Some combinatorial arrays related to the Lotka-Volterra system

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Abstract

The purpose of this paper is to investigate several context-free grammars suggested by the Lotka-Volterra system. Some combinatorial arrays, involving the Stirling numbers of the second kind and Eulerian numbers, are generated by these context-free grammars. In particular, we present grammatical characterization of some statistics on cyclically ordered partitions.

Keywords: Lotka-Volterra system; Context-free grammars; Cyclically ordered partitions; Eulerian numbers

1 Introduction

Throughout this paper a context-free grammar is in the sense of Chen [4]: for an alphabet $A$, let $\mathbb{Q}[[A]]$ be the rational commutative ring of formal power series in monomials formed from letters in $A$. A context-free grammar over $A$ is a function $G : A \rightarrow \mathbb{Q}[[A]]$ that replace

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a letter in $A$ by a formal function over $A$. The formal derivative $D$ is a linear operator defined with respect to a context-free grammar $G$. More precisely, the derivative $D = D_G: \mathbb{Q}[[A]] \to \mathbb{Q}[[A]]$ is defined as follows: for $x \in A$, we have $D(x) = G(x)$; for a monomial $u$ in $\mathbb{Q}[[A]]$, $D(u)$ is defined so that $D$ is a derivation, and for a general element $q \in \mathbb{Q}[[A]]$, $D(q)$ is defined by linearity.

Let $[n] = \{1, 2, \ldots, n\}$. The Stirling number of the second kind $\{n\}_k$ is the number of ways to partition $[n]$ into $k$ blocks. Let $\mathfrak{S}_n$ be the symmetric group of all permutations of $[n]$. A descent of a permutation $\pi \in \mathfrak{S}_n$ is a position $i$ such that $\pi(i) > \pi(i + 1)$. Denote by $\text{des}(\pi)$ the number of descents of $\pi$. The Eulerian number $\langle n\rangle_k$ is the number of permutations in $\mathfrak{S}_n$ with $k - 1$ descents, where $1 \leq k \leq n$ (see [15, A008292]). Let us now recall two classical results.

**Proposition 1** ([4, Eq. 4.8]). For $A = \{x, y\}$ and $G = \{x \to xy, y \to y\}$, we have

$$D^n(x) = x \sum_{k=1}^{n} \binom{n}{k} y^k \quad \text{for } n \geq 1.$$  

**Proposition 2** ([6, Section 2.1]). For $A = \{x, y\}$ and $G = \{x \to xy, y \to xy\}$, we have

$$D^n(x) = \sum_{k=1}^{n} \langle n\rangle_k x^k y^{n-k+1} \quad \text{for } n \geq 1.$$  

One of the most commonly used models of two species predator-prey interaction is the classical Lotka-Volterra system:

$$\frac{dx}{dt} = x(a - by), \quad \frac{dy}{dt} = y(-c + dx),$$  

where $y(t)$ and $x(t)$ represent, respectively, the predator population and the prey population as functions of time, and $a, b, c, d$ are positive constants. The differential system (1) is ubiquitous and arises often in mathematical ecology, dynamical system theory and other branches of mathematics (see [2, 3]).

Motivated by (1), we shall consider context-free grammars of the form:

$$A = \{x, y\}, \quad G = \{x \to x + p(x, y), y \to y + q(x, y)\},$$  

where $p(x, y)$ and $q(x, y)$ are polynomials in $x$ and $y$. For convenience, we shall call

$$G' = \{x \to p(x, y), y \to q(x, y)\}$$

the ancestor of $G$.

This paper is a continuation of [4, 6, 12]. Throughout this paper, arrays are indexed by $n, i$ and $j$. Call $(a_{n, i, j})$ a combinatorial array if the numbers $a_{n, i, j}$ are nonnegative integers. For any function $H(x, p, q)$, we denote by $H_y$ the partial derivative of $H$ with respect to $y$, where $y \in \{x, p, q\}$. In the next section, we present grammatical characterization of some statistics on cyclically ordered partitions.
2 Some permutation statistics on cyclically ordered partitions

Recall that a partition $\pi$ of $[n]$, written $\pi \vdash [n]$, is a collection of disjoint and nonempty subsets $B_1, B_2, \ldots, B_k$ of $[n]$ such that $\bigcup_{i=1}^k B_i = [n]$, where each $B_i$ $(1 \leq i \leq k)$ is called a block of $\pi$. A cyclically ordered partition of $[n]$ is a partition of $[n]$ whose blocks are endowed with a cyclic order. We always use a canonical representation for cyclically ordered partitions, where the block containing 1 comes first and the integers in each block are in increasing order. For example, $(123)$, $(12)(3)$, $(13)(2)$, $(1)(23)$, $(1)(2)(3)$, $(1)(3)(2)$ are all cyclically ordered partitions of $[3]$. The opener of a block is its least element. For example, the list of openers of $(13)(2)$ and $(1)(3)(2)$ are respectively given by 12 and 132.

In this section, we shall study some statistics on the list of openers.

2.1 Descent statistic

Consider the grammar

$$G = \{ x \rightarrow x + xy, y \rightarrow y + xy \}. \tag{3}$$

The combinatorial context for the ancestor $G'$ of $G$ has been given in Proposition 2. From (3), we have

$$D(x) = x + xy,$$
$$D^2(x) = x + 3xy + xy^2 + x^2 y,$$
$$D^3(x) = x + 7xy + 6xy^2 + xy^3 + 6x^2 y + 4x^2 y^2 + x^3 y.$$

For $n \geq 0$, we define $D^n(x) = \sum_{i,j} a_{n,i,j} x^i y^j$. Since

$$D^{n+1}(x) = D \left( \sum_{i,j} a_{n,i,j} x^i y^j \right)$$
$$= \sum_{i,j} (i + j) a_{n,i,j} x^i y^j + \sum_{i,j} ia_{n,i,j} x^i y^{j+1} + \sum_{i,j} ja_{n,i,j} x^{i+1} y^j,$$

we get

$$a_{n+1,i,j} = (i + j) a_{n,i,j} + ia_{n,i-1,j} + ja_{n,i-1,j} \tag{4}$$

for $i, j \geq 1$, with the initial conditions $a_{0,i,j}$ to be 1 if $(i, j) = (1, 0)$, and to be 0 otherwise. Clearly, $a_{n,1,0} = 1$ and $a_{n,i,0} = 0$ for $i \geq 2$.

Example 3. The following table contains the values of $a_{n,i,j}$.

<table>
<thead>
<tr>
<th>$a_{4,i,j}$</th>
<th>$j = 0$</th>
<th>$j = 1$</th>
<th>$j = 2$</th>
<th>$j = 3$</th>
<th>$j = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 1$</td>
<td>1</td>
<td>15</td>
<td>25</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>$i = 2$</td>
<td>0</td>
<td>25</td>
<td>40</td>
<td>11</td>
<td>0</td>
</tr>
<tr>
<td>$i = 3$</td>
<td>0</td>
<td>10</td>
<td>11</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$i = 4$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Define

\[ A = A(x, p, q) = \sum_{n,i,j \geq 0} a_{n,i,j} \frac{x^n}{n!} p^i q^j. \]

We now present the first main result of this paper.

**Theorem 4.** The generating function \( A \) is given by

\[ A = \frac{p(p - q)e^x}{p - q(e^{p-q}(e^x-1))}. \]

Moreover, for all \( n, i, j \geq 1 \),

\[ a_{n,i,j} = \begin{cases} n + 1 & i + j \\ i + j - 1 & \end{cases} \langle i \rangle. \quad (5) \]

**Proof.** By rewriting (4) in terms of generating function \( A \), we have

\[ A_x = p(1 + q)A_p + q(1 + p)A_q. \quad (6) \]

It is routine to check that the generating function \( \tilde{A} \) satisfies (6). Also, this generating function gives \( \tilde{A}(0, p, q) = p, \tilde{A}(x, 0, q) = 0 \) with \( q \neq 0 \). Hence, \( A = \tilde{A} \). Now let us prove that \( a_{n,i,j} = \langle n + 1 \rangle \langle i+j \rangle \langle i+j-1 \rangle \). Note that

\[
\frac{d}{dx} \sum_{n,i,k \geq 0} a_{n,i,k+1}^{x^n} \frac{1}{(n+1)!} v^i w^k = v \frac{d}{dx} \sum_{k \geq 0} \left( \sum_{n \geq k+1} \binom{n+1}{k+1} \frac{x^n}{(n+1)!} \sum_{i=0}^{k} \binom{k}{i} v^i \right) w^k
\]

\[ = v \frac{d}{dx} \sum_{k \geq 0} \left( \sum_{i=0}^{k} \binom{k}{i} v^i \right) \frac{(e^x - 1)^{k+1}}{(k+1)!} w^k. \]

By using the fact that

\[
\sum_{k \geq 0} \left( \sum_{i=0}^{k} \binom{k}{i} p^i \right) v^k = \int_0^u \frac{p-1}{p - e^{u(p-1)}} du' = \frac{1}{p} (u(p - 1) - \ln(e^{u(p-1)} - p) + \ln(1 - p)),
\]

we obtain that

\[
v \frac{d}{dx} \sum_{n,i,k \geq 0} a_{n,i,k+1}^{x^n} \frac{1}{(n+1)!} v^i w^k = \frac{wv(v-1)e^x}{v - e^{(x-1)w(v-1)}},
\]

which implies

\[ A(x, vw, w) = \frac{wv(v-1)e^x}{v - e^{(x-1)w(v-1)}}, \]

as required. \( \square \)
Let \( a_n = \sum_{i \geq 1, j \geq 0} a_{n,i,j} \). Clearly, \( a_n = \sum_{k=0}^{n} k! \binom{n+1}{k+1} \).

**Proposition 5.** \( \binom{n}{k} \binom{k-1}{i} \) is the number of cyclically ordered partitions of \([n]\) with \( k \) blocks whose list of openers contains \( i - 1 \) descents.

Proof. To form such a cyclically ordered partition, start with a partition of \([n]\) into \( k \) blocks in canonical form, each block increasing and blocks arranged in order of increasing first entries (there are \( \binom{n}{k} \) choices). The first opener is thus 1. Then leave the first block in place and rearrange the \( k - 1 \) remaining blocks so that their openers, viewed as a list, contain \( i - 1 \) descents (there are \( \binom{k-1}{i} \) choices).

We can now conclude the following corollary from the discussion above.

**Corollary 6.** For all \( n, i, j \geq 1 \), \( a_{n,i,j} \) is the number of cyclically ordered partitions of \([n+1]\) with \( i + j \) blocks whose list of openers contains \( i - 1 \) descents.

### 2.2 Peak statistics

The idea of a peak (resp. valley) in a list of integers \((w_i)_{i=1}^{n}\) is an entry that is greater (resp. smaller) than its neighbors. The number of peaks in a permutation is an important combinatorial statistic. See, e.g., [1, 5, 7, 10] and the references therein. However, the question of whether the first and/or last entry may qualify as a peak (or valley) gives rise to several different definitions. In this paper, we consider only left peaks and right valleys. A **left peak index** is an index \( i \in [n-1] \) such that \( w_{i-1} < w_i > w_{i+1} \), where we take \( w_0 = 0 \), and the entry \( w_i \) is a left peak. Similarly, a **right valley** is an entry \( w_i \) with \( i \in [2,n] \) such that \( w_{i-1} > w_i < w_{i+1} \), where we take \( w_{n+1} = \infty \). Thus the last entry may be a right valley but not a left peak. For example, the list 64713258 has 3 left peaks and 3 right valleys. Clearly, left peaks and right valleys in a list are equinumerous: they alternate with a peak first and a valley last. Peaks and valleys were considered in [7]. The left peak statistic first appeared in [1, Definition 3.1].

Let \( P(n,k) \) be the number of permutations in \( S_n \) with \( k \) left peaks. Let \( P_n(x) = \sum_{k \geq 0} P(n,k)x^k \). It is well known [15, A008971] that

\[
P(x, z) = 1 + \sum_{n \geq 1} P_n(x) \frac{z^n}{n!} = \frac{\sqrt{1 - x}}{\sqrt{1 - x \cosh(z \sqrt{1 - x} - \sinh(z \sqrt{1 - x})}}
\]

Let \( D \) be the differential operator \( \frac{d}{d\theta} \). Set \( x = \sec \theta \) and \( y = \tan \theta \). Then

\[
D(x) = xy, D(y) = x^2.
\]

Furthermore, if \( G' = \{ x \rightarrow xy, y \rightarrow x^2 \} \), then

\[
D^{n}_{G'}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} P(n,k)x^{2k+1}y^{n-2k} \quad \text{for } n \geq 1,
\]
which was given in [10, Section 2]. There is a large literature devoted to the repeated differentiation of the secant and tangent functions (see [8, 9, 10, 11] for instance).

Consider the grammar
\[ G = \{ x \rightarrow x + xy, y \rightarrow y + x^2 \}. \]  
(7)

From (7), we have
\[ D(x) = x + xy, \]
\[ D^2(x) = x + 3xy + xy^2 + x^3, \]
\[ D^3(x) = x + 7xy + 6xy^2 + xy^3 + 6x^3 + 5x^3y. \]

Define
\[ D^n(x) = \sum_{i \geq 1, j \geq 0} b_{n,i,j} x^i y^j. \]

Since
\[ D^{n+1}(x) = D \left( \sum_{i \geq 1, j \geq 0} b_{n,i,j} x^i y^j \right) \]
\[ = \sum_{i,j} (i + j)b_{n,i,j}x^i y^j + \sum_{i,j} ib_{n,i,j}x^i y^{j+1} + \sum_{i,j} jb_{n,i,j}x^{i+2} y^{j-1}, \]
we get
\[ b_{n+1,i,j} = (i + j)b_{n,i,j} + ib_{n,i,j-1} + (j + 1)b_{n,i-2,j+1} \]
for \( i \geq 1 \) and \( j \geq 0 \), with the initial conditions \( b_{0,i,j} \) to be 1 if \( (i, j) = (1, 0) \), and to be 0 otherwise. Clearly, \( b_{n,1,0} = 1 \) for \( n \geq 1 \).

**Example 7.** The following table contains the values of \( b_{4,i,j} \).

<table>
<thead>
<tr>
<th>[ b_{4,i,j} ]</th>
<th>( j = 0 )</th>
<th>( j = 1 )</th>
<th>( j = 2 )</th>
<th>( j = 3 )</th>
<th>( j = 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = 1 )</td>
<td>1</td>
<td>15</td>
<td>25</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>( i = 3 )</td>
<td>25</td>
<td>50</td>
<td>18</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( i = 5 )</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Define
\[ B = B(x, p, q) = \sum_{n,i,j \geq 0} b_{n,i,j} p^i q^j x^n / n!. \]

We now present the second main result of this paper.

**Theorem 8.** The generating function \( B \) is given by
\[ B(x, p, q) = \frac{p \sqrt{q^2 - p^2 e^x}}{\sqrt{q^2 - p^2} \cosh(\sqrt{q^2 - p^2 (e^x - 1)}) - q \sinh(\sqrt{q^2 - p^2 (e^x - 1)})}. \]

Moreover, for all \( n, i, j \geq 1 \),
\[ b_{n,2i-1,j} = \left\{ \begin{array}{ll} n + 1 \\ 2i - 1 + j \end{array} \right\} P(2i - 2 + j, i - 1). \]  
(9)
Proof. The recurrence (8) can be written as
\[ B_x = p(1 + q)B_p + (p^2 + q)B_q. \] (10)
It is routine to check that the generating function
\[ \tilde{B} = \tilde{B}(x, p, q) = \frac{p\sqrt{q^2 - p^2}e^x}{\sqrt{q^2 - p^2} \cosh(\sqrt{q^2 - p^2}(e^x - 1)) - q \sinh(\sqrt{q^2 - p^2}(e^x - 1))} \]
satisfies (10)). Also, this generating function gives \( \tilde{B}(0, p, q) = p \) and \( \tilde{B}(x, 0, q) = 0 \). Hence, \( B = \tilde{B} \).

It follows from (8) that \( b_{n,2i,j} = 0 \) for all \((i, j) \neq (0, 0)\). Now let us prove that
\[ b_{n,2i-1,j} = \left\{ \begin{array}{ll} n + 1 \\ 2i - 1 + j \end{array} \right\} P(2i - 2 + j, i - 1). \]

Note that
\[ \sum_{n,i,j \geq 0} b_{n,i,j+1-2i} p^i q^j \frac{x^n}{n!} = \sum_{n \geq 0, i, j \geq 1} b_{n,2i-1,j+1-2i} p^i q^j \frac{x^n}{n!} = p \sum_{n \geq 0, j \geq 1} \left\{ \begin{array}{l} n + 1 \\ j \end{array} \right\} P_{j-1}(p)q^j \frac{x^n}{n!} = pe^x \sum_{j \geq 1} \frac{(e^x - 1)^{j-1}}{(j - 1)!} P_{j-1}(p)q^j = p q e^x P(p, q(e^x - 1)), \]
Hence,
\[ \sum_{n,i,j \geq 0} b_{n,i,j} p^i q^j \frac{x^n}{n!} = pe^x P(p^2 / q^2, q(e^x - 1)) = B(x, p, q), \]
as required. \(\square\)

Let \( b_n = \sum_{i \geq 1, j \geq 0} b_{n,i,j} \). It follows from (9) that \( b_n = a_n \). In the following discussion, we shall present a combinatorial interpretation for \( b_{n,i,j} \).

**Lemma 9.** Suppose that \( (w_i)_{i=1}^k \) is a list of distinct integers containing \( \ell \) right valleys and that \( w_1 = 1 \). Then, among the \( k \) ways to insert a new entry \( m > \max(w_i) \) into the list in a noninitial position, \( 2\ell + 1 \) of them will not change the number of right valleys and \( k - (2\ell + 1) \) will increase it by 1.

Proof. As observed above, peaks and valleys alternate, a peak occurring first, and a valley occurring last. Thus there are \( \ell \) peaks. If \( m \) is inserted immediately before or after a peak or at the very end, the number of valleys is unchanged, otherwise it is increased by 1.

**Proposition 10.** The number \( u_{n,k,\ell} \) of cyclically ordered partitions on \([n]\) with \( k \) blocks and \( \ell \) right valleys in the list of openers satisfies the recurrence
\[ u_{n,k,\ell} = k u_{n-1,k,\ell} + (2\ell + 1) u_{n-1,k-1,\ell} + (k - 2\ell) u_{n-1,k-1,\ell-1} \] (11)
for \( n \geq 2, \ell \geq 0, \ 2\ell + 1 \leq k \leq n. \)
Proof. Each cyclically ordered partition of size \( n \) is obtained by inserting \( n \) into one of size \( n - 1 \), either as the last entry in an existing block or as a new singleton block. Let \( U_{n,k,\ell} \) denote the set of cyclically ordered partitions counted by \( u_{n,k,\ell} \). To obtain an element of \( U_{n,k,\ell} \) we can insert \( n \) into any existing block of an element of \( U_{n-1,k,\ell} \) (this gives \( ku_{n-1,k,\ell} \) choices), or insert \( n \) as a singleton block into an element of \( U_{n-1,k-1,\ell} \) so that the number of right valleys is unchanged (this gives \( (2\ell + 1)u_{n-1,k-1,\ell} \) choices), or insert \( n \) as a singleton block into an element of \( U_{n-1,k-1,\ell-1} \) so that the number of right valleys is increased by 1 (this gives \( (k - 2\ell)u_{n-1,k-1,\ell-1} \) choices). The last two counts of choices follow from Lemma 9.

**Corollary 11.** For all \( n, i,j \geq 1 \), \( b_{n,i,j} \) is the number of cyclically ordered partitions of \( [n+1] \) with \( i+j \) blocks and \( \frac{i-1}{2} \) right valleys (equivalently, \( \frac{j-1}{2} \) left peaks) in the list of openers.

Proof. Comparing recurrence relations (8) and (11), we see that \( b_{n,i,j} = u_{n+1,i+j,(i-1)/2} \).

**Remark 12.** A cyclically ordered partition of size \( n \) with \( k \) blocks and \( \ell \) right valleys in the list of openers is obtained by selecting a partition of \( [n] \) with \( k \) blocks in \( \binom{n}{k} \) ways, and then arranging the blocks suitably, in \( P(k, \ell) \) ways. Hence \( u_{n,k,\ell} = \binom{n}{k} P(k, \ell) \) and we get a combinatorial proof that \( b_{n,2i-1,j} = \binom{n+1}{2i-1+j} P(2i - 2 + j, i - 1) \).

### 2.3 The longest alternating subsequences

Let \( \pi = \pi(1)\pi(2)\cdots\pi(n) \in S_n \). An alternating subsequence of \( \pi \) is a subsequence \( \pi(i_1)\cdots\pi(i_k) \) satisfying

\[
\pi(i_1) > \pi(i_2) < \pi(i_3) > \cdots \pi(i_k).
\]

Let as \((\pi)\) be the length (number of terms) of the longest alternating subsequence of \( \pi \). Denote by \( a_k(n) \) the number of permutations \( \pi \) in \( S_n \) such that as \((\pi) = k \). The study of the distribution of the length of the longest alternating subsequences of permutations was recently initiated by Stanley [16].

Let \( L_n(x) = \sum_{k=0}^{\infty} a_k(n)x^k \), and let

\[
L(x, z) = \sum_{n \geq 0} L_n(x) \frac{zn}{n!}.
\]

Stanley [16, Theorem 2.3] obtained the following closed-form formula:

\[
L(x, z) = (1 - x) \frac{\frac{1 + \rho + 2x\rho^z + (1 - \rho)e^{2\rho z}}{1 + \rho - x^2 + (1 - \rho - x^2)e^{2\rho z}}}
\]

where \( \rho = \sqrt{1 - x^2} \).

Let \( \pi = \pi(1)\pi(2)\cdots\pi(n) \in S_n \). We say that \( \pi \) changes direction at position \( i \) if either \( \pi(i - 1) < \pi(i) > \pi(i + 1) \), or \( \pi(i - 1) > \pi(i) < \pi(i + 1) \), where \( i \in \{2, 3, \ldots, n - 1\} \). We say that \( \pi \) has \( k \) alternating runs if there are \( k - 1 \) indices \( i \) such that \( \pi \) changes direction
at these positions. The *up-down runs* of a permutation $\pi$ are the alternating runs of $\pi$ endowed with a 0 in the front. For example, the permutation $\pi = 514632$ has 3 alternating runs and 4 up-down runs. One can easily verify that $a_k(n)$ also counts permutations in $\mathfrak{S}_n$ with $k$ up-down runs. It follows from [13, Corollary 8] that

$$L(x, z) = -\sqrt{\frac{x - 1}{x + 1}} \left( \frac{\sqrt{x^2 - 1} + x \sin(z \sqrt{x^2 - 1})}{1 - x \cos(z \sqrt{x^2 - 1})} \right).$$ (12)

Set $P_0(x) = L_0(x) = 1$. There is a closely connection between the polynomials $P_n(x)$ and $L_n(x)$ (see [13, Corollary 7]):

$$L_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} L_k(x) P_{n-k}(x^2).$$

We now present a grammatical characterization of the numbers $a_k(n)$.

**Proposition 13** ([13, Theorem 6]). For $A = \{w, x, y\}$ and $G' = \{w \rightarrow wx, x \rightarrow xy, y \rightarrow x^2\}$, we have

$$D^n_{G'}(w) = w \sum_{k=0}^{n} a_k(n) x^k y^{n-k}.$$

Consider the grammar

$$G = \{w \rightarrow w + wx, x \rightarrow x + xy, y \rightarrow y + x^2\},$$ (13)

which is the descendant of $G'$ introduced in Proposition 13. From (13), we have

$$D(w) = w(1 + x),$$

$$D^2(w) = w(1 + 3x + xy + x^2);$$

$$D^3(w) = w(1 + 7x + 6xy + xy^2 + 6x^2 + 3x^2y + 2x^3).$$

Define

$$D^n(w) = w \sum_{i,j \geq 0} t_{n,i,j} x^i y^j.$$

Since

$$D^{n+1}(w) = D \left( w \sum_{i,j \geq 0} t_{n,i,j} x^i y^j \right)$$

$$= \sum_{i,j} (1 + i + j)t_{n,i,j}x^i y^j + \sum_{i,j} t_{n,i,j}x^{i+1} y^j + \sum_{i,j} it_{n,i,j}x^i y^{j+1} + \sum_{i,j} jt_{n,i,j}x^{i+2} y^{j-1},$$

we get

$$t_{n+1,i,j} = (1 + i + j)t_{n,i,j} + t_{n,i-1,j} + it_{n,i,j-1} + (j + 1)t_{n,i-2,j+1}$$ (14)

for $i, j \geq 0$, with the initial conditions $t_{0,i,j}$ to be 1 if $(i, j) = (0, 0)$ or $(i, j) = (1, 0)$, and to be 0 otherwise. Clearly, $t_{n,0,0} = 1$ for $n \geq 0$. 

\[\text{the electronic journal of combinatorics 22(2) (2015), \#P2.22} \]
Example 14. The following table contains the values of $t_{4,i,j}$.

<table>
<thead>
<tr>
<th>$t_{4,i,j}$</th>
<th>$j = 0$</th>
<th>$j = 1$</th>
<th>$j = 2$</th>
<th>$j = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 0$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$i = 1$</td>
<td>15</td>
<td>25</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>$i = 2$</td>
<td>25</td>
<td>30</td>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>$i = 3$</td>
<td>20</td>
<td>11</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$i = 4$</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Define

$$T = T(x, p, q) = \sum_{n,i,j \geq 0} t_{n,i,j} p^i q^j x^n / n!.$$ 

We now present the following.

Theorem 15. The generating function $T$ is given by

$$T(x, p, q) = e^{x} \sqrt{\frac{p - q}{p + q} \sqrt{p^2 - q^2} + p \sin((e^x - 1)\sqrt{p^2 - q^2})} \frac{p \cos((e^x - 1)\sqrt{p^2 - q^2}) - q}{p \cos((e^x - 1)\sqrt{p^2 - q^2}) - q}. $$

Moreover, for all $n \geq 1, i \geq 1$ and $j \geq 0$,

$$t_{n,i,j} = \left\{ n + 1 \atop i + j + 1 \right\} a_i(i + j). \quad (15)$$

Proof. The recurrence (14) can be written as

$$T_x = T + p(1 + q)T_p + (p^2 + q)T_q. \quad (16)$$

It is routine to check that the generating function

$$\tilde{T} = \tilde{T}(x, p, q) = e^{x} \sqrt{\frac{p - q}{p + q} \sqrt{p^2 - q^2} + p \sin((e^x - 1)\sqrt{p^2 - q^2})} \frac{p \cos((e^x - 1)\sqrt{p^2 - q^2}) - q}{p \cos((e^x - 1)\sqrt{p^2 - q^2}) - q}$$

satisfies (16)). Also, this generating function gives $\tilde{T}(0, p, q) = 1$ and $\tilde{T}(x, 0, q) = e^x$. Hence, $T = \tilde{T}$.

Now let us prove that $t_{n,2i-1,j} = \left\{ n+1 \atop i+j+1 \right\} a_i(i + j)$. Note that

$$\sum_{n,i,j \geq 0} t_{n,i,j} p^i q^j x^n / n! = \sum_{n,i,j \geq 0} t_{n,i,j} p^i q^j x^n / n! = \sum_{n,j \geq 0} \left\{ n + 1 \atop j + 1 \right\} L_j(p) q^j x^n / n! = e^{x} \sum_{j \geq 0} \frac{(e^x - 1)^j}{(j)!} L_j(p) q^j = e^{x} L(p, q(e^x - 1)), $$

Hence,

$$\sum_{n,i,j \geq 0} t_{n,i,j} p^i q^j x^n / n! = e^{x} L(p/q, q(e^x - 1)) = T(x, p, q),$$

as required. \qed
Let \( t_n = \sum_{i \geq 1, j \geq 0} t_{n,i,j} \). It follows from (15) that \( t_n = \sum_{k=0}^{n} k! \binom{n+1}{k+1} \). Along the same lines as the proof of Corollary 6, we get the following.

**Corollary 16.** For all \( n \geq 1, i \geq 1 \) and \( j \geq 0 \), \( t_{n,i,j} \) is the number of cyclically ordered partitions of \([n + 1]\) having \( i + j + 1 \) blocks such that the list of openers has the longest alternating subsequence of length \( i \).

### 3 Concluding remarks

In this paper, we explore some context-free grammars suggested by (1). In fact, there are many other extensions of (1). For example, many authors investigated the following generalized Lotka-Volterra system (see [14]):

\[
\frac{dx}{dt} = x(Cy + z), \quad \frac{dy}{dt} = y(Az + x), \quad \frac{dz}{dt} = z(Bx + y).
\]

Consider the grammar

\[ G = \{ x \rightarrow x(y+z), y \rightarrow y(z+x), z \rightarrow z(x+y) \}. \]

Define

\[ D^n(x) = \sum_{i \geq 1, j \geq 0} g_{n,i,j} x^i y^j z^{n+1-i-j}. \]

By induction, one can easily verify the following: for all \( n \geq 1, i \geq 1 \) and \( j \geq 0 \), we have

\[ g_{n,i,0} = \binom{n}{i}, \quad g_{n,i,n+1-i} = \binom{n}{i}, \quad g_{n,1,j} = \binom{n+1}{j+1}. \]

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### References


