On obstacle numbers

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Abstract

The obstacle number is a new graph parameter introduced by Alpert, Koch, and Laison (2010). Mukkamala et al. (2012) show that there exist graphs with $n$ vertices having obstacle number in $\Omega(n/\log n)$. In this note, we up this lower bound to $\Omega(n/(\log \log n)^2)$. Our proof makes use of an upper bound of Mukkamala et al. on the number of graphs having obstacle number at most $h$ in such a way that any subsequent improvements to their upper bound will improve our lower bound.

1 Introduction

The obstacle number is a new graph parameter introduced by Alpert, Koch, and Laison [2]. Let $G = (V, E)$ be a graph, let $\varphi : V \rightarrow \mathbb{R}^2$ be a one-to-one mapping of the vertices of $G$ onto $\mathbb{R}^2$ (hereafter called a drawing of $G$), and let $S$ be a set of connected subsets of $\mathbb{R}^2$. The pair $(\varphi, S)$ is an obstacle representation of $G$ when, for every pair of vertices $u, w \in V$, the edge $uw$ is in $E$ if and only if the closed line segment with endpoints $\varphi(u)$ and $\varphi(w)$ does not intersect any obstacle in $S$. An obstacle representation $(\varphi, S)$ is an

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*h*-obstacle representation if \(|S| = h\). The **obstacle number** of a graph \(G\), denoted by \(\text{obs}(G)\), is the minimum value of \(h\) such that \(G\) has an \(h\)-obstacle representation.\(^1\)

Note that obstacle representations of planar graphs using few obstacles often require drawings of those graphs that are far from crossing free. For example, any crossing-free drawing of the \(5 \times 5\) grid, \(G_{5 \times 5}\) shown in the left part of Figure 1 requires at least one obstacle in each of the sixteen internal faces (each of which has at least four sides).

It is somewhat surprising, therefore, that \(G_{5 \times 5}\) has obstacle number 1. The obstacle representation, illustrated on the right part of Figure 1 was given to us by Fabrizio Frati. In this figure, the single obstacle is drawn as a shaded region. Since at least one obstacle is clearly necessary to represent any graph other than a complete graph, this proves that \(\text{obs}(G_{5 \times 5}) = 1\). (A similar drawing can be used to show that the \(a \times b\), grid graph has obstacle number 1, for any integers \(a, b > 1\).)

![Figure 1: The 5 × 5 grid graph has obstacle number 1.](image)

Since their introduction, obstacle numbers have generated significant research interest \([4, 5, 6, 7, 8, 9, 10]\). A fundamental—and far from answered—question about obstacle numbers is that of determining the **worst-case obstacle number**;

\[
\text{obs}(n) = \max\{\text{obs}(G) : G \text{ is a graph with } n \text{ vertices}\},
\]

of a graph with \(n\) vertices.

For a graph \(G = (V, E)\), we call elements of \(\binom{V}{2} \setminus E\) **non-edges** of \(G\). The worst-case obstacle number \(\text{obs}(n)\) is obviously upper bounded by \(\binom{n}{2} \in O(n^2)\) since, by mapping the vertices of \(G\) onto a point set in sufficiently general position, one can place a small obstacle—even a single point—on the mid-point of each non-edge of \(G\). No upper bound on \(\text{obs}(n)\) that is asymptotically better than \(O(n^2)\) is known.

More is known about lower bounds on \(\text{obs}(n)\). Alpert, Koch, and Laison \([2]\) initially show that the worst-case obstacle number is \(\Omega\left(\sqrt{\log n / \log \log n}\right)\) and posed as an open problem the question of determining if \(\text{obs}(n) \in \Omega(n)\). Mukkamala et al. \([7]\) showed that \(\text{obs}(n) \in \Omega(n / \log^2 n)\) and Mukkamala et al. \([6]\) later increased this to \(\text{obs}(n) \in \Omega(n / \log \log n)\).

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\(^1\)Note that this definition of obstacle representation is more generous than that of Alpert, Koch, and Laison \([2]\), which requires that the obstacles be polygonal and that the set of points determined by vertices of the obstacles and the image of \(\varphi\) not contain 3 collinear points. Since the current paper proves a lower bound on the obstacle number, this lower bound also applies to the original definition.
Ω(n/\log n). In the current paper, we up the lower bound again by proving the following theorem:

**Theorem 1.** For every integer \( n > 0 \), \( \text{obs}(n) \in \Omega(n/(\log\log n)^2) \), that is, there exists a sequence, \( \langle G_n \rangle_{n=1}^{\infty} \), such that \( G_n \) is a graph with \( n \) vertices and such that \( \text{obs}(G) \in \Omega(n/\log\log n)^2) \).

The proof of Theorem 1 makes use of an upper bound of Mukkamala et al. [6, Theorem 1] on the number of graphs having obstacle number at most \( h \) in such a way that any subsequent improvements on their upper bound will result in an improved lower bound on \( \text{obs}(n) \).

Although some aspects of our proof are a little technical, the main idea is quite simple: Mukkamala et al. [6] show that, with probability at least \( 1 - 2^{-\Omega(n^2)} \), a random graph on \( n \) vertices has obstacle number at least \( \Omega(n/\log\log n)^2) \). Our proof trades off a lower probability for a higher obstacle number. When all is said and done, our proof shows that, with probability at least \( 1 - 2^{-\Omega(n\log n)} \), a random graph on \( n \) vertices has obstacle number at least \( \Omega(n/\log\log n)^2) \).

2 The Proof

Our proof strategy is an application of the probabilistic method [1]. We fix an arbitrary ordering, \( \pi \), on the vertices of an Erdős–Rényi random graph, \( G = G_{n,1/2} \). We then show that it is very unlikely that there is an obstacle representation, \( (\varphi, S) \) of \( G \) such that \( |S| \in o(n/(\log\log n)^2) \) and the lexicographic ordering of the points assigned to vertices by \( \varphi \) agrees with the ordering given by \( \pi \). Here, “very unlikely” means that this occurs with probability \( p < 1/n! \). Since there are only \( n! \) possible orderings of \( G \)'s vertices, we then apply the union bound which shows that with probability \( 1-pn! > 0 \), there is no obstacle representation of \( G \) using \( o(n/(\log\log n)^2) \) obstacles, that is, \( \text{obs}(G) \in \Omega(n/(\log\log n)^2) \).

2.1 A Random Graph with a Fixed Ordering

We make use of the following theorem, due to Mukkamala, Pach, and Pálvölgyi [6, Theorem 1] about the number of \( n \)-vertex graphs with obstacle number at most \( h \):

**Theorem 2** (Mukkamala, Pach, and Pálvölgyi 2012). For any \( h \geq 1 \), the number of graphs having \( n \) vertices and obstacle number at most \( h \) is at most \( 2^{O(hn\log^2 n)} \).

Recall that an Erdős-Rényi random graph \( G_{n,1/2} \) is a graph with \( n \) vertices and each pair of vertices is chosen as an edge or non-edge with equal probability and independently of every other pair of vertices [3]. The following lemma shows that, for random graphs, a fixed drawing is very unlikely to yield an obstacle representation with few obstacles. Recall that the lexicographic ordering, \( < \), for points in the plane is defined as

\[
(x_1, y_1) < (x_2, y_2) \text{ iff } x_1 < x_2 \text{ or } (x_1 = x_2 \text{ and } y_1 < y_2).
\]
Lemma 1. Let $G = (V, E)$ be an Erdős–Rényi random graph $G_{n,p}$, let $v_1, \ldots, v_n$ be an ordering of the vertices in $V$ that is independent of the choices of edges in $G$, and let $(\varphi, S)$ be an obstacle representation of $G$ using the minimum number of obstacles subject to the constraint that

$$\varphi(v_1) < \varphi(v_2) < \cdots < \varphi(v_n),$$

where $<$ denotes the lexicographic ordering of points. Then, for any constant $c > 0$,

$$\Pr\{|S| \in \Omega(n/(\log \log n)^2)\} \geq 1 - e^{-cn \log n}.$$  

Proof. Fix some integer $k = k(n) \in \omega_n(1)$ to be specified later and first consider the subgraph $G_0$ of $G$ induced by the vertices $v_1, \ldots, v_k$. Applying Theorem 2 with $n = k$ and $h = \alpha k/\log^2 k$, we obtain

$$\Pr\{\text{obs}(G_0) \leq \alpha k/\log^2 k\} \leq \frac{2^{O(\alpha k^2)}}{2^{(\beta k)^2}} \leq e^{-\beta k^2},$$

where $\beta > 0$ for a sufficiently small constant $\alpha > 0$, and sufficiently large $k$. Note that, if $\text{obs}(G_0) \geq h$, then, in the obstacle representation $(\varphi, S)$, there must be at least $h - 1$ obstacles of $S$ that are contained in the convex hull of $\varphi(v_1), \ldots, \varphi(v_k)$; this is because the obstacle representation $(\varphi, S)$ can be turned into an obstacle representation of $G_0$ by merging all obstacles that are not contained in the convex hull of $\varphi(v_1), \ldots, \varphi(v_k)$.

Let $m = \lfloor n/k \rfloor$ and notice that the preceding argument applies to any subset $V_i = \{v_{ki+1}, \ldots, v_{(k+1)i}\}$ of vertices, for any $i \in \{0, \ldots, m - 1\}$. That is, Equation (1) shows that, with probability at least $1 - 2^{-\Omega(k^2)}$, the obstacle number of the subgraph $G_i$ induced by $V_i$ is $\Omega(k/\log^2 k)$. If this occurs, then $S$ has $\Omega(k/\log^2 k)$ obstacles that are completely contained in the convex hull of $V_i$. In particular, the obstacles contained in the convex hull of $V_i$ are different from the obstacles contained in the convex hull of $V_j$, for all $j \neq i$.

We are proving a lower bound on the number of obstacles, so we are worried about the case where the number of convex hulls that do not contain at least $\alpha k/\log^2 k$ obstacles exceeds $m/e$. The number of convex hulls, $M$, not containing at least $\alpha k/\log^2 k$ obstacles is dominated by a binomial$(m, e^{-\beta k^2})$ random variable. Using Chernoff’s bound on the tail of a binomial random variable, we have that

$$\Pr\{M \geq m/e\} = \Pr\{M \geq (1 + \delta)\mu\} \quad \text{(where } \mu = me^{-\beta k^2} \text{ and } \delta = e^{\beta k^2 - 1})$$

$$\leq \left(\frac{e^{\delta}}{(1 + \delta)^{1+\delta}}\right)^\mu$$

$$\leq \left(\frac{e^{e^{\beta k^2}}}{(e^{\beta k^2 - 1})^{e^{\beta k^2 - 1}}}\right)^me^{-\beta k^2}.  \quad \text{(2)}$$

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2Euler’s constant $e = \lim_{n \to \infty}(1 - 1/n)^n$ is just a convenient constant to use here.

3Chernoff’s Bound: For any binomial$(m, p)$ random variable, $B$, any $\delta > 0$ and $\mu = mp,$

$$\Pr\{B \geq (1 + \delta)\mu\} \leq \left(\frac{e^{\delta}}{(1 + \delta)^{1+\delta}}\right)^\mu.$$
\[
\begin{align*}
&= \left( \frac{e^{\beta k^2}}{(e^{(\beta k^2 - 1)e^{\beta k^2 - 1}})} \right)^{m e^{-\beta k^2}} \\
&= \frac{e^m}{e^{m(\beta k^2 - 1)e^{\beta k^2 - 1}e^{-\beta k^2}}} \\
&= \frac{e^m(\beta k^2 - 1)e^{-\beta k^2}}{e^{-\Omega(mk^2)}}.
\end{align*}
\]

Taking \( k = \delta' \log n \), for a sufficiently large constant, \( \delta' \), and recalling that \( m = \lceil n/k \rceil \), we obtain the desired result. In particular,

\[
|S| \geq \Omega \left( \left( \frac{k}{\log^2 k} \cdot (m - m/e) \right) = \Omega \left( n/(\log \log n)^2 \right) \right)
\]

with probability at least

\[
1 - e^{-\Omega(mk^2)} = 1 - e^{-\Omega(\delta' n \log n)} \geq 1 - e^{-cn \log n},
\]

for all \( n \) greater than some sufficiently large constant \( n_0 \). For \( n \in \{1, \ldots, n_0\} \), the lemma is trivially satisfied since \( |S| \geq 0 \) with probability \( 1 \geq 1 - e^{-cn \log n} \).

\[\square\]

**2.2 Finishing Up**

For completeness, we now spell out the proof of Theorem 1.

**Proof of Theorem 1.** Let \( G = (V, E) \) be an Erdős-Rényi random graph with \( n \) vertices with vertex set \( V = \{1, \ldots, n\} \). For every obstacle representation \((\varphi, S)\) of \( G \), there is an ordering on \( V \) given by the lexicographic ordering of the points \( \{\varphi(v) : v \in V\} \).

By Lemma 1, the probability that a particular such ordering, \( v_1, \ldots, v_n \), allows an obstacle representation using \( o(n/(\log \log n)^2) \) obstacles is at most \( p \leq e^{-cn \log n} \) for every constant \( c > 0 \). In particular, for sufficiently large \( c \), we have \( p < 1/n! \). By the union bound the probability that there is any ordering that supports an obstacle representation of \( G \) with \( o(n/(\log \log n)^2) \) obstacles is at most

\[
\hat{p} = p \cdot n! < 1.
\]

We deduce that there exists some graph, \( G' \), with \( \text{obs}(G') \in \Omega(n/(\log \log n)^2) \).

\[\square\]

**3 Remarks**

Our proof of Theorem 1 relates the problem of counting the number of \( n \)-vertex graphs with obstacle number at most \( h \) to the problem of determining the worst-case obstacle number of a graph with \( n \) vertices. Currently, we use Theorem 2 of Mukkamala et al. [7], which proves an upper bound of \( e^{O(hn \log^2 n)} \) on the number of \( n \)-vertex graphs with obstacle number at most \( h \).
Any improvement on the upper bound for the counting problem will immediately translate into an improved lower bound on the worst-case obstacle number. Let \( f(h, k) \) denote the number of \( k \)-vertex graphs with obstacle number at most \( h \) and let

\[
\hat{h}(k) = \max \left\{ h : f(h, k) \leq 2^{k^2/4} \right\}.
\]

The quantity \( \hat{h}(k) \) is chosen so that a random graph on \( k \) vertices has probability at most \( 2^{-\Omega(k^2)} \) of having obstacle number less than \( \hat{h}(k) \); Theorem 2 shows that \( \hat{h}(k) \in \Omega(k/(\log k)^2) \). Our proof of Lemma 1 shows that there exist graphs with obstacle number at least \( \Omega(nh(c \log n)/(c \log n)) \).

We note that our technique gives an improved lower bound until someone is able to prove that \( \hat{h}(n) \in \Omega(n) \). At this point, our approach gives a lower bound worse than the trivial lower bound \( \hat{h}(n) \).

We conjecture that improved upper bounds on \( f(h, n) \) that reduce the dependence on \( h \) are the way forward:

**Conjecture 1.** \( f(h, n) \leq 2^{g(n) \cdot O(h)} \), where \( g(n) \in O(n \log^2 n) \).

In support of this conjecture, we observe that an upper bound of the form \( f(1, n) \leq 2^{g(n)} \) is sufficient to give the crude upper bound \( f(h, n) \leq 2^{h \cdot g(n)} \) since any graph with an \( h \)-obstacle representation is the common intersection of \( h \) graphs that each have a 1-obstacle representation. That is, if \( \text{obs}(G) \leq h \), then there exists \( E_1, \ldots, E_h \) such that \( G = (V, \bigcap_{i=1}^{h} E_i) \) and \( \text{obs}(V, E_i) = 1 \) for all \( i \in \{1, \ldots, h\} \). This seems like a very crude upper bound in which many graphs are counted multiple times. Conjecture 1 asserts that this argument overestimates the dependence on \( h \).

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A previous draft of this article proved a version Lemma 1 for a fixed drawing, \( \varphi \), and then went to great lengths to argue that the number of combinatorially distinct drawings was at most \( 2^{O(n \log n)} \). We are grateful to an anonymous referee who pointed out that the proof of Lemma 1 also holds when only the lexicographic ordering of the vertices is fixed, thereby eliminating the need to bound the number of combinatorially equivalent drawings.

**References**


