ that is congruent to $r_1 + r_2, r_1 + r_3$ or $r_2 + r_3$ modulo $M$.
In this case, we see that $\mu \in \mathcal{B}(n; s)$, and we set $\lambda = \mu$.

Case 2: There exists an integer $j$ such that $\mu_j$ is congruent to $r_1 + r_2, r_1 + r_3$ or $r_2 + r_3$ modulo $M$. We choose $j_1$ to be the maximum integer subject to this condition. Using Lemma 2.1 for $u = \mu_{j_1}$ and $t = 0$, we get $\mu_{j_1} = a_0 + b_0$, where $a_0$ and $b_0$ are given by (2.5) or (2.6). We can transform $\mu$ into a partition, denoted $\beta^{(1)}$, such that the number of parts congruent to $r_1 + r_j$ modulo $M$ in $\beta^{(1)}$ is one less than the number of parts congruent to $r_1 + r_j$ modulo $M$ in $\mu$. There are two cases.

Case (i): $0 \leq \mu_{j_1+1} < b_0$. Let
$$\beta^{(1)} = (\mu_1, \ldots, \mu_{j_1-1}, a_0, b_0, \mu_{j_1+1}, \ldots, \mu_v).$$
We claim that $\beta^{(1)}$ is a partition. Let $\beta^{(1)}_t = (\beta^{(1)}_1, \ldots, \beta^{(1)}_v)$. Since $\mu_1 > \mu_2 > \cdots > \mu_v > 0$, by (2.5) and (2.6), we see that $\mu_1 > \mu_2 > \cdots > \mu_{j_1-1} > a_0 > b_0 > 0$ if $\mu_{j_1+1} = 0$, and $\mu_1 > \mu_2 > \cdots > \mu_{j_1-1} > a_0 > b_0 > \mu_{j_1+1} > \cdots > \mu_v > 0$ if $\mu_{j_1+1} > 0$. It follows that $\beta^{(1)}_1 > \beta^{(1)}_2 > \cdots > \beta^{(1)}_v > 0$.

As $j_1$ is the maximum integer such that $\mu_{j_1}$ is congruent to $r_1 + r_2, r_1 + r_3$ or $r_2 + r_3$ modulo $M$, for $j_1 < t \leq v + 1$, we have $\beta^{(1)}_t \equiv r_1, r_2$ or $r_3$ (mod $M$) since all parts of $\mu$ are congruent to $r_1, r_2, r_3, r_1 + r_2, r_1 + r_3$ or $r_2 + r_3$ modulo $M$. So the number of parts congruent to $r_1 + r_j$ modulo $M$ in $\beta^{(1)}$ is one less than the number of parts congruent to $r_1 + r_j$ modulo $M$ in $\mu$.

Case (ii): $\mu_{j_1+1} \geq b_0$. The following procedure generates a partition $\beta^{(1)}$ from $\mu$. Using Lemma 2.1 for $u = \mu_{j_1}$ and $t \geq 1$ with $\mu_{j_1} - tM \geq r_1 + r_j$, we obtain a unique expression $\mu_{j_1} - tM = a_t + b_t$, where $a_t$ and $b_t$ are given by (2.5) or (2.6). Since $\mu_{j_1+1} \geq b_0$, there is a unique integer $1 \leq k_1 \leq v - j_1$ such that
$$\mu_{j_1+t+1} \geq b_t$$
for $0 \leq t \leq k_1 - 1$, and
$$0 \leq \mu_{j_1+k_1+1} < b_{k_1}.$$  
(2.13)

Let
$$\beta^{(1)} = (\mu_1, \ldots, \mu_{j_1-1}, \mu_{j_1+1} + M, \ldots, \mu_{j_1+k_1} + M, a_{k_1}, b_{k_1}, \mu_{j_1+k_1+1}, \ldots),$$
and denote $\beta^{(1)}$ by $(\beta^{(1)}_1, \ldots, \beta^{(1)}_{v+1})$. Note that $a_{k_1}$ and $b_{k_1}$ are congruent to $r_1, r_2$ or $r_3$ modulo $M$. Recall that $j_1$ is the maximum integer such that $\mu_{j_1}$ is congruent to $r_1 + r_2,$
Since all parts of $\mu$ are congruent to $r_1, r_2, r_3, r_1 + r_2, r_1 + r_3$ or $r_2 + r_3$ modulo $M$, for $j_1 \leq t \leq v + 1$, we have $\beta_1^{(1)} \equiv r_1, r_2$ or $r_3 \pmod{M}$. Hence the number of parts congruent to $r_1 + r_j$ modulo $M$ in $\beta^{(1)}$ is one less than the number of parts congruent to $r_1 + r_j$ modulo $M$ in $\mu$.

It remains to show that $\beta^{(1)}$ is a partition. First, if $j_1 \geq 2$, we need to verify that
\[
\mu_{j_1 - 1} > \mu_{j_1 + 1} + M. 
\]
(2.15)
Since $\mu = (\mu_1, \mu_2, \ldots, \mu_v)$ is a partition in $\mathbb{C}(n; s)$, we have $\mu_i - \mu_{i+1} \geq M$ for $1 \leq i \leq v - 1$. It follows that
\[
\mu_{j_1 - 1} - (\mu_{j_1 + 1} + M) = (\mu_{j_1 - 1} - \mu_{j_1}) + (\mu_{j_1} - \mu_{j_1 + 1}) - M > 0,
\]
which yields (2.15). Next, we prove that
\[
\mu_{j_1 + k_1} + M > a_{k_1}. 
\]
(2.16)
We claim that
\[
a_{k_1} = b_{k_1 - 1}. 
\]
(2.17)
To derive (2.17), we note that $\mu_{j_1} - k_1 M = a_{k_1} + b_{k_1}$ and $\mu_{j_1} - (k_1 - 1) M = a_{k_1 - 1} + b_{k_1 - 1}$, where $a_{k_1}, b_{k_1}, a_{k_1 - 1}$ and $b_{k_1 - 1}$ are given by (2.5) or (2.6). If $\mu_{j_1} - k_1 M$ can be represented by $2M + r_i + r_j$, where $\ell$ is a nonnegative integer, then we have $\mu_{j_1} - (k_1 - 1) M = (2\ell + 1) M + r_i + r_j$. By (2.5) and (2.6) we deduce that
\[
a_{k_1} = \ell M + r_j \quad \text{and} \quad b_{k_1 - 1} = \ell M + r_j,
\]
as required. Similarly, it can be shown that (2.17) also holds if $\mu_{j_1} - k_1 M$ can be represented by $(2\ell + 1) M + r_i + r_j$ for a nonnegative integer $\ell$. So (2.17) is confirmed.

Setting $t = k_1 - 1$ in (2.13) gives
\[
\mu_{j_1 + k_1} \geq b_{k_1 - 1}. 
\]
(2.18)
Combining (2.18) and (2.17), we find that $\mu_{j_1 + k_1} \geq a_{k_1}$. It follows that
\[
\mu_{j_1 + k_1} + M \geq a_{k_1} + M > a_{k_1}.
\]
This proves (2.16). So we have shown that $\beta^{(1)}$ is a partition. Since $\mu$ is a partition in $\mathbb{C}(n; s)$, it has distinct parts. Thus we have reached the conclusion that $\beta^{(1)}$ has distinct parts. This completes the proof in Case (ii).

For either case (i) or case (ii), if each part of $\beta^{(1)}$ is congruent to $r_1, r_2$ or $r_3$ modulo $M$, then we set $\lambda = \beta^{(1)}$. Otherwise, we can find a maximum integer $j_2$ such that $j_2 < j_1$ and $\beta_j^{(1)}$ is congruent to $r_1 + r_2, r_1 + r_3$ or $r_2 + r_3$ modulo $M$. Then we may iterate the above process until we obtain a partition $\lambda$ with all parts congruent to $r_1, r_2$ or $r_3$ modulo $M$.

Moreover, it can be seen that the number of parts of $\lambda$ is equal to the number of parts of $\mu$ with the convention that the parts congruent to $r_1 + r_2, r_1 + r_3$ or $r_2 + r_3$ modulo $M$ are counted twice. Thus we have $\lambda \in \mathbb{B}(n, s)$, and so $\Phi$ is surjective.
Due to the uniqueness of the expression of a positive integer congruent to \( r_i + r_j \) modulo \( M \) in Lemma 2.1, we see that every step of \( \Phi \) is reversible. Hence \( \Phi \) is a bijection between \( B(n,s) \) and \( C(n,s) \). So we have \( B(n,s) = C(n,s) \). This completes the proof. \( \square \)

The following example gives an illustration of the map \( \Phi \). Let \( M = 6, r_1 = 2, r_2 = 4 \) and \( r_3 = 5 \), for which the conditions in (1.2) are satisfied. Let \( \lambda = (92, 70, 64, 53, 52, 46, 38, 35, 23, 17, 4, 2) \), which is a partition in \( B(496; 12) \). In the construction of \( \Phi(\lambda) \), the intermediate partitions \( \alpha^{(1)}, \alpha^{(2)} \) and \( \alpha^{(3)} \) are given below:

\[
\alpha^{(1)} = (123, 86, 64, 58, 46, 38, 35, 23, 17, 4, 2), \\
\alpha^{(2)} = (123, 97, 80, 58, 52, 40, 23, 17, 4, 2), \\
\alpha^{(3)} = (123, 97, 80, 58, 52, 40, 23, 17, 6). 
\]

Note that condition (ii) in Theorem 1.3 holds for all consecutive parts of \( \alpha^{(3)} \), that is, for \( 1 \leq i \leq 8 \), we have \( \alpha^{(3)}_i - \alpha^{(3)}_{i+1} \geq M \) with strict inequality if \( \alpha^{(3)}_i \) is congruent to \( r_1 + r_2 \), \( r_1 + r_3 \) or \( r_2 + r_3 \) modulo \( M \). Moreover, there are only three parts, 123, 97 and 6, which are congruent to \( r_1 + r_2 \), \( r_1 + r_3 \) or \( r_2 + r_3 \) modulo \( M \), and therefore should be counted twice. Hence \( \mu = \alpha^{(3)} = (123, 97, 80, 58, 52, 40, 23, 17, 6) \), which belongs to \( C(496; 12) \).

The following example gives an illustration of the inverse map \( \Phi^{-1} \). Let \( M = 6, r_1 = 2, r_2 = 4 \) and \( r_3 = 5 \), for which the conditions in (1.2) are satisfied. Let \( \mu = (123, 97, 80, 58, 52, 40, 23, 17, 6) \), which is a partition in \( C(496; 12) \). The intermediate partitions \( \beta^{(1)}, \beta^{(2)} \) and \( \beta^{(3)} \) are given below:

\[
\beta^{(1)} = (123, 97, 80, 58, 52, 40, 23, 17, 4, 2), \\
\beta^{(2)} = (123, 86, 64, 58, 46, 38, 35, 23, 17, 4, 2), \\
\beta^{(3)} = (92, 70, 64, 53, 52, 46, 38, 35, 23, 17, 4, 2). 
\]

Clearly, all the parts of \( \beta^{(3)} \) are congruent to 2, 4 or 5 modulo \( M \). Hence \( \lambda = \beta^{(3)} = (92, 70, 64, 53, 52, 46, 38, 35, 23, 17, 4, 2) \), which belongs to \( B(496; 12) \).

**Acknowledgments**

I am grateful to Arthur L.B. Yang for valuable suggestions. I also wish to thank the referee for helpful comments. This work was supported by the 973 Project and the National Science Foundation of China.
References


