Regular graphs are antimagic

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Abstract

An undirected simple graph \( G = (V, E) \) is called antimagic if there exists an injective function \( f : E \to \{1, \ldots, |E|\} \) such that \( \sum_{e \in E(u)} f(e) \neq \sum_{e \in E(v)} f(e) \) for any pair of different nodes \( u, v \in V \). In this note we prove – with a slight modification of an argument of Cranston et al. – that \( k \)-regular graphs are antimagic for \( k \geq 2 \).

Keywords: antimagic labelings; regular graphs

1 Introduction

Throughout the note graphs are assumed to be simple. Given an undirected graph \( G = (V, E) \) and a subset of edges \( F \subseteq E \), \( F(v) \) denotes the set of edges in \( F \) incident to node \( v \in V \), and \( d_F(v) := |F(v)| \) is the degree of \( v \) in \( F \). A labeling is an injective function \( f : E \to \{1, 2, \ldots, |E|\} \). Given a labeling \( f \) and a subset of edges \( F \), let \( f(F) = \sum_{e \in F} f(e) \). A labeling is antimagic if \( f(E(u)) \neq f(E(v)) \) for any pair of different nodes \( u, v \in V \). A graph is said to be antimagic if it admits an antimagic labeling.

Hartsfield and Ringel conjectured [5] that all connected graphs on at least 3 nodes are antimagic. The conjecture has been verified for several classes of graphs (see e.g. [4]), but is widely open in general. In [3] Cranston et al. proved that every \( k \)-regular graph is antimagic if \( k \geq 3 \) is odd. Note that 1-regular graphs are trivially not antimagic. We have observed that a slight modification of their argument also works for even regular graphs\(^1\), hence we prove the following.

\(^1\)The same result has been recently proved independently by Chang et al. [2].
Theorem 1. For $k \geq 2$, every $k$-regular graph is antimagic.

It is worth mentioning the following conjecture of Liang [6]. Let $G = (S,T;E)$ be a bipartite graph. A path $P = \{uw,vw\}$ of length 2 with $u,w \in S$ is called an $S$-link.

Conjecture 2. Let $G = (S,T;E)$ be a bipartite graph such that each node in $S$ has degree at most 4 and each node in $T$ has degree at most 3. Then $G$ has a matching $M$ and a family $\mathcal{P}$ of node-disjoint $S$-links such that every node $v \in T$ of degree 3 is incident to an edge in $M \cup (\bigcup_{P \in \mathcal{P}} P)$.

Liang showed that if the conjecture holds then it implies that every 4-regular graph is antimagic. The starting point of our investigations was proving Conjecture 2. As Theorem 1 provides a more general result, we leave the proof of Conjecture 2 for a forthcoming paper [1].

2 Proof of Theorem 1

A trail in a graph $G = (V,E)$ is an alternating sequence of nodes and edges $v_0, e_1, v_1, \ldots, e_t, v_t$ such that $e_i$ is an edge connecting $v_{i-1}$ and $v_i$ for $i = 1, 2, \ldots, t$, and the edges are all distinct (but there might be repetitions among the nodes). The trail is open if $v_0 \neq v_t$, and closed otherwise. The length of a trail is the number of edges in it. A closed trail containing every edge of the graph is called an Eulerian trail. It is well known that a graph has an Eulerian trail if and only if it is connected and every node has even degree.

Lemma 3. Given a connected graph $G = (V,E)$, let $T = \{v \in V : d_E(v) \text{ is odd}\}$. If $T \neq \emptyset$, then $E$ can be partitioned into $|T|/2$ open trails.

Proof. Note that $|T|$ is even. Arrange the nodes of $T$ into pairs in an arbitrary manner and add a new edge between the members of every pair. Take an Eulerian trail of the resulting graph and delete the new edges to get the $|T|/2$ open trails.

The main advantage of Lemma 3 is that the edge set of the graph can be partitioned into open trails such that at most one trail starts at every node of $V$. Indeed, there is a trail starting at $v$ if and only if $v$ has odd degree in $G$. This is how we see the Helpful Lemma of [3].

Corollary 4 (Helpful Lemma of [3]). Given a bipartite graph $G = (U,W;E)$ with no isolated nodes in $U$, $E$ can be partitioned into subsets $E^u, T_1, T_2, \ldots, T_l$ such that $d_{E^u}(u) = 1$ for every $u \in U$, $T_i$ is an open trail for every $i = 1, 2, \ldots, l$, and the endpoints of $T_i$ and $T_j$ are different for every $i \neq j$.

Proof. Take an arbitrary $E' \subseteq E$ with the property $d_{E'}(u) = 1$ for every $u \in U$. A component of $G - E'$ containing more than one node is called nontrivial. If there exists a nontrivial component of $G - E'$ that only contains even degree nodes then let $uw_1 \in E - E'$ be an edge in this component with $u \in U$ and $w_1 \in W$, and let $uw_2 \in E'$. Replace $uw_2$ with $uw_1$ in $E'$. After this modification, the component of $G - E'$ that
contains \( u \) has an odd degree node, namely \( w_1 \). Iterate this step until every nontrivial component of \( G - E' \) has some odd degree nodes. Let \( E^o = E' \) and apply Lemma 3 to get the decomposition of \( E - E^o \) into open trails.

\[ \square \]

In what follows we prove that regular graphs are antimagic: for sake of completeness we include the odd regular case, too. We emphasize the differences from the proof appearing in [3].

**Proof of Theorem 1.** Note that it suffices to prove the theorem for connected regular graphs. Let \( G = (V, E) \) be a connected \( k \)-regular graph and let \( v^* \in V \) be an arbitrary node. Denote the set of nodes at distance exactly \( i \) from \( v^* \) by \( V_i \) and let \( q \) denote the largest distance from \( v^* \). We denote the edge-set of \( G[V_i] \) by \( E_i \). Apply Corollary 4 to the induced bipartite graph \( G[V_{i-1}, V_i] \) with \( U = V_i \) to get \( E_i^o \) and the trail decomposition of \( G[V_{i-1}, V_i] - E_i^o \) for every \( i = 1, \ldots, q \). The edge set of \( G[V_{i-1}, V_i] - E_i^o \) is denoted by \( E_i' \).

Now we define the antimagic labeling \( f \) of \( G \) as follows. We reserve the \(|E_q|\) smallest labels for labeling \( E_q \), the next \(|E_q^o|\) smallest labels for labeling \( E_q^o \), the next \(|E_q'|\) smallest labels for labeling \( E_q' \), the next \(|E_{q-1}^o|\) smallest labels for labeling \( E_{q-1}^o \), etc. There is an important difference here between our approach and that of [3] as we switched the order of labeling \( E_i^o \) and \( E_i' \), and we don’t yet define the labels, we only reserve the intervals to label the edge sets. Next we prove a claim that tells us how to label the edges in \( E_i' \).

**Claim 5.** Assume that we have to label the edges of \( E_i' \) from interval \( s, s+1, \ldots, \ell \) (where \(|E_i'| = \ell - s + 1| \), and that we are given a trail decomposition of \( E_i' \) into open trails. We can label \( E_i' \) so that successive labels (in a trail) incident to a node \( v_i \in V_i \) have sum at most \( s + \ell \), and successive labels (in a trail) incident to a node \( v_{i-1} \in V_{i-1} \) have sum at least \( s + \ell \).

**Proof.** Our proof of this claim is essentially the same as the proof in [3]: we merely restate it for self-containment. Let \( T \) be the trail decomposition of \( E_i' \) into open trails. Take an arbitrary trail \( T = u_0, e_1, u_1, \ldots, e_t, u_t \) of length \( t \) from \( T \) and consider the following two cases (see Figure 1 for an illustration).

- **Case A:** If \( u_0 \in V_{i-1} \) then label \( e_1, \ldots, e_t \) by \( \ell, s, \ell - 1, \ldots \) in this order. In this case the sum of 2 successive labels is \( s + \ell \) at a node in \( V_i \), and it is \( s + \ell + 1 \) at a node in \( V_{i-1} \).

- **Case B:** If \( u_0 \in V_i \) then label \( e_1, \ldots, e_t \) by \( \ell, s, \ell - 1, s + 1, \ldots \) in this order. In this case the sum of 2 successive labels is \( s + \ell - 1 \) at a node in \( V_i \), and it is \( s + \ell \) at a node in \( V_{i-1} \).

We prove by induction on \(|T|\). The proof is finished by the following cases.

1. If \( T \) contains a trail of even length, then let \( T \) be such a trail (and again \( t \) denotes the length of \( T \)). If the endpoints of \( T \) fall in \( V_{i-1} \) then apply Case A. On the other hand, if the endpoints of \( T \) fall in \( V_i \) then apply Case B. In both cases we use \( \frac{1}{2} \) labels from the lower end of the interval, and \( \frac{1}{2} \) labels from the upper end,
therefore we can label the edges of the trails in $T - T$ from the (remaining) interval $s + \ell, s + \ell + 1, \ldots, \ell - \frac{\ell}{2}$, so that the lower bound $s + \frac{\ell}{2} + \ell - \frac{\ell}{2} = s + \ell$ holds for the sum of two successive labels at every $v_{i-1} \in V_{i-1}$, and the same upper bound holds at each node $v_i \in V_i$.

2. Every trail in $T$ has odd length. If $T$ contains only one trail then label it using either of the two cases above and we are done. Otherwise let $T_1$ and $T_2$ be two trails from $T$, and let $t_i$ be the length of $T_i$ for both $i = 1, 2$. Label first the edges of $T_1$ using Case A (starting at the endpoint of $T_1$ that lies in $V_{i-1}$). Note that the remaining labels form the interval $s + \frac{t_1 + 1}{2}, \ldots, \ell - \frac{t_1 - 1}{2}$. Next label the edges of $T_2$ using Case B (starting at the endpoint of $T_2$ that lies in $V_i$). Note that the sum of successive labels in the trail $T_2$ becomes $s + \frac{t_1 + 1}{2} + (\ell - \frac{t_2 - 1}{2}) - 1 = s + \ell$ at a node in $V_i$, and it is $s + \frac{t_1 + 1}{2} + (\ell - \frac{t_2 - 1}{2}) = s + \ell + 1$ at a node in $V_{i-1}$, which is fine for us. Finally, the remaining labels form the interval $s + \frac{t_1 + 1}{2} + \frac{t_2 - 1}{2}, \ldots, \ell - \frac{t_1 - 1}{2} - \frac{t_2 + 1}{2}$, therefore we can label the edges of the trails in $T - \{T_1, T_2\}$ from the remaining interval so that the lower bound $s + \frac{t_1 + 1}{2} + \frac{t_2 - 1}{2} + \ell - \frac{t_1 - 1}{2} - \frac{t_2 + 1}{2} = s + \ell$ holds for the sum of two successive labels at every node of $V_{i-1}$, and the same upper bound holds at every node of $V_i$.

Now we specify how the labels are determined to make sure $f(E(u)) \neq f(E(v))$ for every $u \neq v$. We label the edges of every $E_i$ arbitrarily from their dedicated intervals. Label the edges of every $E'_i$ in the manner described by Claim 5. For any node $v \in V_i$ with $i > 0$, let $\sigma(v)$ denote the unique edge of $E'_i$ incident to $v$. Let $p(v) = f(E(v)) - f(\sigma(v))$ for every $v \in V - V^*$. We label the edges in $E^*_q, E^*_q - 1, \ldots, E^*_1$ as in [3]: if we already labeled $E^*_q, E^*_q - 1, \ldots, E^*_1$ then $p(v_i)$ is already determined for every $v_i \in V_i$. So we order the nodes of $V_i$ in an increasing order according to their $p$-value and assign the label to their $\sigma$ edge in this order. This ensures that $f(E(u)) \neq f(E(v))$ for an arbitrary pair $u, v \in V_i$. 

Figure 1: An illustration for labeling trails.
Claim 6. For arbitrary $v \in V$, in the bounds needed. Assume first that $p(v) \leq \frac{k-2}{2}(s+\ell) + \ell$ and $p(v_{i-1}) \geq \frac{k-2}{2}(s+\ell) + s$, if $k$ is even, and

2. $p(v_i) \leq \frac{k-1}{2}(s+\ell)$ and $p(v_{i-1}) \geq \frac{k-1}{2}(s+\ell)$, if $k$ is odd.

Proof. Assume first that $k$ is even. In this case $p(v)$ is the sum of an odd number of labels. We pair up all but one of these labels using the trail decomposition of $E'$ to get the bounds needed.

1. Take a node $v_i \in V_i$. Note that $f(e) < s$ for every $e \in E(v_i) - E'_i$. Let $t = d_{E'_i}(v_i)$.

(a) If $t$ is even then $\sum_{e \in E'_i \cap E(v_i)} f(e) \leq \frac{t}{2}(s+\ell)$ by Claim 5, giving $p(v_i) \leq \frac{t}{2}(s+\ell) + (k-1-t)s \leq \frac{k-2}{2}(s+\ell) + \ell$.

(b) If $t$ is odd then $\sum_{e \in E'_i \cap E(v_i)} f(e) \leq \frac{t-1}{2}(s+\ell) + \ell$ by Claim 5, giving $p(v_i) \leq \frac{t-1}{2}(s+\ell) + \ell + (k-1-t)s \leq \frac{k-2}{2}(s+\ell) + \ell$.

2. Now take a node $v_{i-1} \in V_{i-1}$. Note that $f(e) > \ell$ for every $e \in E(v_{i-1}) - E'_i$. Let again $t = d_{E'_i}(v_{i-1})$.

(a) If $t$ is even then $\sum_{e \in E'_i \cap E(v_{i-1})} f(e) \geq \frac{t}{2}(s+\ell)$ by Claim 5, giving $p(v_{i-1}) \geq \frac{t}{2}(s+\ell) + (k-1-t)\ell \geq \frac{k-2}{2}(s+\ell) + s$.

(b) If $t$ is odd then $\sum_{e \in E'_i \cap E(v_{i-1})} f(e) \geq \frac{t-1}{2}(s+\ell) + s$ by Claim 5, giving $p(v_{i-1}) \geq \frac{t-1}{2}(s+\ell) + s + (k-1-t)\ell \geq \frac{k-2}{2}(s+\ell) + s$.

This concludes the proof of $(i)$.

Although the proof of $(ii)$ can be found in [3], we also present it here to make the paper self contained. The proof is very similar to the even case. So assume that $k$ is odd. In this case $p(v)$ is the sum of an even number of labels. We pair up these labels using the trail decomposition of $E'$ to get the bounds needed.

1. Take a node $v_i \in V_i$. Note that $f(e) < s$ for every $e \in E(v_i) - E'_i$. Let $t = d_{E'_i}(v_i)$.

(a) If $t$ is even then $\sum_{e \in E'_i \cap E(v_i)} f(e) \leq \frac{t}{2}(s+\ell)$ by Claim 5, giving $p(v_i) \leq \frac{t}{2}(s+\ell) + (k-1-t)s \leq \frac{k-1}{2}(s+\ell)$.

(b) If $t$ is odd then $\sum_{e \in E'_i \cap E(v_i)} f(e) \leq \frac{t-1}{2}(s+\ell) + \ell$ by Claim 5, giving $p(v_i) \leq \frac{t-1}{2}(s+\ell) + \ell + (k-1-t)s \leq \frac{k-1}{2}(s+\ell)$. 


2. Now take a node $v_{i-1} \in V_{i-1}$. Note that $f(e) > \ell$ for every $e \in E(v_{i-1}) - E'_{i}$. Let again $t = d_{E'}(v_{i-1})$.

(a) If $t$ is even then $\sum_{e \in E' \cap E(v_{i-1})} f(e) \geq \frac{t}{2}(s + \ell)$ by Claim 5, giving $p(v_{i-1}) \geq \frac{t}{2}(s + \ell) + (k - 1 - t)\ell \geq \frac{k-1}{2}(s + \ell)$. 

(b) If $t$ is odd then $\sum_{e \in E' \cap E(v_{i-1})} f(e) \geq \frac{t-1}{2}(s + \ell) + s$ by Claim 5, giving $p(v_{i-1}) \geq \frac{t-1}{2}(s + \ell) + s + (k - 1 - t)\ell \geq \frac{k-1}{2}(s + \ell)$.

This concludes the proof of (ii), and we are done.

Remark 7. Observe that the proof of Theorem 1 does not really use the regularity of the graph, it merely relies on the fact that the degree of a node $v_{i} \in V_{i}$ is not smaller than that of a node $v_{j} \in V_{j}$ where $i < j$. Hence the following result immediately follows.

Theorem 8. Assume that a connected graph $G = (V, E)$ $(|V| \geq 3)$ has a node $v^{*} \in V$ of maximum degree such that $d_{E}(v_{i}) \geq d_{E}(v_{j})$ whenever $v_{i} \in V_{i}, v_{j} \in V_{j}$ and $i < j$, where $V_{\ell}$ denotes the set of nodes at distance exactly $\ell$ from $v^{*}$. Then $G$ is antimagic.

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