Sprague-Grundy Values of the $\mathcal{R}$-Wythoff Game

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Abstract
We examine the Sprague-Grundy values of the game of $\mathcal{R}$-Wythoff, a restriction of Wythoff’s game introduced by Ho, where each move is either to remove a positive number of tokens from the larger pile or to remove the same number of tokens from both piles. Ho showed that the $P$-positions of $\mathcal{R}$-Wythoff agree with those of Wythoff’s game, and found all positions of Sprague-Grundy value 1. We describe all the positions of Sprague-Grundy value 2 and 3, and also conjecture a general form of the positions of Sprague-Grundy value $g$.

Keywords: Wythoff’s Game; Sprague-Grundy values

1 Introduction

Wythoff’s Game is a two-player impartial game played with two piles of tokens. Players alternate turns and for each move a player can remove either a positive number of tokens from one pile, or the same positive number of tokens from both piles. The last player to move wins.

$\mathcal{R}$-Wythoff is a restriction of Wythoff’s game introduced by Ho [3] where each move is either to remove a positive number of tokens from the larger pile or to remove the same number of tokens from both piles.

From here on, we assume that both players play optimally - that is, every move leads to the best possible outcome for that player regardless of his opponent’s responses. An $N$-position of the game is one where the next player to move wins, and a $P$-position is one where the previous player wins. A generalization of these concepts is given by the Sprague-Grundy function $\mathcal{G}$, defined as follows:

- The terminal position has Sprague-Grundy value 0.
Let $\mathbb{N}_0$ be the set of non-negative integers. Given a finite subset $S \in \mathbb{N}_0$, the minimal excludant of $S$ is $\text{mex}(S) = \min(\mathbb{N}_0 \setminus S)$, or the smallest non-negative integer not in $S$. The Sprague-Grundy value of a position $p$ is defined recursively as $G(p) = \text{mex}\{G(q) : q \in F\}$, where $F$ is the set of all positions reachable in one move from $p$.

This function generalizes $P$- and $N$- positions because the $P$-positions of any game are exactly the positions with Sprague-Grundy value 0. Additionally, knowing the Sprague-Grundy function of individual combinatorial games allows fast calculation of the Sprague-Grundy function, and hence winning strategy, of the sum of these games.

1.1 Previous Results

Wythoff gave a simple closed form for the $P$-positions of his game.

**Theorem 1** ([6]). The $P$-positions of Wythoff’s game are $(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor)$ and $(\lfloor \phi^2 n \rfloor, \lfloor \phi n \rfloor)$ for $n \geq 0$, where $\phi = \frac{1 + \sqrt{5}}{2}$ is the golden ratio.

For the game of $R$-Wythoff, Ho proved the remarkable fact that the positions of Sprague-Grundy value 0 are exactly the same set as those of Wythoff’s game. He additionally showed that the positions of Sprague-Grundy value 1 are exactly the translations of the $P$-positions by $-1$ in both dimensions, with finite exceptions [3].

However, aside from these positions with Sprague-Grundy value 0, the Sprague-Grundy values of Wythoff’s game are quite chaotic; for example, no polylogarithmic algorithm has been found to determine the Sprague-Grundy value of a given position [5]. Some authors analyzed positions of a fixed Sprague-Grundy value. Blass and Fraenkel looked at all positions of Sprague-Grundy value 1 [1], and Nivasch analyzed positions of value $g$ for an arbitrary fixed value [5].

Here we analyze the set of positions of a fixed Sprague-Grundy value for the game $R$-Wythoff. We determine all positions of Sprague-Grundy value 2 and 3, and conjecture that for any constant $g$, the set of all positions having Sprague-Grundy value $g$ has a form similar to that of $2-$ and $3-$ positions. If this conjecture is true, then all positions of value $g$ can be characterized after finite computation, in contrast to existing results on Wythoff’s game.

1.2 Notation

Let $(a, b)$ represent a position in $R$-Wythoff. If $G(a, b) = g$, we call $(a, b)$ a $g$-position.

A position $q$ is a follower of $p$ if $p \rightarrow q$ is a valid move. If $G(q) = g$ we will sometimes call $q$ a $g$-follower of $p$.

When plotting values of $G$ we will use the following graphical representation. The first coordinate is plotted vertically, increasing upwards, and the second coordinate is plotted horizontally (see Figure 1). Consequently we call row $r$ the set of points $(r, x)$ for $x \geq 0$ and column $c$ the set of points $(x, c)$ for $x \geq 0$. Also, we call diagonal $d$ the set of points $(x, x + d)$ for $x \geq 0$. In general, when talking about a position $(a, b)$ we assume $a \leq b$. 


Let \( T_g = ((a^g_0, b^g_0), (a^g_1, b^g_1), \ldots) \) denote the sequence of \( g \)-values having \( a^g_n \leq b^g_n \), in increasing order of first coordinate. For convenience let \( p^g_n = (a^g_n, b^g_n) \) and \( d^g_n = b^g_n - a^g_n \). When it is clear that we are talking about a specific \( g \), the superscript will sometimes be dropped.

## 2 Computing Sprague-Grundy Values

### 2.1 Computing \( g \)-positions

We first present an algorithm, based off one of Blass and Frankel’s for Wythoff’s game [1], which computes the sequence \( T_g \) for any positive integer \( g \). Suppose we have already know \( T_h \) for \( 0 \leq h < g \). Now suppose we have used Algorithm \( \text{RWSG} \) to compute \( p^g_i \) for \( i = 0, \ldots, k - 1 \). We run \( \text{RWSG} \) again to compute \( p^g_k \). Intuitively, it is greedily putting \( p^g_k \) into the smallest viable row and then the earliest column that does not lie on the same diagonal as an existing \( g \)-position.

**Algorithm \( \text{RWSG} \)**

1. \( p \leftarrow \text{mex}\{a^g_i, b^g_i : 0 \leq i < k\} \)
2. \( d \leftarrow \text{smallest non-negative integer such that}
   \begin{align*}
   &\text{(i) } d \not\in \{d^g_i : 0 \leq i < k\} \text{ and} \\
   &\text{(ii) } (p, p + d) \not\in T_h \text{ for } 0 \leq h < g
   \end{align*}
3. \( (a^g_k, b^g_k) \leftarrow (p, p + d) \)

We will now prove the correctness of \( \text{RWSG} \) on computing \( T_g \) given that \( T_0, \ldots, T_{g-1} \) are known.

**Theorem 2.** The sequence determined by Algorithm \( \text{RWSG} \) is exactly the sequence \( T_g \) of positions with Sprague-Grundy value \( g \).
Proof. We proceed by induction on \(k\), where we have used the algorithm to compute \(p_k^g\). For \(k = 0\) the algorithm computes \(p_0^g\) correctly as \((0, g)\), since \(p_0^h = (0, h)\) for \(0 \leq h < g\). Now suppose we have correctly computed \(p_0^g, \ldots, p_{k-1}^g\). Let \((p, p + d)\) be the position \(\text{RWSG}\) computes next; we must show it is \(p_k^g\).

First, note that \(a_k^g \geq p\). This follows from the fact that \(a_k^g \neq a_i^g\) and \(a_k^g \neq b_i^g\) for any \(0 \leq i < k\).

Next, each of the positions \((p, p + i) : 0 \leq i < d\) must have not satisfied either 2(i) or 2(ii), or the algorithm would have selected position \((p, p + i)\) instead of \((p, p + d)\). Thus each of these positions either lies in \(T_h\) for some \(h < g\), or has a \(g\)-position as a follower along a diagonal. In particular, \(G(p, p + i) \neq g\) for all \(0 \leq i < d\).

Now by Step 2(ii), \((p, p + d) \notin T_0, \ldots, T_{g-1}\), so \(G(p, p + d) \geq g\).

Finally, we deduce that \(G(p, p + d) = g\). Otherwise, \((p, p + d)\) has a follower that is a \(g\)-position from the definition of mex. But we just observed that no position \((p, p + i) : 0 \leq i < d\) is a \(g\)-position, so it has no \(g\)-position left of it along its row. Also, by Step 2(i) it has no \(g\)-position on its diagonal. So \((p, p + d)\) does not have a \(g\)-follower, implying that \(G(p, p + d) \leq g\).

We conclude that \(G(p, p + d) = g\), and since \(a_k^g \geq p\) we must have \(p_k^g = (a_k^g, b_k^g) = (p, p + d)\).

In practice, when using \(\text{RWSG}\) to compute \(T_g\) it suffices to pick a large upper bound \(U\) and compute \(T_h\) for \(0 \leq h < g\) up to all \(a_i^h \leq U\), and then compute \(T_g\) up to \(a_i^g \leq U\).

### 2.2 Alternate Proofs of Previous Results

As a direct consequence of Step 1 of this algorithm, we have an alternate proof of two theorems of Ho.

**Theorem 3** (Theorem 7 of [3]). For integers \(a\) and \(c\), there exists an integer \(b\) such that \(G(a, b) = c\).

**Theorem 4** (Theorem 8 of [3]). For nonnegative integers \(a\) and \(c\), there exists a unique \(b\) such that \(G(b, a + b) = c\).

*Proof.* Uniqueness is immediate since no \(g\)-position can be a follower of another. For existence, it suffices to show that the value \(a\) is chosen in Step 2 at some iteration in \(\text{RWSG}\). Step 2 will only choose a value greater than \(a\) if \(a\) fails (i) or (ii). If it fails (i), then some \(g\)-position already lies on diagonal \(a\) and we are done. It can only fail (ii) a finite number of times, once for each \(h < g\). Therefore Step 2 will set \(d = a\) after a finite number of iterations. \(\square\)

### 3 Characterization of \(g\)-Positions

Ho determined the positions of Sprague-Grundy value 0 and 1.

**Theorem 5** (Theorem 2 of [3]). The position \((a, b)\) with \(a \leq b\) is a \(P\)-position if and only if it is of the form \((\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor)\) for \(n \geq 0\).
Theorem 6 (Theorem 4 of [3]). The position \((a, b)\) with \(a \leq b\) has Sprague-Grundy value 1 if and only if \((a, b)\) is an element of the set
\[
\{(2, 2), (4, 6), ([\phi n] - 1, [\phi^2 n] - 1) \mid n \geq 1, n \neq 2\}
\]

3.1 Positions of Sprague-Grundy value 2 and 3

We begin by giving all positions of Sprague-Grundy value 2. Recall that \(p_n^0\) denotes the \(n\)th 0-position and for shorthand define \(p_n^0 - (x, y) = ([\phi n] - x, [\phi^2 n] - y)\); that is, subtracting positions is done by coordinate-wise subtraction.

Theorem 7. The first 10 values of \(T_2\) are \((0, 2), (1, 1), (3, 4), (5, 8), (6, 11), (7, 11), (9, 16), (10, 16), (12, 21), (13, 21)\).

Define the sequences
\[
\begin{align*}
(m_1^k)_{k \geq 0} & : m_1^0 = 10, m_1^1 = 17, m_1^{k+2} = m_1^{k+1} + m_1^k \\
(m_2^k)_{k \geq 0} & : m_2^0 = 11, m_2^1 = 18, m_2^{k+2} = m_2^{k+1} + m_2^k \\
(m_3^k)_{k \geq 0} & : m_3^0 = 12, m_3^1 = 20, m_3^{k+2} = m_3^{k+1} + m_3^k \\
(m_4^k)_{k \geq 0} & : m_4^0 = 15, m_4^1 = 24, m_4^{k+2} = m_4^{k+1} + m_4^k
\end{align*}
\]

For \(n \geq 10\), \(p_n^2 =
\[
\left\{
\begin{array}{ll}
p_n^0 - (2, 2) & \text{if } n = m_1^k, \ k \text{ even} \\
p_n^0 - (4, 4) & \text{if } n = m_1^k, \ k \text{ odd} \\
p_n^0 - (2, 2) & \text{if } n = m_2^k, \ k \text{ even} \\
p_n^0 - (4, 4) & \text{if } n = m_2^k, \ k \text{ odd} \\
p_n^0 - (2, 2) & \text{if } n = m_3^k, \ k \text{ even} \\
p_n^0 - (4, 4) & \text{if } n = m_3^k, \ k \text{ odd} \\
p_n^0 - (4, 4) & \text{if } n = m_4^k, \ k \text{ even} \\
p_n^0 - (2, 2) & \text{if } n = m_4^k, \ k \text{ odd} \\
p_n^0 - (3, 3) & \text{otherwise.}
\end{array}
\right.
\]

We first show that the above formula is well-defined because no \(n\) can fall into multiple cases.

Proposition 1. The \((m_k^i)\) are disjoint.

Proof. We can inductively show that \(m_k^1 < m_k^2 < m_k^3 < m_k^4 < m_{k+1}^1\). This is true for \(k = 0\) and \(k = 1\) by definition of the sequences (note that \(m_2^1 = 27 > m_4^1\)).

Then if
\[
\begin{align*}
m_k^1 & < m_k^2 < m_k^3 < m_k^4 < m_{k+1}^1 \\
m_{k+1}^1 & < m_{k+1}^2 < m_{k+1}^3 < m_{k+1}^4 < m_{k+2}^1
\end{align*}
\]

adding these inequality chains together yields \(m_{k+2}^1 < m_{k+2}^2 < m_{k+2}^3 < m_{k+2}^4 < m_{k+3}^1\). \(\square\)
Proposition 3. Define distribution of the elements of $T$ show that $T$.

Proposition 2. Consider a positive integer sequence $a_0, a_1, a_2, \ldots$ satisfying $a_{n+2} = a_{n+1} + a_n$ for $n \geq 0$. Then we can find constants $c_1$ and $c_2$ such that $a_n = c_1\phi^n + c_2\psi^n$, where $\phi = \frac{1 + \sqrt{5}}{2}$ and $\psi = -\frac{1}{\phi}$. Furthermore, if $|c_2| < \frac{1}{\psi^{-1} + \psi^1} = 1/\sqrt{5} \approx 0.447$, then

$$\left(\lfloor c_n \rfloor, \lfloor \phi^2 a_n \rfloor \right) = \begin{cases} (a_{n+1}, a_{n+2}) & \text{if } c_2(-1)^n \geq 0 \\ (a_{n+1} - 1, a_{n+2} - 1) & \text{if } c_2(-1)^n < 0 \end{cases}$$

Proof. The fact that we can write $a_n = c_1\phi^n + c_2\psi^n$ is a standard result of linear recurrences because $\phi, \psi$ are the roots of the characteristic equation $x^2 - x - 1 = 0$.

Therefore $\phi a_n = \phi(c_1\phi^n + c_2\psi^n) = c_1\phi^{n+1} - c_2\psi^{n+1} = c_1\phi^{n+1} + c_2\psi^{n+1} - (c_2\psi^{n+1} + c_2\psi^{n-1}) = a_{n+1} + c_2\psi^n(-\psi^{-1} - \psi^1)$. Then the expression for $\lfloor \phi a_n \rfloor$ follows immediately, and $\lfloor \phi^2 a_n \rfloor = \lfloor \phi a_n \rfloor + a_n$ which simplifies using $a_n + a_{n+1} = a_{n+2}$.

Let $T_2'$ denote the sequence of positions described in Theorem 7; our final goal is to show that $T_2' = T_2$, the sequence of 2-positions.

We use Proposition 2 to show the following proposition, which tells us about the distribution of the elements of $T_2'$.

Proposition 3. Define $p_n' = (a'_n, b'_n)$ to be the $n$th element of $T_2'$. Consider the sequence $S' := (a'_{10}, b'_{10}, a'_{11}, b'_{11}, a'_{12}, \ldots)$. Then this sequence contains no duplicates and covers the set

$$\mathbb{N}_0 \setminus \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 21\}.$$ 

Proof. We will compare the elements of $S'$ to the sequence

$$S := (\lfloor 10\phi \rfloor - 3, \lfloor 10\phi^2 \rfloor - 3, \lfloor 11\phi \rfloor - 3, \lfloor 11\phi^2 \rfloor - 3, \ldots)$$

Because $(\lfloor n\phi \rfloor)_{n \geq 1}$ and $(\lfloor n\phi^2 \rfloor)_{n \geq 1}$ are complementary Beatty sequences [2], the elements of $S$ are distinct and consist of the integers

$$\mathbb{N} \setminus \{\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor : 1 \leq n \leq 9\} = \mathbb{N}_0 \setminus \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 17, 20\}$$

By definition of $T_2'$, $S'$ and $S$ agree everywhere except along $(m^1), (m^2), (m^3), (m^4)$. It remains to compare these exceptions.

More precisely, let

$$M^1 := (\lfloor \phi m^1 \rfloor - 3, \lfloor \phi^2 m^1 \rfloor - 3, \lfloor \phi m^1 \rfloor - 3, \lfloor \phi^2 m^1 \rfloor - 3, \ldots)$$

$$M'^1 := (\lfloor \phi' m^1 \rfloor - 2, \lfloor \phi^2 m^1 \rfloor - 2, \lfloor \phi m^1 \rfloor - 4, \lfloor \phi^2 m^1 \rfloor - 4, \lfloor \phi m^1 \rfloor - 2, \ldots)$$

be the subsequences of $S$ and $S'$, respectively, along the indices given by $(m^1)$. Define $M^2, M'^2, M^3, M'^3, M^4, M'^4$ similarly.
Note that we can write $m_k^1 = (5 + \frac{12}{\sqrt{5}})\phi^k + (5 - \frac{12}{\sqrt{5}})\psi^k$ because it satisfies the Fibonacci recurrence, and $5 - \frac{12}{\sqrt{5}} \approx -0.367$. Therefore by Proposition 2, we can rewrite these sequences as

$$M^1 = (m_1^1 - 3, m_2^1 - 3, m_3^1 - 4, m_4^1 - 4, m_5^1 - 3, \ldots)$$
$$M^1 = (m_1^1 - 4, m_2^1 - 4, m_3^1 - 3, m_4^1 - 3, m_5^1 - 4, \ldots)$$

Thus $\{M^1\} = \{M^1\} \cup (m_1^1 - 3) \setminus (m_1^1 - 4) = \{M^1\} \cup 14 \setminus 13$.

Similarly,

$$m_k^2 = \left(\frac{11}{2} + \frac{25}{2\sqrt{5}}\right)\phi^k + \left(\frac{11}{2} - \frac{25}{2\sqrt{5}}\right)\psi^k$$
$$m_k^3 = \left(6 + \frac{14}{\sqrt{5}}\right)\phi^k + \left(6 - \frac{14}{\sqrt{5}}\right)\psi^k$$
$$m_k^4 = \left(\frac{15}{2} + \frac{33}{2\sqrt{5}}\right)\phi^k + \left(\frac{15}{2} - \frac{33}{2\sqrt{5}}\right)\psi^k$$

and $\frac{11}{2} - \frac{25}{2\sqrt{5}} \approx -0.090, 6 - \frac{14}{\sqrt{5}} \approx -0.261, \frac{15}{2} - \frac{33}{2\sqrt{5}} \approx 0.121$. Applying Proposition 2 in the same way as on $(m^1)$ gives $\{M^2\} = \{M^2\} \cup 15 \setminus 14, \{M^3\} = \{M^3\} \cup 17 \setminus 16$, and $\{M^4\} = \{M^4\} \cup 20 \setminus 21$.

Therefore

$$\{S'\} = \{S\} \setminus (\{M^1\} \cup \{M^2\} \cup \{M^3\} \cup \{M^4\}) \cup \{M^1\} \cup \{M^2\} \cup \{M^3\} \cup \{M^4\}$$
$$= \mathbb{N}_0 \setminus \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 21\}. \tag*{$\square$}$$

**Corollary 1.** The elements of $T_2'$ (and their reflections) cover every row.

**Proof.** It suffices to show that $\{a'_n, b'_n : n \geq 0\}$ covers $\mathbb{N}_0$. Using the notation of Proposition 3, this set can be written as $\{a'_n, b'_n : 0 \leq n < 10\} \cup \{S'\}$. The result follows immediately from the values defined in Theorem 7 and Proposition 3. \tag*{$\square$}

As a tool for proving Theorem 7, we will use the following characterization of the sequence $T_g$ due to Jiao [4].

**Proposition 4** (Lemma 8 in [4]). Every $T_g$ consists exactly of the positions having Sprague-Grundy value $g$ if and only if every $T_g$ satisfies

1. $T_g \cap T_h = \emptyset$ for $h < g$.
2. If $p \in T_g$, then $p$ has no follower in $T_g$.
3. If $p \not\in T_0 \cup \cdots \cup T_g$, then $p$ has a follower in $T_g$.

Now we can prove Theorem 7.
Proof of Theorem 7. It suffices to show that $T_2'$ satisfies the properties of Proposition 4. First, direct computation using Algorithm RWSG gives $p^2_0, \ldots, p^2_9$, and we can manually check that these values satisfy the appropriate properties (see Figure 2).

It suffices to check the properties for the rest of $T_2'$.

1. Note that the $n$th element of $T_2'$, $n \geq 10$ lies on diagonal $n$. By Theorem 4, every diagonal contains exactly one 0- and one 1-position. By Theorem 5 and Theorem 6 these positions are given by $p^0_n - (0, 0)$ and $p^1_n - (1, 1)$, respectively. In other words, they are offset from $p^0_n$ along the diagonal by 0, 1 respectively. However, the corresponding position in $T_2'$ is offset by 2 or 4. Therefore the positions in $T_2'$ do not collide with positions in $T_0$ and $T_1$.

2. Note that $\{b'_n - a'_n : 0 \leq n \leq 9\} = \{0, 1, 2, \ldots, 9\}$. Furthermore $b'_n - a'_n = n$ for $n \geq 10$. Therefore no two positions in $T_2'$ are diagonal followers. Finally, Proposition 3 and noting that $\{S'\}$ and $\{a'_n, b'_n : 0 \leq n \leq 9\}$ are disjoint imply that no two positions are row followers.

3. Let $q \notin T_0 \cup T_1 \cup T_2'$. We will find a $p \in T_2'$ that is a follower of $q$. Without loss of generality let $q$ be below the main diagonal and write $q$ as $(r, r + d)$ with $d \geq 0$. Let
$p \in T_2'$ be the unique element on diagonal $d$. If $p$ is below $q$ we are done, so assume otherwise. If $d \leq 9$ it is clear from inspecting Figure 2 that the claim holds. Suppose $d \geq 10$.

Then $p = p'_k$. By Corollary 1, there exists $k$ such that $a'_k = r$ or $b'_k = r$. In the latter case $(b'_k, a'_k)$ is a 2-follower of $q$. In the former case, $a'_k = r < a'_q$. Since $d \geq 10$ and $(a'_n : n \geq 10)$ is increasing, $k < d$. Then the diagonal $p'_d$ lies on is less than the diagonal $p'_d$ lies on, so $b'_k < r + d$, and $p'_k$ is a row follower of $q$.

Similarly to the positions of Sprague-Grundy value 2, we may determine positions of Sprague-Grundy value 3. To save space we let $F^{(a,b)}$ denote the sequence $(m_k)_{k \geq 0}: m_0 = a, m_1 = b, m_{k+2} = m_{k+1} + m_k$.

**Theorem 8.** The first 36 values of $T_3$ are $(0, 3)$, $(1, 3)$, $(2, 3)$, $(4, 4)$, $(5, 10)$, $(6, 12)$, $(7, 14)$, $(8, 12)$, $(9, 17)$, $(11, 20)$, $(13, 23)$, $(15, 27)$, $(16, 29)$, $(18, 29)$, $(19, 34)$, $(21, 37)$, $(22, 39)$, $(24, 38)$, $(25, 44)$, $(26, 44)$, $(28, 49)$, $(30, 50)$, $(31, 53)$, $(32, 55)$, $(33, 57)$, $(35, 60)$, $(36, 62)$, $(40, 67)$, $(41, 69)$, $(42, 71)$, $(43, 73)$, $(45, 76)$, $(46, 78)$, $(47, 80)$, $(48, 82)$, $(51, 86)$.

For $n \geq 36$,

$$p^3_n = p^0_n - (5, 5) + \begin{cases} ((-1)^{k+1}, (-1)^{k+1}) & \text{if } n = F^{(36, 58)}_k \\ ((-1)^k, (-1)^k) & \text{if } n = F^{(42, 68)}_k \\ ((-1)^{k+1}, (-1)^{k+1}) & \text{if } n = F^{(44, 71)}_k \\ ((-1)^k, (-1)^{k+1}) & \text{if } n = F^{(45, 72)}_k \\ ((-1)^{k+1}, (-1)^k) & \text{if } n = F^{(46, 74)}_k \\ ((-1)^{k+1}, (-1)^{k+1}) & \text{if } n = F^{(47, 76)}_k \\ ((-1)^k, (-1)^k) & \text{if } n = F^{(53, 86)}_k \\ ((-1)^k, (-1)^k) & \text{if } n = F^{(54, 88)}_k \\ ((-1)^k, (-1)^k) & \text{if } n = F^{(55, 89)}_k \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Follows similarly to the proof of Theorem 7. \(\square\)

### 3.2 Positions of Sprague-Grundy Value $g$

Next we turn out attention to greater $g$-positions. For the remainder of this section, we assume $g$ is a fixed integer greater than 3. When talking about positions $p^g_n = (a^g_n, b^g_n)$ we will drop the superscripts when it is clear we are using $g$.

We conjecture that in general, the positions of Sprague-Grundy value $g$ follow a similar pattern to that of the 2- and 3-positions.
Conjecture 1. There exists constants $N, C, k$ and disjoint sequences $F^{(m_1,n_1)}, F^{(m_2,n_2)}, \ldots, F^{(m_k,n_k)}$ and bits $x_i \in \{0, 1\}$ for $1 \leq i \leq k$ such that for all $n \geq N$,

$$(a_n, b_n) = \begin{cases} p_0^0 - (C + (-1)^{k+x_i}, C + (-1)^{k+x_i}) & \text{if } n = F^{(m_i,n_i)}_k \\ p_0^0 - (C,C) & \text{otherwise} \end{cases}$$

Essentially, there is a offset $C$ such that outside of a finite number of positions in $T_g$, the rest lie at $p_0^0 - (C,C)$ except along a finite number of Fibonacci-recurrence sequences, where they alternate above and below.

We reduce this to a weaker conjecture about the $g$-positions.

Conjecture 2. There exists a constant $M$ such that for all $n \geq M$, $d_n = n$.

We will show that Conjecture 2 implies Conjecture 1, and express $C$ in a computable way.

Definition. In a sequence $(c_n)_{n \geq 0}$, let an element $c_j$ be called a repeat if there exists some $i < j$ such that $c_i = c_j$.

Proposition 5. Suppose Conjecture 2 is true. Then Conjecture 1 is true and $C$ is equal to the total number of repeats in $(b_n)_{n \geq 0}$.

Proof. We start off with the following useful fact.

Claim 1. The sequence $b_M, b_{M+1}, b_{M+2}, \ldots$ is increasing and does not contain two consecutive integers.

Proof. For $n \geq M$, $b_{n+1} = a_{n+1} + n + 1 \geq a_n + 1 + n + 1 = b_n + 2$. □

Claim 1 implies the sequence $(b_k)_{k \geq 0}$ has a finite number of repeats. Let $R$ denote the total number of repeats. We will show Conjecture 1 holds and $C$ is this quantity $R$.

Without loss of generality, we also make the assumption that none of $b_M, b_{M+1}, \ldots$ are repeats of earlier elements; if this is not true, increase $M$ until it is true, and Conjecture 2 and Claim 1 still hold for this higher value of $M$.

For $n > M$, let $f(n)$ be defined such that $a_{f(n)} = b_n - 1$ and $a_{f(n)+1} = b_n + 1$. This is well-defined because Claim 1 implies that $b_n - 1$ and $b_n + 1$ are not in the sequence $(b_k)$, so they must be in the sequence $(a_k)$. Furthermore note that

$$f(n) = |\{a_0, a_1, \ldots \} \cap \{0, 1, \ldots, b_n - 2\}|$$

because $a_0, \ldots, a_{f(n)-1}$ are the only terms of $(a_k)$ that lie in the set $\{0, 1, \ldots, b_n - 2\}$.

The sequence $b_0, b_1, \ldots, b_{n-1}$ contains $n-R$ distinct values, so of the values $\{0, 1, \ldots, b_n - 2\}$ exactly $n-R$ of them appear in the sequence $(b_n)$ and the rest appear in $(a_n)$. Finally, there is exactly one $k$ such that $a_k = b_k$ by Theorem 4. Combining these facts gives

$$f(n) = b_n - n + R = a_n + R.$$
Now consider the “error” quantity \( x_n = a_n - (\phi n - R) \), similar to a technique used by Nivasch [5]. Using the above equations we can determine that

\[
x_{f(n)+1} = a_{f(n)+1} - \phi (f(n) + 1) + R = b_n + 1 - \phi(a_n + R + 1)
\]

\[
= a_n(1 - \phi) + n + R(1 - \phi) + 1 - \phi
\]

\[
= -\frac{1}{\phi} x_n - \frac{1}{\phi}
\]

and similarly

\[
x_{f(n)} = -\frac{1}{\phi} x_n - 1.
\]

Define the functions \( g_1(x) = -\frac{1}{\phi} x - \frac{1}{\phi} \) and \( g_2(x) = -\frac{1}{\phi} x - 1 \).

Recall that we are trying to show that the \( a_n \) tend to \( \phi n + R \), so it suffices to show that \( x_n \) is close to 0. To do this we will show that for large \( n \), \( x_n \) can be written in terms of many compositions of \( g_1 \) or \( g_2 \).

**Claim 2.** For every integer \( k \geq 1 \), there exists a constant \( M_k \) such that for all \( n \geq M_k \), there is a sequence \( n = n_1 > n_2 > \cdots > n_\ell \) such that \( \ell \geq k \), for each \( i \) either \( n_i = f(n_{i+1}) \) or \( n_i = f(n_{i+1}) + 1 \), and \( M_k \leq n_\ell \leq b_M + R \).

**Proof.** Consider some \( m > b_M + R \). Then there exists \( n \) such that \( m = f(n) \) or \( m = f(n) + 1 \): otherwise, neither \( m - R \) nor \( m - R - 1 \) are in \( (a_n) \), so they are in \( (b_n) \), contradicting Claim 1.

Now we induct on \( k \). For \( k = 1 \), we claim the constant \( M_1 = M \) works. Consider any \( n > M \). If \( n \leq b_M + R \) the singleton sequence consisting of itself works. Otherwise, choose \( n_2 \) such that \( n = f(n_2) \) or \( f(n_2) + 1 \), and continue until we reach \( n_\ell \leq b_M + R \).

Finally, suppose the statement is true for \( 1, \ldots, k-1 \). We claim \( M_k = f(M_{k-1}) + 1 \) works. Consider any \( n > M_k \). Let \( n_1 = n \) and let \( n_2 \) be such that \( n_1 = f(n_2) \) or \( f(n_2) + 1 \). Noting that \( n_1 > M_k \) implies \( n_2 > M_{k-1} \), and applying the inductive hypothesis finishes the claim. \( \square \)

Now consider the function \( h(x) = \min\{ |x - \alpha| : \alpha \in [-1, 0] \} = \max(x - 0, -1 - x, 0) \). We establish a few properties of \( h \) in relation to \( g_1 \) and \( g_2 \).

**Claim 3** (Properties of \( h \)).

(a) \( h(g_1(x)) \leq \frac{1}{\phi} h(x) \) and \( h(g_2(x)) \leq \frac{1}{\phi} h(x) \).

(b) If \( h(x) = 0 \), then \( h(g_1(x)) = h(g_2(x)) = 0 \).

(c) If \( 0 < h(x) < \frac{1}{\phi} \), then

i. If \( x > 0 \), then \( h(g_1(x)) = 0 \) and \( g_2(x) < -1 \).

ii. If \( x < -1 \), then \( h(g_2(x)) = 0 \) and \( g_1(x) > 0 \).
Proof. (a) We case on $x$. First suppose $x < 0$. Note that $g_i(x) > -1$. If $g_i(x) \leq 0$ then $h(g_i(x)) = 0$ and the inequality is true. In particular, if $x \geq -1$ then both $g_1(x), g_2(x) \leq 0$. So assume $x < -1$, and at least one $g_i(x) > 0$. Since $h(x) = -1 - x$, we have $h(g_2(x)) \leq h(g_1(x)) = -\frac{1}{\phi}(x + 1) = \frac{1}{\phi} h(x)$.

Next assume $x > 0$. Clearly $g_i(x) < 0$. If both $g_i(x) \geq -1$, then $h(g_i(x)) = 0$ and we are done. Otherwise, $h(g_2(x)) \leq h(g_1(x)) = -1 - (-\frac{1}{\phi} x - 1) = \frac{1}{\phi} h(x)$.

(b) If $h(x) = 0$, then $-1 \leq x \leq 0$. Then $0 \leq -\frac{1}{\phi} x \leq \frac{1}{\phi}$. This implies $-1 \leq g_i(x) \leq 0$, so $h(g_i(x)) = 0$.

(c) i. Assume $0 < x < \frac{1}{\phi}$. Then $-1 < g_1(x) < -\frac{1}{\phi}$ so $h(g_1(x)) = 0$, and $x > 0 \implies g_2(x) = -\frac{1}{\phi} x - 1 < 0$.

ii. Assume $-1 - \frac{1}{\phi} < x < -1$. Then $-1 + \frac{1}{\phi} < g_2(x) < 0$ so $h(g_2(x)) = 0$, and $-\frac{1}{\phi} x > \frac{1}{\phi} \implies g_1(x) > 0$.

Let $P = \max\{h(x_n) : M \leq n \leq b_M + R\}$. For every $n \geq M_k$, apply Claim 2 to find a sequence $n_1 > \cdots > n_\ell$. For each $1 \leq i < \ell$, $n_i = f(n_{i+1})$ or $f(n_{i+1}) + 1$, so $x_{n_i} = g_1(x_{n_{i+1}})$ or $x_{n_i} = g_2(x_{n_{i+1}})$. By Claim 3(a), $h(x_{n_i}) \leq \frac{1}{\phi} h(x_{n_{i+1}})$. Therefore $h(x_{n_1}) \leq \frac{1}{\phi^{\ell-1}} h(x_{n_\ell}) \leq \frac{1}{\phi^{\ell-1}} P$.

In particular, there exists a large $M'$ such that for all $n \geq M'$, $h(x_n) < \frac{1}{\phi}$.

Claim 4. For every $n \geq M'$, $h(x_n) < \frac{1}{\phi}$, and there is a sequence $n = n_1 > n_2 > \cdots > n_\ell$ such that for each $i$ either $n_i = f(n_{i+1})$ or $n_i = f(n_{i+1}) + 1$, and $M' \leq n_\ell \leq b_{M'} + R$.

Proof. Analogous to the proof of Claim 2.

Thus for every $n \geq M'$, by working backwards we can find a sequence $n = n_1 > n_2 > \cdots > n_\ell$ such that $x_{n_i} = g_1(x_{n_{i+1}})$ or $x_{n_i} = g_2(x_{n_{i+1}})$. Furthermore we can ensure that $n_k \leq b_{M'} + R$ and $n_{k-1} \leq f(b_{M'} + R) + 1$.

Finally, suppose that $h(x_n) > 0$. By Claim 3(b), none of the $n_i$ satisfy $h(x_{n_i}) = 0$. By Claim 3(c), the sequence $(n_i)$ satisfies either

$$x_{n_1} = g_1(x_{n_2}) = g_1(g_2(x_{n_3})) = g_1(g_2(g_1(x_{n_4}))) = \cdots$$

$$x_{n_1} = g_2(x_{n_2}) = g_2(g_1(x_{n_3})) = g_2(g_1(g_2(x_{n_4}))) = \cdots$$

This implies that any three consecutive terms of $(n_i)$ has the form $m, f(m), f(f(m)) + 1$ or $m, f(m) + 1, f(f(m) + 1)$. But by the definition and formula for $f$, we have

$$f(f(m)) + 1 = a_{f(m)} + R + 1 = b_m + R = m + f(m)$$

and similarly $f(f(m) + 1) = a_{f(m)+1} + R = b_n + 1 + R = n + (f(n) + 1)$. Therefore the sequence $(n_i)$ satisfies the Fibonacci recurrence.
To relate everything back to the \( g \)-positions \((a_n, b_n)\), knowing \( x_n \) tells us \( a_n \) since 
\[
x_n = a_n - (\phi n - R).
\]
For example, if \( x_n \in (0, 1] \), then 
\[
a_n = x_n + \phi n - R = 1 + \lfloor \phi n \rfloor - R.
\]
The three relevant cases are 
\[
0 < x_n \leq 1 \iff a_n = (\lfloor \phi n \rfloor - R) + 1
\]
\[
-1 < x_n \leq 0 \iff a_n = \lfloor \phi n \rfloor - R
\]
\[
-2 < x_n \leq -1 \iff a_n = \lfloor \phi n \rfloor - R - 1.
\]
In summary we have shown that for all \( n > M' \), 
\[
h(x_n) < 1/\phi \implies -2 < x_n < 1,
\]
which implies that \((a_n, b_n)\) has the form 
\[
p^g_n - (R, R) \text{ or } p^g_n - (R \pm 1, R \pm 1)
\]
according to the above cases. If it has the latter form, then we can find a sequence \( n = n_1 > n_2 > \cdots > n_\ell \) satisfying the Fibonacci recurrence where the terms alternate between the form \( a_{n_i} = p^g_{n_i} - (R + 1, R + 1) \) and \( a_{n_i} = p^g_{n_i} - (R - 1, R - 1) \). Since we put bounds on \( n_\ell \) and \( n_\ell - 1 \) there are a finite number of choices for this pair, but the sequence is uniquely determined by the two smallest values, so there are a finite number of these Fibonacci-recurrence “exception” sequences. We conclude that Conjecture 1 follows Conjecture 2.

4 Additional Comments

The best result we have shown toward Conjecture 2 is a bound on how far away \( b_n^g - a_n^g \) can be from \( n \) as a constant depending on \( g \).

**Proposition 6.** For all \( g \) and \( n \geq 0 \), 
\[
|d_n^g - n| \leq g.
\]

**Proof.** According to step 2 of Algorithm RWSG, a diagonal is only skipped when a previous \( g \)-position already occupies it, or any \( h \)-position for \( h < g \). When calculating \( p_n^g \), the former can occur \( n \) times, and the latter \( g \) times since there is at most one \( h \)-position per row. Thus \( d_n^g \leq n + g \).

Now suppose that \( d_n^g < n - g \). Then out of the \( n \) previous \( g \)-positions, more than \( g \) of them skipped diagonal \( d_n^g \). Since it had no \( g \)-positions up to this point, it could only have been skipped by step 2(ii) of RWSG because an \( h \)-position was already there. This can happen at most \( g \) times since there can be at most one \( h \)-position per diagonal, which is a contradiction.

Therefore \( n - g \leq d_n^g \leq n + g \) as desired.

4.1 Empirical Data and Further Directions

We have verified Conjecture 2, and hence Conjecture 1, up to \( g = 20 \) using Algorithm RWSG with computer assistance. Figure 3 give values for constants in the conjectures, where \( C_g \) denotes the value of \( C \) in Conjecture 1 for the \( g \)-positions and \( M_g \) denotes the value of \( M \) in Conjecture 2.

It would be helpful to determine more properties of these values. The following conjectures were attempted as progress towards Conjecture 1 and Conjecture 2.
Conjecture 3. For every \( g \), the sequence \( B^g = b^g_0, b^g_1, \ldots \) has a finite number of repeats.

Conjecture 3 would imply that the values \( C_g \) would exist independently of whether Conjecture 1 is true.

Intuitively, our work shows that having \( R \) repeats in \( B^g \) means the \( g \)-positions lie approximately at positions \( p_n^g - (R, R) \). If \( R \) grows much larger than any of the values \( C_0, C_1, \ldots, C_{g-1} \), this position will not collide with any positions of Sprague-Grundy value \( 0, 1, \ldots, g-1 \). Then Algorithm RWSG will greedily select the first available diagonal, which leads to the property in Conjecture 2, which bounds the total number of repeats.

Other partial results that may help are showing that no two \( C_g \) are consecutive, aside from \( C_0 \) and \( C_1 \) (this ensures that their positions are consistent with Conjecture 1), or showing some type of growth behavior of the \( C_g \). Indeed, any further information about \( (C_g) \) would be of interest.

Another interesting statistic is the number of \( n \) such that \( b^g_n - a^g_n \neq n \), or the number of \( g \)-positions which lie on the wrong diagonals. We can call \( p^g_n \) a wrong diagonal position if it does not lie on diagonal \( n \).

In fact, from viewing the empirical data, most of the repeats come from “single skipped diagonals,” which occur when \( p^g_n \) skips a diagonal and \( p^g_{n+1} \) fills it in; more precisely, when \( d^g_n = n + 1 \) and \( d^g_{n+1} = n \). Notice that when \( a^g_n \) and \( a^g_{n+1} \) are consecutive, \( b^g_n = b^g_{n+1} \). Furthermore this phenomenon occurs at a roughly fixed ratio: it is known that the sequence \( A^0 \) of P-positions Wythoff’s game has a \( 1/\phi \) proportion of consecutive \( a_n \) values. Looking at the empirical data of number of repeats \( C_g \) versus number of wrong diagonals positions \( M_g \), their ratio is a relatively stable constant that seems to hover between 4 and 7 (Figure 4). This supports the close relationship between repeats and wrong diagonals positions.

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