A note on maxima in random walks

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Submitted: Jun 13, 2015; Accepted: Jan 13, 2016; Published: Jan 22, 2016
Mathematics Subject Classifications: 60G50, 05A19

Abstract

We give a combinatorial proof that a random walk attains a unique maximum with probability at least $1/2$. For closed random walks with uniform step size, we recover Dwass’s count of the number of length $\ell$ walks attaining the maximum exactly $k$ times. We also show that the probability that there is both a unique maximum and a unique minimum is asymptotically equal to $1/4$ and that the probability that a Dyck word has a unique minimum is asymptotically $1/2$.

Keywords: Random walk, Dyck Words, Catalan Numbers

1 Introduction

A length $\ell$ walk is a sequence $w : \{1, \ldots, \ell\} \to \{\pm 1\}$. The trajectory of $w$ is the sequence $\bar{w} : \{0, \ldots, \ell\} \to \mathbb{Z}$ defined by $\bar{w}(j) = \sum_{i=1}^{j} w(i)$. We define $\max(w) = \sup \{\bar{w}(j) : 0 \leq j \leq \ell\}$. The length $\ell$ walk $w$ is closed if $\bar{w}(\ell) = 0$. Let $C(n)$ denote the set of length $2n$ closed walks and let $M(n) \subset C(n)$ denote the subset consisting of those walks $w$ for which there is a unique $i \in \{1, \ldots, 2n\}$ such that $\bar{w}(i) = \max(w)$.

The paper centers around a combinatorial proof of the following theorem which was first proven by Dwass in [5]:

Theorem 2.4 $|M(n)| = \frac{1}{2} |C(n)|$ for each $n \geq 1$.

In [5], this is proven by a method that computes the probabilities of events in finite random walks by relating them to events in infinite random walks for which probabilities are more readily computed. This is a general analytic method used to compute a large

∗Research supported by NSERC
number of quantities including $\mathcal{M}(n)$ as well as the more general $\mathcal{M}(n, r)$ which we discuss in Section 3.

As is often the case, a combinatorial proof offers other intuition and insight. In this case, we will see that our method generalizes to certain cases not amenable to Dwass’s method.

In Section 2 we give a first proof of Dwass’s result, which uses a method we will employ fundamentally in the text. A second more transparent proof is given in Section 6. This second proof uses that the number of Dyck words is the corresponding Catalan number.

In Section 3 we recover Dwass’s stronger result that there are precisely $\frac{2n-r}{r-1}$ length $2n$ closed walks attaining their maximum exactly $r$ times. In Section 4 we explain that the combinatorial proof generalizes to show that more general types of finite random walks have probability $\geq \frac{1}{2}$ of attaining a unique maximum. This conclusion does not assume that the walks are closed and allows an arbitrary distribution of step sizes. In Section 5 we show that the probability of having both a unique minimum and a unique maximum approaches $\frac{1}{4}$ as the length of a uniform closed walk increases. In Section 6 we show that the probability that a length $n$ Dyck word has a unique minimum approaches $\frac{1}{2}$ as $n \to \infty$.

2 Dyck Words and Leads

A Dyck word of length $2n$ is a closed walk $w$ such that $\max(w) = 0$. Let $\mathcal{D}(n)$ denote the set of length $2n$ Dyck words. The number of Dyck words of a given length is the corresponding Catalan number:

\begin{equation}
|\mathcal{D}(n)| = \frac{1}{n+1} \binom{2n}{n}.
\end{equation}

The lead of $w \in \mathcal{C}(k)$ is the number of values $i \in \{1, \ldots, 2k\}$ with both $w(i) > 0$ and $\bar{w}(i) > 0$. For $0 \leq e \leq k$ let $\mathcal{L}(k, e) \subset \mathcal{C}(k)$ be the set of lead $e$ walks (so that $\mathcal{L}(k, 0) = \mathcal{D}(k)$). Thus $\mathcal{C}(k) = \bigsqcup_{e=0}^{k} \mathcal{L}(k, e)$.

Since $|\mathcal{C}(n)| = \binom{2n}{n}$, Theorem 2.1 follows from the Chung-Feller Theorem which states that $|\mathcal{L}(k, e)|$ is independent of $e$. Among the many proofs of the Chung-Feller theorem is a bijective explanation given in [1, 6] the former of which traced the explanation to [4]. We now recount the bijection:

Lemma 2.2. For each $1 \leq e \leq k$ there is a bijection $\psi : \mathcal{L}(k, e-1) \to \mathcal{L}(k, e)$.

Proof. Let $w \in \mathcal{L}(k, e-1)$. Let $p > 0$ be maximal such that $w(p) = -1$ and $\bar{w}(p) = -1$. Regard $w$ as a string in $\{-1\}$, and express $w$ as the concatenation $axb$ where $a$ is the initial length $p$ subpath and $x$ is a single symbol (which is necessarily $-1$). Define $\psi(w)$ to be the sequence corresponding to $bxa$. This lies in $\mathcal{L}(k, e)$ since $a, xb, bx$ are all closed and the lead of $bx$ is one greater than that of $b$.

The map $\psi^{-1}$ is defined by recognizing the decomposition of $w \in \mathcal{L}(k, e)$ as a concatenation $bxa$ by declaring $b$ to be the length $n$ subword where $n$ is minimal such that $w(n+1) = -1$ and $\bar{w}(n+1) = 0$. \qed
Figure 1: Performing a swap

\[
\begin{array}{cccccccc}
M(1,2) & 1 \\
M(1,1) & M(2,3) & 1 & 1 \\
M(2,2) & M(3,4) & 2 & 1 \\
M(2,1) & M(3,3) & M(4,5) & 3 & 3 & 1 \\
M(3,2) & M(4,4) & M(5,6) & 6 & 4 & 1 \\
M(3,1) & M(4,3) & M(5,5) & M(6,7) & 10 & 10 & 5 & 1 \\
M(4,2) & M(5,4) & M(6,6) & M(7,8) & 20 & 15 & 6 & 1 \\
M(4,1) & M(5,3) & M(6,5) & M(7,7) & M(8,9) & 35 & 35 & 21 & 7 & 1 \\
\end{array}
\]

Figure 2: The base cases (1) are the entries 1, 3, 10, 35, . . . and the 1 at the top. The identity (2) states that an entry in Pascal’s triangle is the sum of all the numbers in the diagonal path above it, e.g. 15=10+4+1.

Remark 2.3. We utilize $\psi : \mathcal{L}(k,0) \rightarrow \mathcal{L}(k,1)$ whose crucial property is that $\psi(w) \in M(k)$ for $w \in \mathcal{L}(k,0)$.

Theorem 2.4. $|M(n)| = \frac{1}{2} |C(n)|$ for each $n \geq 1$.

Proof. We describe a bijection $\Psi : C(n) - M(n) \rightarrow M(n)$. Let $w \in C(n) - M(n)$. As rank($w$) $\geq 2$, we let $a$ be the nontrivial subsequence of $w$ with domain $\{p, \ldots, q\} \subset \mathbb{N}$ where $p-1, q$ are the minimal and maximal values of $\bar{w}^{-1}(\max(w))$. Note that $a \in \mathcal{L}(q-p+1,0)$ and let $\psi(a) \in \mathcal{L}(q-p+1,1)$ be as provided by Lemma 2.2. Define $\Psi(w)$ to be the sequence obtained from $w$ by substituting $\psi(a)$ for $a$ as in Figure 1. Note that $\Psi(w) \in M(n)$ by Remark 2.3.

An alternate proof is given in Section 6.

3 Counting walks of arbitrary rank

The rank of a walk $w$ is $|\bar{w}^{-1}(\max(w))|$. For $r \geq 1$ let $M(n,r) \subset C(n)$ denote the subset of rank $r$ length $2n$ closed walks. As $M(n,1) = M(n)$, Theorem 2.4 states that $|M(n,1)| = \frac{1}{2} \binom{2n}{n} = \binom{2n-1}{n-1}$. We now extend this result and count each $M(n,r)$:

Theorem 3.1. For $n, r \geq 1$ we have $|M(n,r)| = \binom{2n-r}{n-1}$.

Proof. Let $M(n,r) = \binom{2n-r}{n-1}$. We first observe that the numbers $M(n,r)$ satisfy the following recursive definition (see Figure 2).
The base cases are:

\[ M(n, 1) = \binom{2n - 1}{n - 1} \quad \text{and} \quad M(1, 2) = 1 \quad (1) \]

The inductive step for \( n, r \geq 2 \) follows by iterating Pascal’s identity:

\[ M(n, r) = \sum_{j=r-1}^{n} M(n-1, j) \quad (2) \]

We will show that Equations (1) and (2) are satisfied with \( M(a, b) \) replaced by \( |M(a, b)| \), from which it follows that \( |M(n, r)| = M(n, r) \) for all \( n, r \geq 1 \) as desired.

Equation (1) is easy as Theorem 2.4 asserts that \( |M(n, 1)| = \binom{2n - 1}{n - 1} \) and obviously \( |M(1, 2)| = 1 \).

To verify Equation (2), for each \( n, r \geq 2 \), we describe a bijection \( \chi_r \) between \( \bigcup_{j=r-1}^{n} M(n-1, j) \) and \( M(n, r) \). Let \( w \in \bigcup_{j=r-1}^{n} M(n-1, j) \). As \( \text{rank}(w) \geq r - 1 \), we may consider the \((r-1)\)th maximal peak of \( w \), counted from the right. As in Figure 3, we insert a peak at this point to obtain a path \( \hat{w} \in M(n) \). We then apply the map \( \Psi^{-1} : M(n) \to \mathcal{C}(n) \setminus M(n) \) to \( \hat{w} \) to obtain \( \chi_r(w) \).

We must show that \( \chi_r \) is injective and that its image is \( M(n, r) \). The former holds since a left-inverse to \( \chi_r \) is obtained by first applying \( \Psi \) and then removing the single maximal peak. The latter holds by Lemma 3.2.

**Lemma 3.2.** Let \( n, r \geq 2 \), let \( w \in M(n, r) \) and let \( w' \) be obtained by removing the single maximal peak from \( \Psi(w) \). Then \( \text{rank}(w') \geq r - 1 \).

**Proof.** The general case follows from the case where \( w \in \mathcal{D}(m) \cap M(m, r) \) so that \( w' \in \mathcal{D}(m-1) \). We express \( w \) as \( axb \) where the length \( i \) of \( a \) is maximal such that \( i < m \) and \( \bar{w}(i) = 0 \). Then \( \Psi(w) = bx a \) by definition. As \( \text{rank}(w) = r \), we have \( \text{rank}(a) = r - 1 \). Hence \( \text{rank}(w') \geq r - 1 \) as desired. \( \square \)

### 4 Variable step lengths

In this section we provide a different generalization of Theorem 2.4: we prove that for a random closed walk of variable step size, and with a nonzero fixed number of each type of step, the probability of attaining a unique maximum is at least \( \frac{1}{2} \).

**Definition 4.1.** Let \( S \) be a finite set. A length \( \ell \) \( S \)-walk \( w \) is a sequence \( \{1, \ldots, \ell\} \to S \). Let \( \mathcal{W}_S(n) \) denote the set of length \( n \) \( S \)-walks. Let \( v : S \to \mathbb{R} \). The \( v \)-trajectory of \( w \) is...
\[ \bar{w}_v(j) = \sum_{i=1}^j v(w(i)). \] The \textit{v-maximum} of \( w \) is \( \max_v(w) = \max\{\bar{w}_v(j) : 0 \leq j \leq \ell\} \). Let \( M_v(n) \subset W_S(n) \) denote the subset consisting of those walks \( w \) for which there is a unique \( i \in \{1, \ldots, n\} \) such that \( \bar{w}(i) = \max(w) \). The length \( \ell \) \( S \)-walk \( w \) is \( v \)-\textit{closed} if \( \bar{w}_v(\ell) = 0 \).

Let \( D_v(n) \subset W_s(n) \) denote the set of length \( N \) \( v \)-closed \( S \)-walks with \( v \)-maximum 0.

The proof of Lemma 2.2 can be carried over to this more general context:

**Lemma 4.2.** There is an injection \( \psi_v : D_v(n) \rightarrow M_v(n) \)

**Proof.** Let \( w \in D_v(n) \). Let \( p < n \) be maximal such that \( \bar{w}_v(p) = 0 \). Regard \( w \) as a string in \( S \), and express \( w \) as the concatenation \( axb \) where \( a \) is the initial length \( p \) subpath and \( x \) is a single symbol (which necessarily satisfies \( v(x) < 0 \)). Define \( \psi_v(w) \) to be the sequence corresponding to \( bxa \). This lies in \( M_v(n) \) since \( a \) has \( v \)-maximum 0 and \( b_x \) obtains its unique maximum at the end.

To see that \( \psi_v \) is injective, we describe its (left-)inverse. Any walk \( w \) in the image of \( \psi_v \) has the form \( bxa \), where \( \max_v(a) = 0 \), \( b \) has length \( q \), \( \bar{w}_v(q) > 0 \), and \( \bar{w}_v(q + 1) = 0 \). Moreover there can clearly be at most one such representation of \( w \). The unique pre-image of \( w \) under \( \psi_v \) is then \( axb \). \( \square \)

**Theorem 4.3.** Let \( S \) be a finite set and let \( v : S \rightarrow \mathbb{R} \). Then \( |M_v(n)| \geq \frac{1}{2} |W_S(n)| \).

**Proof.** The proof is the same as that of Theorem 2.4: we use \( \psi_v \) in the same way to define a map \( \Psi_v : W_S(n) - M_v(n) \rightarrow M_v(n) \) and this is an injection. \( \square \)

**Remark 4.4.** Let \( X \subset W_S(n) \) be any \( \Psi_v \)-invariant subset. Then by restricting \( \Psi_v \) to \( X \cap (W_S(n) - M_v(n)) \), we see that \( |X \cap M_v(n)| \geq \frac{1}{2} |X| \). For example, we could take \( X \) to be the set of \( v \)-closed walks, since \( \Psi_v \) takes \( v \)-closed walks to \( v \)-closed walks. Also, note that \( \Psi_v \) preserves the cardinality of \( S \)-fibers in the sense that \( |w^{-1}(s)| = |(\Psi_v)w^{-1}(s)| \) for each \( s \in S \). Hence, we could take \( X \) to be the set of walks whose \( S \)-fibers have some prescribed cardinalities.

**Remark 4.5.** We can also consider weighted random walks. That is, we let \( \mu \) be a probability measure on \( S \), and consider the induced measure \( \mu \) on \( W(n) \) that assigns to a walk \( w \) the probability \( \frac{1}{n} \sum_{s \in S} |w^{-1}(s)| \mu(s) \). Theorem 4.3 also generalizes to this case: with respect to this measure, the measure of \( M_v(n) \) is at least \( \frac{1}{2} \). This works because the measure is \( \Psi_v \)-invariant, so

\[
\mu_v(W_S(n) - M_v(n)) = \sum_{w \in W_S(n) - M_v(n)} \mu(w) = \sum_{w \in W_S(n) - M_v(n)} \mu(\Psi(w)) \leq \sum_{w \in M_v(n)} \mu(\Psi(w)) = \mu_v(M_v(n))
\]

5 Estimating the probability of a unique max and a unique min

The goal of this section is to prove Theorem 5.9 which gives a \( \frac{1}{4} \) asymptotic probability that a random walk with uniform step size has both a unique minimum and a unique maximum. The strategy of the proof is to show that there is a dense subset having a partition into four equal cardinality parts lying in:

\[
M \cap -M \quad M^c \cap -M \quad M \cap (-M)^c \quad M^c \cap (-M)^c
\]
Definition 5.1. Let \( w \in \mathcal{C}(n) \). If \( w \notin \mathcal{M}(n) \), we define the max-interval of \( w \) to be the largest subsequence \( \{p, \ldots, q\} \) of \( \{1, \ldots, 2n\} \) such that \( \bar{w}(p-1) = \max(w) = \bar{w}(q) \). For \( w \in \mathcal{M}(n) \), we define the max-interval of \( w \) to be the max-interval of \( \Psi^{-1}(w) \). Note that the max-interval subsumes the part of \( w \) which is modified by \( \Psi \) (or \( \Psi^{-1} \)). We define the min-interval of \( w \) to be the max-interval of \(-w\). The size of the max-interval is its cardinality and likewise for the min-interval.

The max-interval and min-interval are generically small in the following sense:

Lemma 5.2. Let \( \mathcal{U}_+(n,k) \subset \mathcal{C}(n) \) be the set of walks whose max-interval has size \( 2k \). Similarly, \( \mathcal{U}_-(n,k) \) denotes the walks with a size \( 2k \) min-interval. For any \( \epsilon > 0 \), there exists \( N \) such that for all \( n \geq N \) we have:

\[
\frac{1}{|\mathcal{C}(n)|} \left| \bigcup_{k=N}^{n} \mathcal{U}_+(n,k) \right| = \sum_{k=N}^{n} \frac{|\mathcal{U}_+(n,k)|}{|\mathcal{C}(n)|} < \epsilon
\]

and similarly for \( \mathcal{U}_- \).

Proof. We will prove the claim for \( \mathcal{U}_+ \) as the proof for \( \mathcal{U}_- \) is identical.

For \( 1 \leq k \leq n \), let \( \mathcal{U}_+(n,k) = \mathcal{U}_+(n,k) \setminus \mathcal{M}(n) \). From the definitions we have \( |\mathcal{U}_+(n,k)| = \frac{1}{2} |\mathcal{U}_+(n,k)| \) so it suffices to prove the claim with \( \mathcal{U}_+ \) replaced by \( \mathcal{U}_+^c \).

Let \( \epsilon > 0 \). For any \( 1 \leq k \leq n \),

\[
|\mathcal{U}_+(n,k)| = |\mathcal{D}(k)| \cdot |\mathcal{M}(n-k)| = \frac{1}{k+1} \frac{2k}{k} \cdot \frac{1}{2} \left( \frac{2(n-k)}{n-k} \right)
\]

Indeed, each \( w \in \mathcal{U}_+(n,k) \) corresponds to a pair \((d,m)\) with \( d \in \mathcal{D}(k) \) and \( m \in \mathcal{M}(n-k) \). The correspondence arises by inserting \( d \) at the maximum of \( m \).

We now have the following inequality which proves the claim. Its first part holds since \( \bigcup_{k=1}^{n} \mathcal{U}_+(n,k) = \mathcal{C}(n) \setminus \mathcal{M}(n) \) and \( |\mathcal{M}(n)| = \frac{1}{2} |\mathcal{C}(n)| \) by Theorem 2.4. Its second part holds by Lemma 5.3 and its last part holds for \( N \) sufficiently large by Lemma 5.4

\[
\sum_{k=N}^{n} \frac{|\mathcal{U}_+(n,k)|}{|\mathcal{C}(n)|} = \frac{1}{2} - \sum_{k=1}^{N-1} \frac{|\mathcal{U}_+(n,k)|}{|\mathcal{C}(n)|} \leq \frac{1}{2} - \sum_{k=1}^{N-1} \frac{1}{2k+1} \left( \frac{2k}{k} \right) 4^{-k} < \epsilon \]

Lemma 5.3. \( \binom{2(n-k)}{n-k} / \binom{2n}{n} \geq 4^{-k} \) for \( 0 \leq k \leq n \).

Proof. For each fixed \( n \), we prove this by induction on \( k \).

Base case: \( \binom{2(n-0)}{n-0} / \binom{2n}{n} = 1 \geq 4^0 \)

Inductive step: For \( 0 \leq k < n \)

\[
\frac{\binom{2(n-k)}{n-k}}{\binom{2n}{n}} / \left( \frac{2n}{n} \right) = \frac{(2(n-k)-2)}{(2(n-k)-1)} = \frac{(n-k)^2}{(2(n-k) - 1)(2(n-k))} > 4^{-1}
\]

\[\square\]
Lemma 5.4. \( \sum_{k=1}^{\infty} \frac{1}{2^k+1} \binom{2k}{k} 4^{-k} = \frac{1}{2} \).

Proof. The well-known generating function for the Catalan numbers is

\[
\sum_{k=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} x^k = \frac{1 - \sqrt{1 - 4x}}{2x}
\]

where this equality holds for \( |x| < 1/4 \). Setting \( x = 1/4 \), the left-hand side converges by the elementary estimate \( \binom{2k}{k} \leq \frac{4^k}{\sqrt{3k+1}} \) of the central binomial coefficient. We thus obtain the following by applying \( \lim_{x \to \frac{1}{4}} \) to each side, and note that Abel’s theorem ensures the convergence of this limit on the left.

\[
\sum_{k=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} 4^{-k} = 2
\]

The conclusion follows since the 0-th term of this series is 1. \( \square \)

Lemma 5.5. If the max-interval and min-interval of \( w \) intersect and have size \( s_1 \) and \( s_2 \), then:

\[
\text{max}(w) - \text{min}(w) \leq \frac{1}{2} \sup(s_1, s_2)
\]

Proof. Let \( \{a, \ldots, b\} \) be the max-interval of \( w \). By hypothesis, there is \( c \in \{a, \ldots, b\} \) with \( \bar{w}(c) = \min(w) \). Clearly, the difference between the maximum and minimum on a size \( s \) interval is at most \( s \). Since \( \max(w) \) and \( \min(w) \) are both attained on \( \{a, \ldots, c\} \) and on \( \{c, \ldots, b\} \), and since one of these intervals is of size at most \( \frac{s}{2} \), it follows that \( \max(w) - \min(w) \leq \frac{s}{2} \). Similarly, \( \max(w) - \min(w) \leq \frac{s}{2} \). \( \square \)

The following is a classical fact about random walks, and we refer to [7] for an account of its history:

Lemma 5.6 (Reflection principle). For \( h \geq 0 \), the number of walks \( w \in \mathcal{C}(n) \) with \( \max w \geq h \) is equal to \( \binom{2n}{n+h} \).

Proof. For any \( w \in \mathcal{C}(n) \) with \( \max(w) \geq h \), define \( Rw \) by

\[
(Rw)(i) = \begin{cases} w(i); & i \leq I(w) \\ -w(i); & i > I(w) \end{cases}
\]

where \( I(w) \) is minimal such that \( \bar{w}(i) = h \). Then \( R \) is an injection onto the set of walks \( w \in \mathcal{W}(2n) \) such that \( \bar{w}(2n) = 2h \). The cardinality of the latter set is \( \binom{2n}{n+h} \). \( \square \)

Lemma 5.7 (Generically Disjoint). Let \( \mathcal{J}(n) \subset \mathcal{C}(n) \) be the subset of walks whose max-interval and min-interval are disjoint. Then \( \lim_{n \to \infty} \frac{\mathcal{J}(n)}{\mathcal{C}(n)} = 1 \).
Proof. Fix \( \epsilon > 0 \). Let \( N \) be as in Lemma 5.2 and let \( n \geq N \). Let \( O(n) = C(n) \setminus J(n) \) consist of those walks whose max-interval and min-interval overlap. Let

\[
K_1(n) = O(n) \cap \left( \bigcup_{k=N}^{n} U_+(n, k) \cup \bigcup_{k=N}^{n} U_-(n, k) \right)
\]

(so that \( \frac{|K_1(n)|}{|C(n)|} \leq 2\epsilon \) by Lemma 5.2) and let

\[
K_2(n) = O(n) \cap \left( \bigcup_{k=1}^{N-1} U_+(n, k) \cap \bigcup_{k=1}^{N-1} U_-(n, k) \right)
\]

By Lemma 5.5 for any \( w \in K_2(n) \) we have \( \max(w) < N \). Hence, by Lemma 5.6, we have

\[
\frac{|K_2(n)|}{|C(n)|} \leq \frac{(2n)^2}{(2n)^2} - 1 - \frac{(n-N+1) \cdots (n-N+N)}{(n+1) \cdots (n+N)} \xrightarrow{n \to \infty} 0
\]

and so

\[
\lim_{n \to \infty} \frac{|O(n)|}{|C(n)|} = \lim_{n \to \infty} \frac{|K_1(n)|}{|C(n)|} + \lim_{n \to \infty} \frac{|K_2(n)|}{|C(n)|} < 2\epsilon + \lim_{n \to \infty} \frac{|K_2(n)|}{|C(n)|} = 2\epsilon.
\]

Hence, \( \frac{|J(n)|}{|C(n)|} \geq 1 - 2\epsilon \) for every \( \epsilon > 0 \), which proves the claim.

\[\square\]

**Lemma 5.8.** \( J(n) \) is partitioned into 4 subsets of equal cardinality according to whether there is a unique max and/or unique min.

**Proof.** Since the max-interval and min-interval of elements of \( J(n) \) are disjoint, it is easily seen that the restrictions of the map \( \Psi \) to \( J(n) \) leaves \( J(n) \cap -M(n) \) invariant and hence provides bijections

\[
(J(n) \cap -M(n)) \setminus M(n) \rightarrow (J(n) \cap -M(n)) \cap M(n)
\]

and

\[
(J(n) \setminus -M(n)) \setminus M(n) \rightarrow (J(n) \setminus -M(n)) \cap M(n).
\]

Similarly, the map \( w \mapsto -\Psi(-w) \) provides bijections

\[
(J(n) \setminus -M(n)) \cap M(n) \rightarrow (J(n) \cap -M(n)) \cap M(n)
\]

and

\[
(J(n) \setminus -M(n)) \setminus M(n) \rightarrow (J(n) \cap -M(n)) \setminus M(n).
\]

Combining these gives the desired one-to-one-to-one correspondence. \[\square\]

**Theorem 5.9.** \( \lim_{n \to \infty} \frac{|M(n) \cap -M(n)|}{|C(n)|} = \frac{1}{4} \).

**Proof.** Combine Lemma 5.7 and Lemma 5.8. \[\square\]
Remark 5.10. The first few terms of the sequence \(|M(n) \cap -M(n)|\) are:

0, 2, 4, 18, 64, 230, 852, 3206, 12144, 46188, \ldots

The convergent sequence \(|M(n) \cap -M(n)| / |C(n)|\) has the following initial terms, where \(d = 29099070\):

\[
\begin{array}{cccccccc}
0 & 9699690 & 5819814 & 7482618 & 7390240 & 7243275 & 7248766 & 7274610 \\
\frac{d}{d} & \frac{d}{d} & \frac{d}{d} & \frac{d}{d} & \frac{d}{d} & \frac{d}{d} & \frac{d}{d} & \frac{d}{d} \\
\end{array}
\]

It is not monotonic, but we have verified that its terms are \(\geq 1/4\) for all \(n \geq 1\).

6 Dyck words with a unique maximum

In this section we show that, asymptotically, one half of all Dyck words have a unique maximum. We refer to [3] for a variety of other elegant counts of frequencies of various configurations within Dyck words.

We begin by describing a second, more straightforward, proof of Theorem 2.4.

Cyclic Permutation Proof. We describe a map \(M(n) \to D(n-1)\). We cyclically permute so that the maximum appears at the beginning and end. This yields a \(2n-1\)-to-1 map to length \(2n\) Dyck words whose trajectories are negative except at the endpoints. After removing the first and last edges, we obtain a \((2n-1)\)-to-1 map from \(M(n) \to D(n-1)\). Since \(|D(n-1)| = \frac{1}{n} \binom{2n-2}{n-1}\) by Theorem 2.1, we have: \(M(n) = \frac{2n-1}{n} \binom{2n-2}{n-1} = \frac{1}{2} \binom{2n}{n}\).

Theorem 6.1. \(\lim_{n \to \infty} \frac{|D(n) \cap -M(n)|}{|D(n)|} = \frac{1}{2}\).

Proof. We employ the \((2n-1)\)-to-1 map \(M(n) \to D(n-1)\) from the above proof of Theorem 2.4. Observe that an element of \(D(n-1)\) has a unique minimum if and only if \((2n-2)\) of its \((2n-1)\) pre-images have a unique minimum.

Thus:

\(|D(n-1) \cap -M(n-1)| = \frac{1}{2n-2} |M(n) \cap -M(n)|\)

and hence

\[\lim_{n \to \infty} \frac{|D(n-1) \cap M(n-1)|}{|D(n-1)|} = \lim_{n \to \infty} \frac{1}{2n-2} \frac{1}{|M(n)|} \frac{|M(n) \cap -M(n)|}{|D(n)|} = \frac{1}{2}\]

where the last equality is by Theorem 5.9.

Remark 6.2. As in Remark 5.10, we note that \(\frac{|D(n) \cap -M(n)|}{|D(n)|} \geq \frac{1}{2}\) for all \(n\).
References


