A special case of a combinatorial theorem of De Bruijn and Erdős asserts that every noncollinear set of \( n \) points in the plane determines at least \( n \) distinct lines. Chen and Chvátal suggested a possible generalization of this assertion in metric spaces with appropriately defined lines. We prove this generalization in all metric spaces induced by connected chordal graphs.

**Keywords:** Combinatorial geometry; Metric space; Extremal combinatorics
1 Introduction

It is well known that

(i) every noncollinear set of \( n \) points in the plane determines at least \( n \) distinct lines.

As noted by Erdős [12], theorem (i) is a corollary of the Sylvester–Gallai theorem (asserting that, for every noncollinear set \( S \) of finitely many points in the plane, some line goes through precisely two points of \( S \)); it is also a special case of a combinatorial theorem proved later by De Bruijn and Erdős [11].

Theorem (i) involves neither measurement of distances nor measurement of angles: the only notion employed here is incidence of points and lines. Such theorems are a part of ordered geometry [7], which is built around the ternary relation of betweenness: point \( b \) is said to lie between points \( a \) and \( c \) if \( b \) is an interior point of the line segment with endpoints \( a \) and \( c \). It is customary to write \( [abc] \) for the statement that \( b \) lies between \( a \) and \( c \). In this notation, a line \( \overline{uv} \) is defined — for any two distinct points \( u \) and \( v \) — as

\[
\{u, v\} \cup \{p : [puv] \vee [upv] \vee [uwp]\}.
\]

In terms of the Euclidean metric \( \text{dist} \), we have

\[
[abc] \iff a, b, c \text{ are three distinct points and } \text{dist}(a, b) + \text{dist}(b, c) = \text{dist}(a, c).
\]

In an arbitrary metric space, equivalence (2) defines the ternary relation of metric betweenness introduced in [14] and further studied in [1, 3, 8]; in turn, (1) defines the line \( \overline{uv} \) for any two distinct points \( u \) and \( v \) in the metric space. The resulting family of lines may have strange properties. For instance, a line can be a proper subset of another: in the metric space with points \( u, v, x, y, z \) and

\[
\begin{align*}
\text{dist}(u, v) &= \text{dist}(v, x) = \text{dist}(x, y) = \text{dist}(y, z) = \text{dist}(z, u) = 1, \\
\text{dist}(u, x) &= \text{dist}(v, y) = \text{dist}(x, z) = \text{dist}(y, u) = \text{dist}(z, v) = 2,
\end{align*}
\]

we have

\[
\overline{vy} = \{v, x, y\} \quad \text{and} \quad \overline{xy} = \{v, x, y, z\}.
\]

Chen [4] proved, using a definition of \( \overline{uv} \) different from (1), that the Sylvester–Gallai theorem generalizes in the framework of metric spaces. Chen and Chvátal [5] suggested that theorem (i), too, might generalize in this framework:

(ii) True or false? Every metric space on \( n \) points, where \( n \geq 2 \), either has at least \( n \) distinct lines or else has a line that consists of all \( n \) points.

They proved that

- every metric space on \( n \) points either has at least \( \lg n \) distinct lines or else has a line that consists of all \( n \) points
and noted that the lower bound \( \lg n \) can be improved to
\[
\lg n + \frac{1}{2} \lg \lg n + \frac{1}{2} \lg \frac{n}{2} - o(1).
\]
(Here, as usual, \( \lg x \) stands for \( \log_2 x \).)

Every connected undirected graph induces a metric space on its vertex set, where
\( \text{dist}(u, v) \) is the familiar graph-theoretic distance between vertices \( u \) and \( v \), defined as the smallest number of edges in a path from \( u \) to \( v \). (Some people call this the ‘hop distance’.) Chiniforooshan and Chvátal [6] proved that

• every metric space induced by a connected graph on \( n \) vertices either has \( \Omega(n^{2/7}) \) distinct lines or else has a line that consists of all \( n \) vertices;

we will prove that the answer to (ii) is ‘true’ for all metric spaces induced by connected chordal graphs. (We follow the graph-theoretic terminology of Bondy and Murty [2]. In particular, a chordal graph is a graph that contains no induced cycle of length four or more.)

**Theorem 1.** Every metric space induced by a connected chordal graph on \( n \) vertices, where \( n \geq 2 \), either has at least \( n \) distinct lines or else has a line that consists of all \( n \) vertices.

**2 The proof**

Given an undirected graph, let us write \([abc]\) to mean that \( a, b, c \) are three distinct vertices such that \( \text{dist}(a, b) + \text{dist}(b, c) = \text{dist}(a, c) \); this is equivalent to saying that \( b \) is an interior vertex of a shortest path from \( a \) to \( c \).

**Lemma 2.** Let \( s, x, y \) be vertices in a finite chordal graph such that \([sxy]\). If \( sx = sy \), then \( x \) is a cut vertex separating \( s \) and \( y \).

**Proof.** The set of all vertices \( u \) such that \( \text{dist}(s, u) = \text{dist}(s, x) \) separates \( s \) and \( y \). Among all its subsets that separate \( s \) and \( y \), choose a minimal one and call it \( C \). Since \( x \) is an interior vertex of a shortest path from \( s \) to \( y \), it belongs to \( C \). To prove that \( C \) includes no other vertex, assume, to the contrary, that \( C \) includes a vertex \( u \) other than \( x \).

Our graph with \( C \) removed has distinct connected components \( S \) and \( Y \) such that \( s \in S \) and \( y \in Y \); the minimality of \( C \) guarantees that each of its vertices has at least one neighbour in \( S \) and at least one neighbour in \( Y \). Since each of \( u \) and \( x \) has at least one neighbour in \( S \), there is a path from \( u \) to \( x \) with at least one interior vertex and with all interior vertices in \( S \). Let \( P \) be a shortest such path; note that \( P \) has no chords except possibly the chord \( ux \). Similarly, there is a path \( Q \) from \( u \) to \( x \) with at least one interior vertex, and with all interior vertices in \( Y \), that has no chords except possibly the chord \( ux \). The union of \( P \) and \( Q \) is a cycle of length at least four; since this cycle must have a chord, vertices \( u \) and \( x \) must be adjacent. In turn, the union of \( Q \) and \( ux \) is a chordless cycle, and so \( Q \) has precisely two edges. This means that some vertex \( v \) in \( Y \) is adjacent to both \( u \) and \( x \). (Similarly, some vertex in \( S \) is adjacent to both \( u \) and \( x \); however, this
Write $i = \text{dist}(s, x)$ and $j = \text{dist}(x, y)$. Since all vertices $t$ with $\text{dist}(s, t) < i$ belong to $S$ and since $v$ has no neighbours in $S$, we must have $\text{dist}(s, v) > i$; since $\text{dist}(x, v) = 1$, we conclude that $\text{dist}(s, v) = i + 1$ and that $v \in \overline{sx}$. Since $\overline{sx} = \overline{sy}$, it follows that $v \in \overline{sy}$. Since $\text{dist}(v, x) = 1$ and $\text{dist}(x, y) = j$, we have $\text{dist}(v, y) \leq j + 1$. From $\text{dist}(s, v) = i + 1$, $\text{dist}(s, y) = i + j$, we conclude that $\text{dist}(v, y) = j + 1$, $i \geq 1$, $j \geq 1$, and $v \in \overline{sy}$, we deduce that $\text{dist}(v, y) = i + 1$. Since $\text{dist}(u, v) = 1$, it follows that $\text{dist}(u, y) \leq j$; since $\text{dist}(s, u) = i$ and $\text{dist}(s, y) = i + j$, we conclude that $\text{dist}(u, y) = j$ and $u \in \overline{sy}$. Since $\text{dist}(s, u) = i$, $\text{dist}(s, x) = i$, and $\text{dist}(u, x) = 1$, we have $u \not\in \overline{sx}$. But then $\overline{sx} \neq \overline{sy}$, a contradiction.

A vertex of a graph is called \textit{simplicial} if its neighbours are pairwise adjacent.

**Lemma 3.** Let $s, x, y$ be three distinct vertices in a finite connected chordal graph. If $s$ is simplicial and $\overline{sx} = \overline{sy}$, then $\overline{xy}$ consists of all the vertices of the graph.

**Proof.** Since $\overline{sx} = \overline{sy}$, we have $y \in \overline{sx}$, and so $[ysx]$ or $[syx]$ or $[sxy]$; since $s$ is simplicial, $[ysx]$ is excluded; switching $x$ and $y$ if necessary, we may assume that $[sxy]$. Given an arbitrary vertex $u$, we have to prove that $u \in \overline{xy}$. Let $P$ be a shortest path from $s$ to $u$ and let $Q$ be a shortest path from $u$ to $y$. Lemma 2 guarantees that $x$ is a cut vertex separating $s$ and $y$, and so the concatenation of $P$ and $Q$ must pass through $x$. This means that $[sxu]$ or $[uxy]$ (or both). If $[uxy]$, then $u \in \overline{xy}$; to complete the proof, we may assume that $[sxu]$, and so $u \in \overline{sx}$.

Since $\overline{sx} = \overline{sy}$, we have $[usy]$ or $[suy]$ or $[syu]$; since $s$ is simplicial, $[usy]$ is excluded. If $[suy]$, then $[sxu]$ implies $[xuy]$; if $[syu]$, then $[sxy]$ implies $[xyu]$; in either case, $u \in \overline{xy}$. □

**Proof of Theorem 1.** Consider a connected chordal graph on $n$ vertices where $n \geq 2$. By a theorem of Dirac [10, Theorem 4], this graph has at least two simplicial vertices; choose one of them and call it $s$. We may assume that the lines $\overline{sz}$ with $z \neq s$ are pairwise distinct (else some line consists of all $n$ vertices by Lemma 3). Since the graph is connected and has at least two vertices, $s$ has at least one neighbour; choose one and call it $u$. If $u$ is the only neighbour of $s$, then every path from $s$ to another vertex must pass through $u$, and so $\overline{su}$ consists of all $n$ vertices. If $s$ has a neighbour $v$ other than $u$, then line $\overline{sv}$ is distinct from all of the $n - 1$ lines $\overline{sz}$ with $z \neq s$: since $s, u, v$ are pairwise adjacent, we have $s \not\in \overline{uv}$.

3 Related theorems

In Theorem 1, ‘connected chordal graph’ can be replaced by ‘connected bipartite graph’:

- every metric space induced by a connected bipartite graph on $n$ vertices, where $n \geq 2$, has a line that consists of all $n$ vertices.
In fact, $xy$ consists of all $n$ vertices whenever $x$ and $y$ are adjacent. To prove this, consider an arbitrary vertex $u$. Since the graph is bipartite, $\text{dist}(u, x)$ and $\text{dist}(u, y)$ have distinct parities; since $\text{dist}(x, y) = 1$, they differ by at most one. We conclude that $\text{dist}(u, x)$ and $\text{dist}(u, y)$ differ by precisely one, and so $u \in xy$.

In Theorem 1, ‘connected chordal graph’ can be also replaced by ‘graph of diameter two’: Chvátal [9] proved that

- every metric space on $n$ points where $n \geq 2$ and each nonzero distance equals 1 or 2 either has at least $n$ distinct lines or else has a line that consists of all $n$ vertices.

Kantor and Patkós [13] proved that

- if no two of $n$ points in the plane share their $x$- or $y$-coordinate, then these $n$ points with the $L_1$ metric either induce at least $n$ distinct lines or else they induce a line that consists of all of them.

(For sets of $n$ points in the plane that are allowed to share their coordinates, [13] provides a weaker conclusion: these $n$ points with the $L_1$ metric either induce at least $n/37$ distinct lines or else they induce a line that consists of all of them.)

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References


