Arithmetic Properties of a Restricted Bipartition Function

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Abstract

A bipartition of $n$ is an ordered pair of partitions $(\lambda, \mu)$ such that the sum of all of the parts equals $n$. In this article, we concentrate on the function $c_5(n)$, which counts the number of bipartitions $(\lambda, \mu)$ of $n$ subject to the restriction that each part of $\mu$ is divisible by 5. We explicitly establish four Ramanujan type congruences and several infinite families of congruences for $c_5(n)$ modulo 3.

Keywords: bipartition, congruence

1 Introduction

In a series of papers [4, 5, 6], Chan studied the arithmetic properties of the cubic partition function $a(n)$, which is defined by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q; q)_{\infty}(q^2; q^2)_{\infty}}.$$
Throughout the paper, we adopt the following standard $q$-series notation
\[(a; q)_\infty = \prod_{n=1}^{\infty} (1 - aq^{n-1}).\]

In [4], Chan proved that

**Theorem 1.** For $n \geq 0$,
\[a(3n + 2) \equiv 0 \pmod{3}.\] (1)


**Theorem 2.** For $k \geq 1$ and $n \geq 0$,
\[a(3^k n + s_k) \equiv 0 \pmod{3^k + \delta(k)},\]
where $s_k$ is the reciprocal modulo $3^k$ of 8 and $\delta(k) = 1$ if $k$ is even, and 0 otherwise.

Chan and Toh [7] also established the following nice congruence, which was also discovered by Xiong [20] independently.

**Theorem 3.** If $k \geq 1$ and $n \geq 0$, then
\[a(5^k n + t_k) \equiv 0 \pmod{5^{\left\lfloor k/2 \right\rfloor}},\]
where $t_k$ is the reciprocal modulo $5^k$ of 8.

Inspired by the work of Ramanujan on the standard partition function $p(n)$, Chan [5] asked whether there are any other congruences of the following form
\[a(\ell n + k) \equiv 0 \pmod{\ell},\]
where $\ell$ is prime and $0 \leq k < \ell$. Later, Sinick [18] answered Chan’s question in the negative by considering the following restricted bipartition function:
\[\sum_{n=0}^{\infty} c_N(n)q^n = \frac{1}{(q; q)_\infty(q^N; q^N)_\infty}.\] (2)

A bipartition of $n$ is an ordered pair of partitions $(\lambda, \mu)$ such that the sum of all of the parts equals $n$. Then we know that $c_N(n)$ counts the number of bipartitions $(\lambda, \mu)$ of $n$ subject to the restriction that each part of $\mu$ is divisible by $N$. Recently, bipartitions with certain restrictions on each partition have been investigated by many authors, see [3, 8, 9, 10, 11, 12, 14, 15, 16, 17, 19] for instance.

In this paper, we investigate the bipartition function $c_5(n)$ from an arithmetic point of view in the spirit of Ramanujan's congruences for the standard partition function $p(n)$. 

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2 Ramanujan type congruences for $c_5(n)$

We first introduce a useful lemma which will be used later.

Lemma 4. We have

$$(q; q)_\infty^2(q^5; q^5)_\infty^2 \equiv (q^3; q^3)_\infty^4 + q(q^3; q^3)_\infty^2(q^{15}; q^{15})_\infty^2 - q^2(q^{15}; q^{15})_\infty^4 \pmod{3}. \quad (3)$$

Proof. From [1, p.28, Entry 1.6.2], we see that

$$(q; q)_\infty^2(q^5; q^5)_\infty^2 = (\psi^2(q) - q\psi^2(q^5))(\psi^2(q) - 5q\psi^2(q^5))$$

$$\equiv \psi(q)\psi(q^3) - q^2\psi(q^5)\psi(q^{15}) \pmod{3}, \quad (4)$$

where

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty}.$$ 

Invoking [2, p.49, Corollary (ii)], we have

$$\psi(q) = A(q^3) + q\psi(q^9), \quad (5)$$

where

$$A(q) = (-q; q^3)_\infty(-q^2; q^3)_\infty(q^3; q^3)_\infty.$$ 

Substituting (5) into (4), we find that

$$(q; q)_\infty^2(q^5; q^5)_\infty^2 \equiv \psi(q^3)(A(q^3) + q\psi(q^9)) - q^2\psi(q^{15})(A(q^{15}) + q^5\psi(q^{45})) \pmod{3}$$

$$= \psi(q^3)A(q^3) - q^2\psi(q^{15})A(q^{15}) + q(\psi(q^3)\psi(q^9) - q^6\psi(q^{15})\psi(q^{45})).$$

On the other hand, applying (4) with $q$ replaced by $q^3$ yields that

$$(q^3; q^3)_\infty^2(q^{15}; q^{15})_\infty^2 \equiv \psi(q^3)\psi(q^9) - q^6\psi(q^{15})\psi(q^{45}) \pmod{3}. \quad (6)$$

Therefore, we arrive at

$$(q; q)_\infty^2(q^5; q^5)_\infty^2 \equiv \psi(q^3)A(q^3) - q^2\psi(q^{15})A(q^{15}) + q(q^3; q^3)_\infty^2(q^{15}; q^{15})_\infty^2 \pmod{3}. \quad (6)$$

In addition, it is easy to see that

$$\psi(q)A(q) \equiv (q; q)_\infty^4 \pmod{3}.$$ 

Utilizing the above congruence in (6), we complete the proof of (3). □

With Lemma 4 in hand, we now move to the dissections of the generating function for $c_5(n)$ modulo 3.
Theorem 5. We have

\[
\sum_{n=0}^{\infty} c_5(3n)q^n \equiv \frac{(q^3; q^3)^\infty}{(q^5; q^5)^\infty} \quad (\text{mod } 3),
\]
(7)

\[
\sum_{n=0}^{\infty} c_5(3n+1)q^n \equiv (q; q)\infty(q^5; q^5)^\infty \quad (\text{mod } 3),
\]
(8)

\[
\sum_{n=0}^{\infty} c_5(3n+2)q^n \equiv -\frac{(q^{15}; q^{15})^\infty}{(q; q)^\infty} \quad (\text{mod } 3).
\]
(9)

Proof. From (2), we can easily deduce that

\[
\sum_{n=0}^{\infty} c_5(n)q^n \equiv \frac{(q; q)^2(q^5; q^5)^2}{(q^3; q^3)(q^{15}; q^{15})^\infty} \quad (\text{mod } 3).
\]

Applying Lemma 4, we obtain

\[
\sum_{n=0}^{\infty} c_5(n)q^n \equiv \frac{(q^9; q^9)^\infty}{(q^{15}; q^{15})^\infty} + q(q^3; q^3)^\infty(q^{15}; q^{15})^\infty - q^2\frac{(q^{45}; q^{45})^\infty}{(q^3; q^3)^\infty} \quad (\text{mod } 3),
\]
from which we get

\[
\sum_{n=0}^{\infty} c_5(3n)q^{3n} \equiv \frac{(q^9; q^9)^\infty}{(q^{15}; q^{15})^\infty} \quad (\text{mod } 3),
\]

\[
\sum_{n=0}^{\infty} c_5(3n+1)q^{3n+1} \equiv q(q^3; q^3)^\infty(q^{15}; q^{15})^\infty \quad (\text{mod } 3),
\]

\[
\sum_{n=0}^{\infty} c_5(3n+2)q^{3n+2} \equiv -q^2\frac{(q^{45}; q^{45})^\infty}{(q^3; q^3)^\infty} \quad (\text{mod } 3),
\]
simplification upon which yields the desired results.

The following is a consequence of Theorem 5.

Corollary 6. We have

\[
\sum_{n=0}^{\infty} c_5(9n+7)q^n \equiv -(q^3; q^3)^\infty(q^{15}; q^{15})^\infty \quad (\text{mod } 3).
\]
(10)
Proof. By (8), we find that
\[
\sum_{n=0}^{\infty} c_5(3n+1)q^n \equiv (q^3; q^3)_{\infty}(q^{15}; q^{15})_{\infty} \frac{1}{(q; q)^2_{\infty}(q^5; q^5)^2_{\infty}} \pmod{3}
\]
\[
= (q^3; q^3)_{\infty}(q^{15}; q^{15})_{\infty} \left( \sum_{n=0}^{\infty} c_5(3n)q^{3n} + \sum_{n=0}^{\infty} c_5(3n+1)q^{3n+1} + \sum_{n=0}^{\infty} c_5(3n+2)q^{3n+2} \right)^2.
\]
Extracting those terms on each side for which the powers of \( q \) are of the form \( 3n+2 \), dividing by \( q^2 \), and replacing \( q^3 \) by \( q \), we obtain
\[
\sum_{n=0}^{\infty} c_5(9n+7)q^n \equiv (q; q)_{\infty}(q^5; q^5)_{\infty} \left( \sum_{n=0}^{\infty} c_5(3n+1)q^n \right)^2
\]
\[
+ 2\sum_{n=0}^{\infty} c_5(3n)q^n \sum_{n=0}^{\infty} c_5(3n+2)q^n \pmod{3}.
\]
It follows from Theorem 5 that
\[
\sum_{n=0}^{\infty} c_5(9n+7)q^n \equiv (q; q)_{\infty}(q^5; q^5)_{\infty} \left( (q; q)^2_{\infty}(q^5; q^5)^2_{\infty} - 2(q^3; q^3)_{\infty}(q^{15}; q^{15})_{\infty} \right)
\]
\[
\equiv -(q^3; q^3)_{\infty}(q^{15}; q^{15})_{\infty} \pmod{3}.
\]
This completes the proof. \(\Box\)

We now establish four Ramanujan type congruences for \( c_5(n) \).

**Theorem 7.** For all \( n \geq 0 \),
\[
c_5(15n+6) \equiv 0 \pmod{3}, \quad (11)
\]
\[
c_5(15n+10) \equiv 0 \pmod{3}, \quad (12)
\]
\[
c_5(15n+12) \equiv 0 \pmod{3}, \quad (13)
\]
\[
c_5(15n+13) \equiv 0 \pmod{3}. \quad (14)
\]

**Proof.** Recall that Euler’s pentagonal number theorem [2, p.36, Entry 22]
\[
(q; q)_{\infty} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}. \quad (15)
\]
Substituting (15) into (7), we have
\[ \sum_{n=0}^{\infty} c_5(3n)q^n \equiv \frac{1}{(q^5; q^5)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(3n+1)/2} \pmod{3}. \]

Extracting those terms on each side whose power of \( q \) is of the form \( 5n + 2 \) or \( 5n + 4 \), and employing the fact that there exist no integers \( n \) such that \( 3n(3n+1)/2 \) is congruent to 2 or 4 modulo 5, we get
\[ \sum_{n=0}^{\infty} c_5(15n + 6)q^{5n+2} \equiv \sum_{n=0}^{\infty} c_5(15n + 12)q^{5n+4} \equiv 0 \pmod{3}, \]
which means that
\[ c_5(15n + 6) \equiv c_5(15n + 12) \equiv 0 \pmod{3}. \]

Similarly, from (8) and the fact that there are no integers \( n \) with \( n(3n+1)/2 \) being congruent to 3 or 4 modulo 5, it is not hard to obtain
\[ c_5(15n + 10) \equiv c_5(15n + 13) \equiv 0 \pmod{3}. \]

This concludes the proof.

3 Two infinite families of congruences for \( c_5(n) \)

We start with investigating a generalization of the congruences (8) and (10).

**Theorem 8.** For \( \alpha \geq 1 \), we have
\[ \sum_{n=0}^{\infty} c_5 \left( 3^{2\alpha-1}n + \frac{3^{2\alpha-1}+1}{4} \right) q^n \equiv (-1)^{\alpha+1}(q^3; q^3)_\infty(q^{15}; q^{15})_\infty \pmod{3}, \]  
\[ \sum_{n=0}^{\infty} c_5 \left( 3^{2\alpha}n + \frac{3^{2\alpha+1}+1}{4} \right) q^n \equiv (-1)^{\alpha}(q^3; q^3)_\infty(q^{15}; q^{15})_\infty \pmod{3}. \]

**Proof.** We proceed by induction on \( \alpha \). The case \( \alpha = 1 \) corresponds to the congruences (8) and (10).

Assume that
\[ \sum_{n=0}^{\infty} c_5 \left( 3^{2\alpha}n + \frac{3^{2\alpha+1}+1}{4} \right) q^n \equiv (-1)^{\alpha}(q^3; q^3)_\infty(q^{15}; q^{15})_\infty \pmod{3} \]
is true for some fixed integer \( \alpha \geq 1 \). Since the terms appearing on the right side of the above congruence are powers of \( q^3 \), we have
\[ \sum_{n=0}^{\infty} c_5 \left( 3^{2\alpha}(3n) + \frac{3^{2\alpha+1}+1}{4} \right) q^{3n} \equiv (-1)^{\alpha}(q^3; q^3)_\infty(q^{15}; q^{15})_\infty \pmod{3}, \]
which yields that

$$\sum_{n=0}^{\infty} c_5 \left( 3^{2\alpha+1}n + \frac{3^{2\alpha+1} + 1}{4} \right) q^n \equiv (-1)^{\alpha+2} (q; q)_\infty (q^5; q^5)_\infty \pmod{3}. $$

Now we suppose that

$$\sum_{n=0}^{\infty} c_5 \left( 3^{2\alpha-1}n + \frac{3^{2\alpha-1} + 1}{4} \right) q^n \equiv (-1)^{\alpha+1} (q; q)_\infty (q^5; q^5)_\infty \pmod{3}$$

is true for some fixed integer $\alpha \geq 1$, to which applying the same argument as in the proof of Corollary 6 yields that

$$\sum_{n=0}^{\infty} c_5 \left( 3^{2\alpha}n + \frac{3^{2\alpha+1} + 1}{4} \right) q^n \equiv (-1)^{\alpha+2} (q^3; q^3)_\infty (q^{15}; q^{15})_\infty \pmod{3}.$$ 

The proof is complete. \(\square\)

As a consequence of Theorem 8, we have the following result.

**Corollary 9.** If $\alpha \geq 1$ and $n \geq 0$,

$$c_5 \left( 3^{2\alpha+1}n + \frac{7 \times 3^{2\alpha} + 1}{4} \right) \equiv 0 \pmod{3}, \quad (18)$$

$$c_5 \left( 3^{2\alpha+1}n + \frac{11 \times 3^{2\alpha} + 1}{4} \right) \equiv 0 \pmod{3}. \quad (19)$$

**Proof.** Note that all the terms on the right hand side of (17) are of the form $q^{3n}$. We can immediately obtain (18) and (19) by equating the coefficients of $q^{3n+1}$ and $q^{3n+2}$ on both sides of (17). \(\square\)

## 4 More infinite families of congruences for $c_5(n)$

To establish new congruences for $c_5(n)$, we need the following lemma.

**Lemma 10.** Let

$$\sum_{n=0}^{\infty} b(n)q^n = (q; q)_\infty (q^5; q^5)_\infty. \quad (20)$$

Then, for a given prime $p \geq 5$ with $\left( \frac{-5}{p} \right) = -1$, we have

$$\sum_{n=0}^{\infty} b \left( pn + \frac{p^2 - 1}{4} \right) q^n = (q^p; q^p)_\infty (q^{5p}; q^{5p})_\infty. \quad (21)$$

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Proof. Applying Euler’s pentagonal number theorem, we have
\[
\sum_{n=0}^{\infty} b(n)q^{n} = \sum_{m,n=-\infty}^{\infty} (-1)^{m+n}q^{m(3m+1)/2+5n(3n+1)/2}.
\] (22)

We now consider
\[
m(3m+1)/2 + 5n(3n+1)/2 \equiv p^2 - 1/4 \pmod{p},
\]
amely,
\[
(6m + 1)^2 + 5(6n + 1)^2 \equiv 0 \pmod{p}.
\]
Since \((-\frac{5}{p}) = -1\), we deduce that
\[
6m + 1 \equiv 6n + 1 \equiv 0 \pmod{p}.
\]
If \(p \equiv 1 \pmod{6}\), then \(m \equiv n \equiv \frac{p-1}{6} \pmod{p}\). Let
\[
m = kp + \frac{p-1}{6} \quad \text{and} \quad n = lp + \frac{p-1}{6},
\]
we have
\[
m(3m+1)/2 + 5n(3n+1)/2 = (p^2 - 1)/4 + p^2(3k^2 + k)/2 + 5p^2(3l^2 + l)/2.
\]
If \(p \equiv -1 \pmod{6}\), then \(m \equiv n \equiv -\frac{p-1}{6} \pmod{p}\). Let
\[
m = -kp - \frac{p+1}{6} \quad \text{and} \quad n = -lp - (p+1)/6,
\]
we also have
\[
m(3m+1)/2 + 5n(3n+1)/2 = (p^2 - 1)/4 + p^2(3k^2 + k)/2 + 5p^2(3l^2 + l)/2.
\]
Extracting the terms whose power of \(q\) is congruent to \(p^2-1/4\) modulo \(p\) from (22), and employing the above analysis, we obtain
\[
\sum_{n=0}^{\infty} b\left(pn + \frac{p^2-1}{4}\right)q^{pn+\frac{p^2-1}{4}} = \sum_{k,l=-\infty}^{\infty} (-1)^{k+l}q^{(p^2-1)/4 + p^2(3k^2+k)/2 + 5p^2(3l^2+l)/2},
\]
which can be simplified to
\[
\sum_{n=0}^{\infty} b\left(pn + \frac{p^2-1}{4}\right)q^{n} = \sum_{k,l=-\infty}^{\infty} (-1)^{k+l}q^{p(3k^2+k)/2 + 5p(3l^2+l)/2}.
\]
Applying Euler’s pentagonal number theorem again, we derive that
\[
\sum_{n=0}^{\infty} b\left(pn + \frac{p^2-1}{4}\right)q^{n} = (q^p; q^p)_{\infty}(q^{5p}; q^{5p})_{\infty},
\]
which completes the proof. \(\square\)
Based on Lemma 10, we can easily obtain the following congruence.

**Theorem 11.** If $p \geq 5$ is a prime with $\left( \frac{-5}{p} \right) = -1$, we have
\[
\sum_{n=0}^{\infty} c_5 \left( 3pm + \frac{p^2 - 1}{4} \right) q^n \equiv (q^p; q^p)_\infty (q^{5p}; q^{5p})_\infty \pmod{3}.
\] (23)

**Proof.** It follows from (8) and (20) that
\[
c_5(3n + 1) \equiv b(n) \pmod{3}.
\]
Applying Lemma 10, we deduce that
\[
\sum_{n=0}^{\infty} c_5 \left( 3 \left( pm + \frac{p^2 - 1}{4} \right) + 1 \right) q^n \equiv (q^p; q^p)_\infty (q^{5p}; q^{5p})_\infty \pmod{3},
\]
which finishes the proof.

One can generalize the above congruence to the form as we show below.

**Theorem 12.** Given a prime $p \geq 5$ with $\left( \frac{-5}{p} \right) = -1$, then for all $\alpha \geq 1$, we have
\[
\sum_{n=0}^{\infty} c_5 \left( 3p^{2\alpha - 1}n + \frac{3p^{2\alpha} + 1}{4} \right) q^n \equiv (q^p; q^p)_\infty (q^{5p}; q^{5p})_\infty \pmod{3}.
\] (24)

**Proof.** The proof follows by induction on $\alpha$. The case $\alpha = 1$ is given in Theorem 11. Assuming the result holds for a positive integer $\alpha = t$, namely,
\[
\sum_{n=0}^{\infty} c_5 \left( 3p^{2t - 1}n + \frac{3p^{2t} + 1}{4} \right) q^n \equiv (q^p; q^p)_\infty (q^{5p}; q^{5p})_\infty \pmod{3}.
\]
Choosing those terms on each side whose power of $q$ is of the form $pn$, and replacing $q^p$ by $q$, we obtain
\[
\sum_{n=0}^{\infty} c_5 \left( 3p^{2t}n + \frac{3p^{2t} + 1}{4} \right) q^n \equiv (q; q)_\infty (q^{5}; q^{5})_\infty \pmod{3},
\]
which implies that
\[
c_5 \left( 3p^{2t}n + \frac{3p^{2t} + 1}{4} \right) \equiv b(n) \pmod{3}.
\]
Furthermore, from Lemma 10 we see that
\[
\sum_{n=0}^{\infty} c_5 \left( 3p^{2t} \left( pm + \frac{p^2 - 1}{4} \right) + \frac{3p^{2t} + 1}{4} \right) q^n \equiv (q^p; q^p)_\infty (q^{5p}; q^{5p})_\infty \pmod{3},
\]
which upon simplification completes the induction on $\alpha$. 

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As an immediate consequence of Theorem 12, we obtain the following infinite families of congruences for \( c_5(n) \).

**Corollary 13.** Given a prime \( p \geq 5 \) with \( \left( \frac{-5}{p} \right) = -1 \), if \( \alpha \geq 1 \) and \( n \geq 0 \), we have

\[
c_5 \left( 3p^{2\alpha} n + 3p^{2\alpha-1} i + \frac{3p^{2\alpha} + 1}{4} \right) \equiv 0 \pmod{3},
\]

where \( i = 1, 2, \ldots, p - 1 \).

**Proof.** Collecting those terms on each side of (24) for which the powers of \( q \) are of the form \( pn + i \), dividing by \( q^i \), and replacing \( q^p \) by \( q \), we obtain that for \( i = 1, 2, \ldots, p - 1 \),

\[
\sum_{n=0}^{\infty} c_5 \left( 3p^{2\alpha-1}(pn + i) + \frac{3p^{2\alpha} + 1}{4} \right) q^n \equiv 0 \pmod{3},
\]

which proves the claim in the corollary. \( \square \)

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