A Deza–Frankl type theorem for set partitions

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Abstract
A set partition of \([n]\) is a collection of pairwise disjoint nonempty subsets (called blocks) of \([n]\) whose union is \([n]\). Let \(B(n)\) denote the family of all set partitions of \([n]\). A family \(A \subseteq B(n)\) is said to be \(m\)-intersecting if any two of its members have at least \(m\) blocks in common. For any set partition \(P \in B(n)\), let \(\tau(P) = \{x : \{x\} \in P\}\) denote the union of its singletons. Also, let \(\mu(P) = [n] \setminus \tau(P)\) denote the set of elements that do not appear as a singleton in \(P\). Let

\[
F_{2t} = \{ P \in B(n) : |\mu(P)| \leq t \}; \\
F_{2t+1}(i_0) = \{ P \in B(n) : |\mu(P) \cap ([n] \setminus \{i_0\})| \leq t \}.
\]

In this paper, we show that for \(r \geq 3\), there exists a constant \(n_0 = n_0(r)\) depending on \(r\) such that for all \(n \geq n_0\), if \(A \subseteq B(n)\) is \((n - r)\)-intersecting, then

\[
|A| \leq \begin{cases} 
|F_{2t}|, & \text{if } r = 2t; \\
|F_{2t+1}(1)|, & \text{if } r = 2t + 1.
\end{cases}
\]

Moreover, equality holds if and only if

\[
A = \begin{cases} 
F_{2t}, & \text{if } r = 2t; \\
F_{2t+1}(i_0), & \text{if } r = 2t + 1,
\end{cases}
\]

for some \(i_0 \in [n]\).

Keywords: \(t\)-intersecting family, Erdős-Ko-Rado, set partitions
1 Introduction

Let \( n = \{1, \ldots, n\} \), and let \( \binom{n}{k} \) denote the family of all \( k \)-subsets of \( [n] \). A family \( \mathcal{A} \) of subsets of \( [n] \) is \( t \)-intersecting if \( |A \cap B| \geq t \) for all \( A, B \in \mathcal{A} \). One of the most beautiful results in extremal combinatorics is the Erdős-Ko-Rado theorem.

**Theorem 1** (Erdős, Ko, and Rado [13], Frankl [15], Wilson [46]). Suppose \( \mathcal{A} \subseteq \binom{[n]}{k} \) is \( t \)-intersecting and \( n > 2k - t \). Then for \( n \geq (k - t + 1)(t + 1) \), we have

\[
|\mathcal{A}| \leq \binom{n - t}{k - t}.
\]

Moreover, if \( n > (k - t + 1)(t + 1) \) then equality holds if and only if \( \mathcal{A} = \{ A \in \binom{[n]}{k} : T \subseteq A \} \) for some \( t \)-set \( T \).

In the celebrated paper [1], Ahlswede and Khachatrian extended the Erdős-Ko-Rado theorem by determining the structure of all \( t \)-intersecting set systems of maximum size for all possible \( n \) (see also [3, 14, 16, 17, 20, 25, 31, 36, 40, 42, 43, 45] for some related results). There have been many recent results showing that a version of the Erdős-Ko-Rado theorem holds for combinatorial objects other than set systems. For example, an analogue of the Erdős-Ko-Rado theorem for the Hamming scheme is proved in [41]. A complete solution for the \( t \)-intersection problem in the Hamming space is given in [2]. Intersecting families of permutations were initiated by Deza and Frankl [10]. Some recent work done on this problem and its variants can be found in [5, 7, 8, 11, 12, 19, 26, 28, 35, 37, 38, 39, 44]. The investigation of the Erdős-Ko-Rado property for graphs started in [23], and gave rise to [4, 6, 21, 22, 24, 47]. The Erdős-Ko-Rado type results also appear in vector spaces [9, 18], set partitions [27, 29, 30] and weak compositions [32, 33, 34].

Let \( S_n \) denote the set of permutations of \( [n] \). A family \( \mathcal{A} \subseteq S_n \) is said to be \( m \)-intersecting if for any \( \sigma, \delta \in \mathcal{A} \), there is an \( m \)-set \( T \subseteq [n] \) such that \( \sigma(j) = \delta(j) \) for all \( j \in T \). Given any \( \sigma \in S_n \), set \( \overline{\mu}(\sigma) = \{ j \in [n] : \mu(j) \neq j \} \), i.e., \( \overline{\mu}(\sigma) \) is the set of all elements in \( [n] \) that are not fixed by \( \sigma \). Let

\[
\mathcal{F}_r = \begin{cases} 
\{ \sigma \in S_n : |\overline{\mu}(\sigma)| \leq t \}, & \text{if } r = 2t; \\
\{ \sigma \in S_n : |\overline{\mu}(\sigma) \cap ([n] \setminus \{1\})| \leq t \}, & \text{if } r = 2t + 1.
\end{cases}
\]

It can be verified easily that \( \mathcal{F}_r \) is \( (n - r) \)-intersecting. Furthermore, Deza and Frankl [10] proved the following theorem.

**Theorem 2** (Deza-Frankl). For \( r \geq 3 \), there exists an \( n_0 = n_0(r) \) such that for all \( n \geq n_0 \), if \( \mathcal{A} \subseteq S_n \) is \( (n - r) \)-intersecting, then

\[
|\mathcal{A}| \leq |\mathcal{F}_r|.
\]

A set partition of \( [n] \) is a collection of pairwise disjoint nonempty subsets (called blocks) of \( [n] \) whose union is \( [n] \). Let \( \mathcal{B}(n) \) denote the family of all set partitions of \( [n] \).
It is well-known that the size of $B(n)$ is the $n$-th Bell number, denoted by $B_n$. A block of size one is also known as a singleton. We denote the number of all set partitions of $[n]$ which are singleton-free (i.e. without any singleton) by $\tilde{B}_n$.

A family $\mathcal{A} \subseteq \mathcal{B}(n)$ is said to be $m$-intersecting if $|P \cap Q| \geq m$ for all $P, Q \in \mathcal{A}$, i.e., any two of its members have at least $m$ blocks in common. Let $I(n, m)$ denote the set of all $m$-intersecting families of set partitions of $[n]$.

For any set partition $P \in \mathcal{B}(n)$, let $\tau(P) = \{ x : \{ x \} \subseteq P \}$ denote the union of its singletons. Also, let $\mu(P) = [n] - \tau(P)$ denote the set of elements that do not appear as a singleton in $P$. For any two partitions $P, Q$, we make the following simple observations:

- $P$ and $Q$ cannot intersect in any singleton $\{ x \}$ where $x \in \mu(P) \triangle \mu(Q)$ (here the operation $\triangle$ denotes the symmetric difference of two sets).
- $P$ and $Q$ must intersect in every singleton $\{ x \}$ where $x \in [n] - (\mu(P) \cup \mu(Q))$.

Let

$$
\mathcal{F}_{2t} = \{ P \in \mathcal{B}(n) : |\mu(P)| \leq t \};
\mathcal{F}_{2t+1}(i_0) = \{ P \in \mathcal{B}(n) : |\mu(P) \cap ([n] \setminus \{i_0\})| \leq t \}.
$$

It can be readily verified that $\mathcal{F}_{2t} \in I(n, n - 2t)$ and $\mathcal{F}_{2t+1}(i_0) \in I(n, n - 2t - 1)$. Moreover,

$$
|\mathcal{F}_{2t}| = \sum_{i=0}^{t} \tilde{B}_i \binom{n}{i},
$$

$$
|\mathcal{F}_{2t+1}(i_0)| = \sum_{i=0}^{t} \tilde{B}_i \binom{n}{i} + \tilde{B}_{t+1} \binom{n-1}{t}.
$$

In this paper, we will prove the following theorem.

**Theorem 3.** For $r \geq 3$, there exists an $n_0 = n_0(r)$ such that for all $n \geq n_0$, if $\mathcal{A} \subseteq \mathcal{B}(n)$ is $(n - r)$-intersecting, then

$$
|\mathcal{A}| \leq \begin{cases} 
|\mathcal{F}_{2t}|, & \text{if } r = 2t; \\
|\mathcal{F}_{2t+1}(1)|, & \text{if } r = 2t + 1.
\end{cases}
$$

Moreover, equality holds if and only if

$$
\mathcal{A} = \begin{cases} 
\mathcal{F}_{2t}, & \text{if } r = 2t; \\
\mathcal{F}_{2t+1}(i_0), & \text{if } r = 2t + 1,
\end{cases}
$$

for some $i_0 \in [n]$.

Note that Theorem 3 can be considered as an analogue of Theorem 2 for set partitions. Let $A_0 = \{ \{ x \} : x \in [n] \}$ and $A_1 = \{ \{ x \} : x \in [n] \setminus \{1, 2\} \} \cup \{\{1, 2\}\}$. Then $\mathcal{F}_2 = \{A_0\}$ and $\{A_0, A_1\} \in I(n, n - 2)$. So, $|\{A_0, A_1\}| = 2 > |\mathcal{F}_2| = 1$. This explains why $r \geq 3$ is required in Theorem 3.
2 Splitting operation

In this section, we summarize some important results regarding the splitting operation for intersecting family of set partitions. We refer the reader to [27] for proofs which are omitted here.

Let \( i, j \in [n], i \neq j, \) and \( P \in \mathcal{B}(n) \). Denote by \( P[i] \) the block of \( P \) which contains \( i \). We define the \((i, j)\)-split of \( P \) to be the following set partition:

\[
s_{ij}(P) = \begin{cases} P \setminus \{P[i]\} \cup \{\{i\}, P[i] \setminus \{i\}\} & \text{if } j \in P[i], \\ P & \text{otherwise.} \end{cases}
\]

For a family \( \mathcal{A} \subseteq \mathcal{B}(n) \), let \( s_{ij}(\mathcal{A}) = \{s_{ij}(P) : P \in \mathcal{A}\} \). Any family \( \mathcal{A} \) of set partitions can be decomposed with respect to given \( i, j \in [n] \) as follows:

\[
\mathcal{A} = (\mathcal{A} \setminus \mathcal{A}_{ij}) \cup \mathcal{A}_{ij},
\]

where \( \mathcal{A}_{ij} = \{P \in \mathcal{A} : s_{ij}(P) \notin \mathcal{A}\} \). Define the \((i, j)\)-splitting of \( \mathcal{A} \) to be the family

\[
S_{ij}(\mathcal{A}) = (\mathcal{A} \setminus \mathcal{A}_{ij}) \cup s_{ij}(\mathcal{A}_{ij}).
\]

Surprisingly, it turns out that for any \( \mathcal{A} \in I(n, m) \), splitting operations preserve the size and the intersecting property.

Lemma 4 ([27], Proposition 3.2). Let \( \mathcal{A} \in I(n, m) \). Then \( S_{ij}(\mathcal{A}) \in I(n, m) \) and \( |S_{ij}(\mathcal{A})| = |\mathcal{A}| \).

A family \( \mathcal{A} \) of set partitions is compressed if for any \( i, j \in [n], i \neq j, \) we have \( S_{ij}(\mathcal{A}) = \mathcal{A} \).

Lemma 5 ([27], Proposition 3.3). Given a family \( \mathcal{A} \in I(n, t) \), by repeatedly applying the splitting operations, we eventually obtain a compressed family \( \mathcal{A}^* \in I(n, t) \) with \( |\mathcal{A}^*| = |\mathcal{A}| \).

Lemma 6. Let \( a, b \) be positive integers with \( a + b \leq n \). Let \( P, Q \in \mathcal{B}(n) \) be such that \( |P \cap Q| \geq n - a \). If \( |\tau(P) \setminus \mu(Q)| \leq n - a - b \), then \( P \) and \( Q \) have at least \( b \) blocks of size at least 2 in common and \( |\mu(P) \cap \mu(Q)| \geq 2b \).

Proof. Since \( |\tau(P) \setminus \mu(Q)| \leq n - a - b \), \( P \) and \( Q \) have at most \( n - a - b \) singletons in common. Now, \( |P \cap Q| \geq n - a \) means that \( P \) and \( Q \) have at least \( n - a \) blocks in common. Therefore, \( P \) and \( Q \) must have at least \( b \) blocks of size at least 2 in common.

Let \( W_1, \ldots, W_b \in P \cap Q \) with \( |W_i| \geq 2 \) for all \( i \). Then \( \bigcup_{i=1}^{b} W_i \subseteq \mu(P) \cap \mu(Q) \) and \( W_i \cap W_j = \emptyset \) for \( i \neq j \). This implies that \( 2b \leq \sum_{i=1}^{b} |W_i| \leq |\mu(P) \cap \mu(Q)| \).

Lemma 7. If \( \mathcal{A} \in I(n, n - r) \), then \( \max_{P \in \mathcal{A}} |\mu(P)| \leq 2r \).

Proof. Suppose \( \max_{P \in \mathcal{A}} |\mu(P)| = 2r + s \) where \( s \geq 1 \). Let \( P_0 \in \mathcal{A} \) with \( |\mu(P_0)| = 2r + s \). Then \( |\tau(P_0)| = n - 2r - s \). Note that \( |P_0| \geq n - r \) for \( \mathcal{A} \in I(n, n - r) \). By Lemma 6 (take \( Q = P = P_0 \) with \( a = r, b = r + s \)), we have \( 2r + s = |\mu(P_0)| \geq 2(r + s) \). Thus, we have \( s \leq 0 \), a contradiction. Hence, the lemma follows.
The following theorem says that the family $F_{2t+1}(i_0)$ is preserved when ‘undoing’ the splitting operations.

**Theorem 8.** If $t \geq 1$, $n \geq 5t + 3$, $A \in I(n,n - 2t - 1)$ and $S_{ij}(A) = F_{2t+1}(i_0)$, then $A = F_{2t+1}(i_0)$.

**Proof.** Suppose $A \not\subseteq F_{2t+1}(i_0)$. Then $\max_{P \in A} |\mu(P) \cap ([n] \setminus \{i_0\})| = t + s$ with $s \geq 1$. Let $P_0 \in A$ with $|\mu(P_0) \cap ([n] \setminus \{i_0\})| = t + s$. Then $|\mu(P_0)| = t + 1 + s$ or $t + s$, depending on whether $i_0 \notin \mu(P_0)$ or not. By Lemma 7, $|\mu(P_0)| \leq 4t + 2$. Since $n \geq 5t + 3$, there is a $t$-set $T \subseteq [n] \setminus (\mu(P_0) \cup \{i_0\})$. Let $A_1 = \{\{x\} : x \in [n] \setminus (T \cup \{i_0\})\} \cup \{T \cup \{i_0\}\}$. Then $A_1 \subseteq F_{2t+1}(i_0) = S_{ij}(A)$.

Now, $|\tau(P_0) \setminus (T \cup \{i_0\})| = n - 2t - 1 - s$, $|T \cup \{i_0\}| = t + 1 \geq 2$ and $(T \cup \{i_0\}) \not\subseteq \mu(P_0)$. The only block in $A_1$ that has size greater than one is $T \cup \{i_0\}$. Since $(T \cup \{i_0\}) \not\subseteq \mu(P_0)$, $T \cup \{i_0\} \not\subseteq P_0$. So, $P_0$ and $A_1$ have singletons in common only. Note that the number of singletons that $P_0$ and $A_1$ have in common is exactly $|\tau(P_0) \setminus (T \cup \{i_0\})| = n - 2t - 1 - s$. Thus, $|P_0 \cap A_1| \leq n - 2t - 1 - s \leq n - 2t - 2$. This means that $A_1 \notin A$ and $A_1 = S_{ij}(C_1)$ for some $C_1 \in A$.

Now, there are two possibilities for $C_1$ depending on whether $j$ is in $T \cup \{i_0\}$ or not:

(i) $C_1 = \{\{x\} : x \in [n] \setminus (T \cup \{i_0\})\} \cup \{T \cup \{i_0\}\}$ and $j \in T \cup \{i_0\}$.

(ii) $C_1 = \{\{x\} : x \in [n] \setminus (T \cup \{i_0, j\})\} \cup \{(T \cup \{i_0\}, \{i, j\}\}$.

If (i) holds, then $(T \cup \{i_0\}) \not\subseteq P_0$ since $(T \cup \{i_0\}) \not\subseteq \mu(P_0)$, and $|\tau(P_0) \setminus (T \cup \{i_0\})| \leq |\tau(P_0) \setminus (T \cup \{i_0\})| = n - 2t - 1 - s$. Thus, $|P_0 \cap C_1| \leq n - 2t - 1 - s \leq n - 2t - 2$, a contradiction.

Suppose (ii) holds. The number of singletons that $P_0$ and $C_1$ have in common is at most $|\tau(P_0) \setminus (T \cup \{i_0, j\})| \leq |\tau(P_0) \setminus (T \cup \{i_0\})| \leq n - 2t - 1 - s$. Recall that $T \cup \{i_0\} \not\subseteq P_0$. If $\{i, j\} \notin P_0$, then $|P_0 \cap C_1| \leq n - 2t - 1 - s \leq n - 2t - 2$, a contradiction.

If $\{i, j\} \in P_0$, then $P_0 \cap C_1 \leq n - 2t - s$. Since $|P_0 \cap C_1| > n - 2t - 1$, we must have $s = 1$, $|P_0 \cap C_1| = n - 2t - 1$, and $C_1 = \{\{x\} : x \in [n] \setminus (T \cup \{i_0, j\})\} \cup \{(T \cup \{i_0\}, \{i, j\}\}$. But then $|\mu(C_1) \setminus ([n] \setminus \{i_0\})| = |T \cup \{i, j\}| = t + 2 > t + 1 = |\mu(P_0) \cap ([n] \setminus \{i_0\})|$, contradicting the choice of $P_0$. Thus, $A \subseteq F_{2t+1}(i_0)$. By Lemma 4, $|A| = |S_{ij}(A)| = |F_{2t+1}(i_0)|$. Hence, $A = F_{2t+1}(i_0)$.

\[ \square \]

3 Main result

**Lemma 9.** Let $t \geq 1$, $A \subseteq B(n)$ and $W \subseteq [n]$. Suppose that $|W| \leq q$ and $|\mu(P) \setminus W| \leq t - 1$ for all $P \in A$. Then there exists an $n_0 = n_0(q,t)$ such that for all $n \geq n_0$,

$$|A| < n^{t - 0.5}.$$  

**Proof.** Note that for each $P \in A$,

$$\mu(P) = C_1 \cup C_2,$$
where \( C_1 \subseteq [n] \setminus W, |C_1| \leq t - 1 \) and \( C_2 \subseteq W \). The number of such \( C_1 \) is at most
\[
\sum_{i=0}^{t-1} \binom{n - |W|}{i} \leq \sum_{i=0}^{t-1} \binom{n}{i},
\]
and the number of such \( C_2 \) is at most \( 2^{|W|} \leq 2^n \). Furthermore, \(|\mu(P)| = |\mu(P) \setminus W| + |\mu(P) \cap W| \leq t - 1 + q \). Therefore the number of \( Q \in \mathcal{A} \) with \( |\mu(Q) = \mu(P)| \) is at most \( \tilde{B}_{|\mu(P)|} \leq \tilde{B}_{t-1+q} \), where \( \tilde{B}_m \) is the number of singleton-free set partitions of \([m]\). Thus
\[
|\mathcal{A}| \leq \tilde{B}_{t-1+q} 2^n \sum_{i=0}^{t-1} \binom{n}{i}.
\]
If \( t = 1 \), then \( |\mathcal{A}| \leq \tilde{B}_q 2^n < n^{0.5} \) provided that \( n \geq (B_q 2^n)^2 \). Suppose \( t \geq 2 \). Then
\[
|\mathcal{A}| \leq \tilde{B}_{t-1+q} 2^n \left( 1 + \sum_{i=1}^{t-1} \frac{n^i}{i!} \prod_{j=1}^{i-1} \left( 1 - \frac{j}{n} \right) \right)
< \tilde{B}_{t-1+q} 2^n \left( 1 + \sum_{i=1}^{t-1} \frac{n^{t-1}}{i!} \right)
= \tilde{B}_{t-1+q} 2^n t^{t-1} < n^{t-0.5},
\]
provided that \( n \geq (\tilde{B}_{t-1+q} 2^n t)^2 \). This completes the proof of the lemma. \( \square \)

**Lemma 10.** For \( t \geq 2 \), there exists an \( n_0 = n_0(t) \) such that for all \( n \geq n_0 \), if \( \mathcal{A} \in I(n, n - 2t) \), then
\[
|\mathcal{A}| \leq |\mathcal{F}_{2t}|.
\]
Moreover, equality holds if and only if \( \mathcal{A} = \mathcal{F}_{2t} \).

**Proof.** Suppose \( \mathcal{A} \not\subseteq \mathcal{F}_{2t} \). Then \( \max_{P \in \mathcal{A}} |\mu(P)| = t + s \) with \( s \geq 1 \). Let \( P_0 \in \mathcal{A} \) with \( |\mu(P_0)| = t + s \). By Lemma 7, \( \max_{P \in \mathcal{A}} |\mu(P)| \leq 4t \).

**Claim*.** \(|\mu(P) \setminus \mu(P_0)| \leq t - 1 \) for all \( P \in \mathcal{A} \).

Suppose there is a \( Q \in \mathcal{A} \) with \(|\mu(Q) \setminus \mu(P_0)| \geq t \). Then \(|\tau(P_0) \setminus \mu(Q)| \leq n - 2t - s \). Since \(|P_0 \cap Q| \geq n - 2t \), by Lemma 6, \(|\mu(P_0) \cap \mu(Q)| \geq 2s \). Therefore \(|\mu(Q)| = |\mu(Q) \setminus \mu(P_0)| + |\mu(P_0) \cap \mu(Q)| \geq t + 2s \). On the other hand, \(|\mu(Q)| \leq |\mu(P_0)| = t + s \) by the choice of \( P_0 \). This implies that \( s \leq 0 \), a contradiction. Hence, the claim follows.

By Claim* and Lemma 9 (take \( W = \mu(P_0) \) and \( q = 4t \)), \( |\mathcal{A}| < n^{t-0.5} \). Note that \( \tilde{B}_t \geq \tilde{B}_2 = 1 \) for \( t \geq 2 \). So, by equation (1),
\[
|\mathcal{F}_{2t}| = \sum_{i=0}^{t} \tilde{B}_i \binom{n}{i} > \tilde{B}_t \binom{n}{t} > \frac{1}{t!} \prod_{j=0}^{t-1} (n - j) \geq \frac{n^t}{t! 2^{t-1}} > n^{t-0.5},
\]
provided that \( n \geq \max \left( \left( t! 2^{t-1} \right)^2, 2t - 2 \right) \). Thus, \( |\mathcal{A}| < |\mathcal{F}_{2t}| \).

Suppose \( \mathcal{A} \subseteq \mathcal{F}_{2t} \). Then \( |\mathcal{A}| \leq |\mathcal{F}_{2t}| \) and equality holds if and only if \( \mathcal{A} = \mathcal{F}_{2t} \). \( \square \)
Lemma 11. For \( t \geq 1 \), there exists an \( n_0 = n_0(t) \) such that for all \( n \geq n_0 \), if \( A \in I(n, n - 2t - 1) \) and \( A \) is compressed, then

\[
|A| \leq |\mathcal{F}_{2t+1}(1)|.
\]

Moreover, equality holds if and only if \( A = \mathcal{F}_{2t+1}(i_0) \) for some \( i_0 \in [n] \).

Proof. Since \( t \geq 1 \), \( \tilde{B}_{t+1} = \tilde{B}_2 = 1 \). Therefore, by equation (2), for all \( a \in [n] \),

\[
|\mathcal{F}_{2t+1}(a)| = \sum_{i=0}^{t} \tilde{B}_i \binom{n}{i} + \tilde{B}_{t+1} \binom{n-1}{t}
\]

\[
\geq \tilde{B}_t \binom{n}{t} + \binom{n-1}{t}
\]

\[
= \tilde{B}_t \binom{n}{t} + \frac{1}{t!} \prod_{j=0}^{t-1} (n-1-j)
\]

\[
\geq \tilde{B}_t \binom{n}{t} + \frac{n^t}{t!2^t}, \tag{3}
\]

provided that \( n \geq 2t \).

Suppose \( \max_{P \in A} |\mu(P)| \leq t \). Then \( |\mu(P) \cap ([n] \setminus \{1\})| \leq t \) for all \( P \in A \). Hence, \( A \subseteq \mathcal{F}_{2t+1}(1) \) and the lemma follows.

Suppose \( \max_{P \in A} |\mu(P)| = t + s \) with \( s \geq 1 \). Let \( P_0 \in A \) with \( |\mu(P_0)| = t + s \). By Lemma 7, \( \max_{P \in A} |\mu(P)| \leq 4t + 2 \).

Claim**. If \( s \geq 2 \), then \( |\mu(P) \setminus \mu(P_0)| \leq t - 1 \) for all \( P \in A \).

Suppose there is a \( Q \in A \) with \( |\mu(Q) \setminus \mu(P_0)| \geq t \). Then \( |\tau(P_0) \setminus \mu(Q)| \leq n - 2t - s = n - 2t - 1 - (s - 1) \). Since \( |P_0 \cap Q| \geq n - 2t - 1 \), by Lemma 6, \( P_0 \) and \( Q \) have at least \((s - 1)\) blocks of size at least 2 in common and \( |\mu(P_0) \cap \mu(Q)| \geq 2(s - 1) \). Therefore

\[
|\mu(Q)| = |\mu(Q) \setminus \mu(P_0)| + |\mu(P_0) \cap \mu(Q)| \geq t + 2(s - 1).
\]

On the other hand, \( |\mu(Q)| \leq |\mu(P_0)| = t + s \) by the choice of \( P_0 \). This implies that \( s \leq 2 \). Since \( s \geq 2 \), we must have \( s = 2 \), \( |\mu(Q)| = |\mu(P_0)| = t + 2 \) and \( P_0 \) and \( Q \) have exactly one block of size 2 in common, say \( \{i, j\} \). Since \( A \) is compressed, \( s_{ij}(Q) \in A \). Note that \( \mu(s_{ij}(Q)) = \mu(Q) \setminus \mu(P_0) \). So,

\[
|\mu(s_{ij}(Q)) \setminus \mu(P_0)| = t \quad \text{and} \quad |\tau(P_0) \setminus \mu(s_{ij}(Q))| = n - 2t - 2 = n - 2t - 1 - 1.
\]

Since \( |P_0 \cap s_{ij}(Q)| \geq n - 2t - 1 \), by Lemma 6,

\[
|\mu(P_0) \cap \mu(s_{ij}(Q))| \geq 2.
\]

This contradicts that \( \mu(s_{ij}(Q)) = \mu(Q) \setminus \mu(P_0) \). Hence, the claim follows.

Suppose \( s \geq 2 \). By Claim** and Lemma 9 (take \( W = \mu(P_0) \) and \( q = 4t + 2 \)), \( |A| < n^{t-0.5} \) for sufficiently large \( n \). It then follows from equation (3) that

\[
|A| < n^{t-0.5} < \frac{n^t}{t!2^t} \leq |\mathcal{F}_{2t+1}(1)|,
\]

if \( n \geq (t!2^t)^2 \).

Suppose \( s = 1 \). Then \( \mu(P) \leq t + 1 \) for all \( P \in A \). Let \( P_0, P_1, \ldots, P_m \in A \) be such that for all \( 1 \leq i \leq m \), we have
We may assume that $m$ is the largest integer in the sense that there is no $R \in \mathcal{A}$ with $|\mu(R)| = t + 1$ and $|\mu(R) \setminus (\bigcup_{j=0}^{m} \mu(P_j))| = t$.

If there is a $Q \in \mathcal{A}$ with $|\mu(Q) \setminus (\bigcup_{j=0}^{m} \mu(P_j))| \geq t + 1$, then $|\mu(Q)| = t + 1$ and $\mu(Q) \cap \mu(P_0) = \emptyset$. So, $|P_0 \cap Q| = |\tau(P_0) \setminus \mu(Q)| = n - 2t - 2 < n - 2t - 1$, a contradiction. Thus, $|\mu(P) \setminus (\bigcup_{j=0}^{m} \mu(P_j))| \leq t$ for all $P \in \mathcal{A}$, and $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ where

$$\mathcal{A}_1 = \left\{ P \in \mathcal{A} : \left| \mu(P) \setminus \left( \bigcup_{j=0}^{m} \mu(P_j) \right) \right| \leq t - 1 \right\},$$

$$\mathcal{A}_2 = \left\{ P \in \mathcal{A} : \left| \mu(P) \setminus \left( \bigcup_{j=0}^{m} \mu(P_j) \right) \right| = t \text{ and } |\mu(P)| = t \right\}.$$

Suppose $m \leq t$. Then $\left| \bigcup_{j=0}^{m} \mu(P_j) \right| \leq \sum_{j=0}^{m} |\mu(P_j)| \leq (t + 1)^2$. By Lemma 9 (take $W = \bigcup_{j=0}^{m} \mu(P_j)$ and $q = (t + 1)^2$), $|\mathcal{A}_1| < n^{t-0.5}$ for sufficiently large $n$. Note that the number of $\mu(R)$ with $R \in \mathcal{A}$ and $|\mu(R)| = t$ is at most $\binom{n}{t}$ and the number of $Q \in \mathcal{A}$ with $\mu(Q) = \mu(R)$ is at most $\tilde{B}_t$. Thus,

$$|\mathcal{A}_2| \leq \tilde{B}_t \binom{n}{t}.$$

It then follows from equation (3) that

$$|\mathcal{A}| \leq |\mathcal{A}_1| + |\mathcal{A}_2| < n^{t-0.5} + \tilde{B}_t \binom{n}{t} < \frac{n^t}{t!2^t} + \tilde{B}_t \binom{n}{t} \leq |\mathcal{F}_{2t+1}(1)|,$$

if $n \geq (t!2)^2$.

Suppose $m \geq t + 1$.

**Claim**. There is a $i_0 \in [n]$ with $i_0 \in P_i$ for $i = 0, 1, 2, \ldots, t + 1$.

Note that if $\mu(P_i) \cap \mu(P_j) = \emptyset$ for $i \neq j$, then $|P_i \cap P_j| = |\tau(P_i) \setminus \mu(P_j)| = n - 2t - 2 < n - 2t - 1$, a contradiction. So, $\mu(P_i) \cap \mu(P_j) \neq \emptyset$ for $i \neq j$. By properties (i) and (ii), we may conclude that $|\mu(P_i) \cap \mu(P_j)| = 1$ for all $i, j$ with $i \neq j$.

Let $\mu(P_i) \cap \mu(P_0) = \{i_0\}$, $\mu(P_i) \cap \mu(P_0) = \{j_1\}$ and $\mu(P_i) \cap \mu(P_0) = \{j_2\}$ where $2 \leq i \leq t + 1$. Since $|\mu(P_i)| = t + 1$ and $|\mu(P_i) \setminus (\bigcup_{j=0}^{m-1} \mu(P_j))| = t$, $j_1 = j_2 \in \mu(P_i) \cap \mu(P_0) = \{i_0\}$. Thus, $i_0 \in P_i$ for $i = 0, 1, 2, \ldots, t + 1$. This completes the proof of the claim.

By Claim***,

$$\mu(P_i) = W_i \cup \{i_0\},$$
for \( i = 0, 1, \ldots, t + 1 \) and \( W_i \cap W_j = \emptyset \) for \( i \neq j \). Suppose \( A \not\subseteq \mathcal{F}_{2t+1}(i_0) \). Then there is a \( Q \in A \) with \( |\mu(Q) \cap ([n] \setminus \{i_0\})| = t + 1 \), i.e., \( |\mu(Q)| = t + 1 \) and \( i_0 \notin \mu(Q) \). Note that \( \mu(Q) \cap \mu(P_i) \neq \emptyset \) for all \( i \), for otherwise, \( |Q \cap P_i| = |\tau(Q) \setminus \mu(P_i)| = n - 2t - 2 < n - 2t - 1 \). Therefore \( \mu(Q) \cap W_i \neq \emptyset \). Since \( W_i \cap W_j = \emptyset \) for \( i \neq j \), \( \mu(Q) \) will have at least \( t + 2 \) elements, a contradiction. Hence, \( A \subseteq \mathcal{F}_{2t+1}(i_0) \), \( |A| \leq \mathcal{F}_{2t+1}(1) \) and equality holds if and only if \( A = \mathcal{F}_{2t+1}(i_0) \).

This completes the proof of the lemma.

Proof of Theorem 3. If \( r = 2t \), then the theorem follows from Lemma 10. Suppose \( r = 2t + 1 \). By repeatedly applying the splitting operations, we eventually obtain a compressed family \( A^* \in I(n, n - 2t - 1) \) with \( |A^*| = |A| \) (Lemma 5). It then follows from Lemma 11 that \( |A| = |A^*| \leq \mathcal{F}_{2t+1}(1) \) and equality holds if and only if \( A^* = \mathcal{F}_{2t+1}(i_0) \) for some \( i_0 \in [n] \). By Theorem 8, we may conclude that \( A^* = \mathcal{F}_{2t+1}(i_0) \) implies that \( A = \mathcal{F}_{2t+1}(i_0) \).

This completes the proof of Theorem 3.

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References


