Subspaces intersecting each element of a regulus in one point, André-Bruck-Bose representation and clubs

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Abstract

In this paper results are proved with applications to the orbits of \((n-1)\)-dimensional subspaces disjoint from a regulus \(R\) of \((n-1)\)-subspaces in \(\text{PG}(2n-1,q)\), with respect to the subgroup of \(\text{PGL}(2n,q)\) fixing \(R\). Such results have consequences on several aspects of finite geometry. First of all, a necessary condition for an \((n-1)\)-subspace \(U\) and a regulus \(R\) of \((n-1)\)-subspaces to be extendable to a Desarguesian spread is given. The description also allows to improve results in [2] on the André-Bruck-Bose representation of a \(q\)-subline in \(\text{PG}(2,q^n)\). Furthermore, the results in this paper are applied to the classification of linear sets, in particular clubs.

Keywords: club; linear set; subplane; André-Bruck-Bose representation; Segre variety

1 Introduction

The \((n-1)\)-dimensional projective projective space over the field \(F\) is denoted by \(\text{PG}(n-1,F)\) or \(\text{PG}(n-1,q)\) if \(F\) is the finite field of order \(q\) (denoted by \(\mathbb{F}_q\)). The set of nonzero elements of a field \(F\) will be denoted by \(F^*\), and similarly, the set of nonzero vectors of a vector space \(V\) by \(V^*\). If \(L\) is an extension field \(\mathbb{F}_q\), then the projective space defined by

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the $F_q$-vector space induced by $L^d$ is also denoted by $\text{PG}_q(L^d)$. For a (sets of) subspace(s) $R$ of a vector space or a projective space, the notation $\langle R \rangle$ is used to denote the subspace generated by (the elements of) $R$. In case there is any ambiguity about the coefficient field, then the notation $\langle R \rangle_q$ will be used, to denote that the considered subspace is the one generated over $F_q$. In this case the terminology of $F_q$-span will sometimes be used. For example, if $S$ is a set of two points on the projective line $\text{PG}(1,q^2)$, then $\langle S \rangle_q$ denotes the $F_q$-subline defined by $S$, while $\langle S \rangle_{q^2}$ coincides with the whole projective line $\text{PG}(1,q^2)$.

For further notation and general definitions employed in this paper the reader is referred to [9, 11, 13].

For more information on Desarguesian spreads see [1].

This paper is structured as follows. In Section 2 subspaces which intersect each element of a regulus in one point are studied and a result from [4] is generalised. Section 3 contains one of the main results of this paper, determining the order of the normal rational curves obtained from $n$-dimensional subspaces on an external $(n-1)$-dimensional subspace with respect to a regulus in $\text{PG}(2n-1,q)$, obtained from a point and a subline after applying the field reduction map to $\text{PG}(1,q^n)$. This leads to a necessary condition on the existence of a Desarguesian spread containing a subspace and regulus (Corollary 7). The André-Bruck-Bose representation of sublines and subplanes of a finite projective plane is studied in Section 4 and improvements are obtained with respect to the known results [3, 14, 16, 2]. The results from the first sections of this paper are then applied to the classification problem for clubs of rank three in $\text{PG}(1,q^n)$ in Section 5. A study of the incidence structure of the clubs in $\text{PG}(1,q^n)$ after field reduction yields to a partial classification, concluding that the orbits of clubs under $\text{PGL}(2,q^n)$ are at least $k-1$, where $k$ stands for the number of divisors of $n$. The paper concludes with an appendix discussing a result motivated by Burau [4] for the complex numbers: the result is extended to general algebraically closed fields; a new proof is provided; and counterexamples are given to some of the arguments used in the original proof.

2 Subspaces intersecting each element of a regulus in one point

Let $\mathcal{R}$ be a regulus of subspaces in a projective space and let $S$ be any subspace of $\langle \mathcal{R} \rangle$. Questions about the properties of the set of intersection points, which for reasons of simplicity of notation we will denote by $S \cap \mathcal{R}$, often turn up while investigating objects in finite geometry. If $S$ intersects each element of the regulus $\mathcal{R}$ in a point, then the intersection $S \cap \mathcal{R}$ is a normal rational curve, see Lemma 1. This was already pointed out in [4, p.173] with a proof originally intended for complex projective spaces, but actually holding in a more general setting. The notation of [4] will be partly adopted.

The Segre variety representing the Cartesian product $\text{PG}(n,F) \times \text{PG}(m,F)$ in $\text{PG}((n+1)(m+1)−1,F)$ is denoted by $\mathcal{S}_{n,m,F}$. It is well known that $\mathcal{S}_{n,m,F}$ contains two families $\mathcal{S}_{n,m,F}^I$ and $\mathcal{S}_{n,m,F}^{II}$ of maximal subspaces of dimensions $n$ and $m$, respectively. When

\footnote{A different study of $F_q^k$-sublines and $F_q^k$-subplanes of $\text{PG}(2,q^n)$ in this representation can be found in [15].}
convenient, the notation $S^I$ or $S^{II}$ will be used for a subspace belonging to the first or second family. The points of $S_{n,m,F}$ may be represented as one-dimensional subspaces spanned by rank one $(m+1) \times (n+1)$ matrices. This is the standard example of a regular embedding of product spaces, see [17]. Note that in the finite case it is possible to embed product spaces in projective spaces of smaller dimension (see e.g. [7]). A regulus $R$ of $(n-1)$-dimensional subspaces can also be defined as $S^{I}_{n-1,1,F}$.

**Lemma 1.** Let $n > 1$ be an integer, and $F$ a field. Let $S_{t}$ be a $t$-subspace of $\text{PG}(2n-1,F)$ intersecting each $S^I \in S_{n-1,1,F}$ in precisely one point. Define $\Phi = S_{t} \cap S_{n-1,1,F}$, and assume $\langle \Phi \rangle = S_{t}$. Then $|F| \geq t$ and the following properties hold.

(i) The set $\Phi$ is a normal rational curve of order $t$.

(ii) Let $\Xi^I \in S_{n-1,1,F}$. Then the set $S(\Phi, \Xi^I)$ of the intersections of $\Xi^I$ with all transversal lines $t^{II}$ such that $t^{II} \cap \Phi = \emptyset$ is a normal rational curve of order $t$ or $t-1$ if $|F| = t$, and of order $t-1$ if $|F| > t$.

(iii) If $\Phi$ is contained in a subvariety $S_{t-1,1,F}$ of $S_{n-1,1,F}$, then homogeneous coordinates can be chosen such that $\Phi$ is represented parametrically by

\[
\left\langle \begin{pmatrix} y_0^t & y_0^{-1} y_1 & \ldots & y_0 y_1^{-1} \\ y_0^{-1} y_1 & y_0^{-2} y_1 & \ldots & y_1^{-1} y_1 \end{pmatrix} \rightrangle, \quad (y_0, y_1) \in (F^2)^*, \tag{1}
\]

and $S(\Phi, \Xi^I)$, for $z_0, z_1$ depending only on $\Xi^I$, by

\[
\left\langle \begin{pmatrix} y_0^{-1} z_0 & y_0^{-2} y_1 z_0 & \ldots & y_1^{-1} z_0 \\ y_0^{-1} z_1 & y_0^{-2} y_1 z_1 & \ldots & y_1^{-1} z_1 \end{pmatrix} \rightrangle, \quad (y_0, y_1) \in (F^2)^*. \tag{2}
\]

**Proof.** (i), (iii) The proof in [4, Sect.41 no.3], which is offered for $F = \mathbb{C}$, works exactly the same provided that $|F| > t$ or, more generally, that $\Phi$ is contained in some subvariety $S_{t-1,1,F}$ of $S_{n-1,1,F}$. In case $|F| \leq t$, the size of $\Phi$ being $|F| + 1$ implies $|F| = t$, so $\Phi$ is just a set of $t+1$ independent points in a subspace isomorphic to $\text{PG}(t,t)$, hence $\Phi$ is a normal rational curve of order $t$.

(ii) The case $|F| > t$ is proved in [4] immediately after the corollary at p. 175. If $|F| \leq t$, then $|F| = t$ and two cases are possible. If $\Phi$ is contained in some $S_{t-1,1,F} \subseteq S_{n-1,1,F}$, Burau’s proof is still valid as was mentioned in case (ii); so, $S(\Phi, \Xi^I)$ is a normal rational curve of order $t-1 = |F| - 1$. Otherwise $S(\Phi, \Xi^I)$ is an independent $(t+1)$-set, hence a normal rational curve of order $|F|$. \hfill \Box

**Remark 2.** If $|F| = t$ both cases in Lemma 1 (ii) can occur. The following two examples use the Segre embedding $\sigma = \sigma_{t-1,1,F}$ of the product space $\text{PG}(t-1,t) \times \text{PG}(1,t)$ in $\text{PG}(2t-1,t)$. Let $\{s_0, s_1, \ldots, s_i\}$ be the set of points on $\text{PG}(1,t)$ and suppose $\{r_0, r_1, \ldots, r_t\}$ is a set of $t+1$ points in $\text{PG}(t-1,t)$. Put $\Xi^I = \sigma(\text{PG}(1,t) \times s_0)$ and $\Phi := \{\sigma(r_i \times s_i) : i = 0, 1, \ldots, t\}$. Then $\Phi$ consists of $t+1$ points on the Segre variety $S_{t-1,1,F}$. Depending on the set $\{r_0, r_1, \ldots, r_t\}$ one obtains the two cases described in Lemma 1 (ii).
a. If \( \{r_0, r_1, \ldots, r_t\} \) is a frame of a hyperplane of \( \text{PG}(t-1,t) \) then \( \Phi \) generates a \( t \)-dimensional subspace of \( \text{PG}(2t-1,t) \) intersecting \( S_{t-1,1,F} \) in \( \Phi \) and \( S(\Phi, \Xi^t) \) is a normal rational curve of order \( t-1 \).

b. If \( \{r_0, r_1, \ldots, r_t\} \) generates \( \text{PG}(t-1,t) \) then \( \Phi \) generates a \( t \)-dimensional subspace of \( \text{PG}(2t-1,t) \) intersecting \( S_{t-1,1,F} \) in \( \Phi \) and \( S(\Phi, \Xi^t) \) is a normal rational curve of order \( t \).

Remark 3. By (1) and (2), the map \( \alpha : \Phi \to S(\Phi, \Xi^t) \) defined by the condition that \( X \) and \( X^\alpha \) are on a common line in \( S_{n-1,1,F}^I \) is related to a projectivity between the parametrizing projective lines. Such an \( \alpha \) is also called a projectivity.

3 The order of normal rational curves contained in \( S_{n-1,1,q} \)

Here \( n \geq 2 \) is an integer. The field reduction map \( F_{m,n,q} \) from \( \text{PG}(m-1,q^n) \) to \( \text{PG}(mn-1,q) \) will also be denoted by \( F \). If \( S \) is a set of points, in \( \text{PG}(m-1,q^n) \), then \( F(S) \) is a subset of subspaces, whose union, as a set of points will be denoted by \( \tilde{F}(S) \). The \( \mathbb{F}_q \)-span of a subset \( b \) of \( \text{PG}(d,q^n) \) is denoted by \( \langle b \rangle_{q^n} \).

Proposition 4. Let \( b \) be a \( q \)-subline of \( \text{PG}(1,q^n) \), and let \( \Theta \) be a point of \( \text{PG}(1,q^n) \). Let \( (1, \zeta) \) and \( (1, \zeta') \) be homogeneous coordinates of \( \Theta \) with respect to two reference frames for \( \langle b \rangle_{q^n} \), each of which consists of three points of \( b \). Then \( \mathbb{F}_q(\zeta) = \mathbb{F}_q(\zeta') \).

Proof. Homogeneous coordinates of a point in both reference frames, say \( (x_0, x_1) \) and \( (x_0', x_1') \), are related by an equation of the form \( \rho(x_0', x_1')^T = A(x_0, x_1)^T, \rho \in \mathbb{F}_q^*, A \in \text{GL}(2,q) \). Hence \( (\rho \rho')^T = A(1 \ 1)^T \) and this implies \( \zeta' \in \mathbb{F}_q(\zeta) \). The proof of \( \zeta \in \mathbb{F}_q(\zeta') \) is similar.

By Proposition 4, given a \( q \)-subline \( b \) in a finite projective space \( \text{PG}(d,q^n) \) and a point \( \Theta \in \langle b \rangle_{q^n}, \) with homogeneous coordinates \( (1, \zeta) \) with respect to a reference frame of \( \langle b \rangle_{q^n} \), consisting of three points of \( b \), the degree of \( \Theta \) over \( b \), denoted by \( [\Theta : b] \), is well-defined in terms of the field extension degree as follows: \( [\Theta : b] = [\mathbb{F}_q(\zeta) : \mathbb{F}_q] \).

This \( [\Theta : b] \) also equals the minimum integer \( m \) such that a subgeometry \( \Sigma \cong \text{PG}(d,q^m) \) exists containing both \( b \) and \( \Theta \).

Proposition 5. Any \( n \)-subspace of \( \text{PG}(2n-1,q) \) containing an \( (n-1) \)-subspace \( S_I \in S_{n-1,1,q}^I \) intersects \( S_{n-1,1,q} \) in the union of \( S_I \) and a line in \( S_{n-1,1,q}^I \).

Theorem 6. Let \( b \) be a \( q \)-subline of \( \text{PG}(1,q^n) \), and \( \Theta \notin b \) a point of \( \text{PG}(1,q^n) \). Then in \( \text{PG}(2n-1,q) \) any \( n \)-subspace \( H \) containing \( F(\Theta) \) intersects the Segre variety \( S_{n-1,1,q} = F(b) \), in a normal rational curve whose order is \( \min\{q, [\Theta : b]\} \).

Proof. Set \( L = \mathbb{F}_{q^2}, F = \mathbb{F}_q \). Without loss of generality, \( \text{PG}(2n-1,q) = \text{PG}_q(L^2), F(b) = \{L(x,y) \mid (x,y) \in (F^2)^*\}^2, \) and \( \Theta = L(1,\xi) \) with \( [F(\xi) : F] = [\Theta : b] \). The

\[ \text{For } x, y \in L, F(x,y) = ((x,y))_q, \] and \( L(x,y) = ((x,y))_{q^n}. \]
$n$-subspace $\mathcal{H}$ intersects $L(1,0)$ in one point $Y$ of the form $Y = F(\theta, 0), \theta \in L^*$. For any $x \in F$, seeking for the intersection $\langle \mathcal{F}(\Theta), Y \rangle_q \cap L(x, 1)$, or $$\langle L(1, \xi), F(\theta, 0) \rangle_q \cap L(x, 1)$$ gives two equations in $\alpha, \beta \in L$:

$$\alpha + \theta = \beta x, \quad \alpha \xi = \beta;$$

whence $\beta = \theta(x - \xi^{-1})^{-1}$. The intersection point is then $F(x\theta(x - \xi^{-1})^{-1}, \theta(x - \xi^{-1})^{-1})$. So, for $\Xi = L(0,1)$, the set of the intersections of $\Xi$ with all lines in $S_{n-1,1,q}^{II}$ which meet $\mathcal{H}$ is

$$S(\mathcal{H} \cap S_{n-1,1,q}, \Xi) = \{ F(0, \theta(x - \xi^{-1})^{-1}) \mid x \in \mathcal{F}_q \} \cup \{ F(0, \theta) \}.$$ 

This $S(\mathcal{H} \cap S_{n-1,1,q}, \Xi)$ is obtained by inversion from the line joining the points $F(0, \theta^{-1})$ and $F(0, \theta\xi^{-1})$. By [10, Theorem 5], $C_\mathcal{Y}$ is a normal rational curve of order $\delta' = \min\{ q, [F(\xi^{-1}) : F] - 1 \} = \min\{ q, [\Theta : b] - 1 \}$. Now apply lemma 1 for $S_t = \langle \mathcal{H} \cap S_{n-1,1,q} \rangle_q$: if $t \geq q$, then $t = q$ and $\delta' = q$ or $\delta' = q - 1$, so $[\Theta : b] \geq q$ and $t = \min\{ q, [\Theta : b] \}$. If on the contrary $t < q$, then $t - 1 = \delta' = [\Theta : b] - 1$, so $t = [\Theta : b]$ and $t = \min\{ q, [\Theta : b] \}$ again. \hfill $\Box$

An important consequence of the above result answers the question of the existence of a Desarguesian spread containing a given regulus $\mathcal{R}$ and a subspace disjoint from $\mathcal{R}$.

**Corollary 7.** If a regulus $\mathcal{R} = S_{n-1,1,q}$ and an $(n-1)$-dimensional subspace $U$, disjoint from $\mathcal{R}$, in $PG(2n - 1, q)$ are contained in a Desarguesian spread then there is an integer $c$ such that any $n$-subspace $\mathcal{H}$ containing $U$ intersects $\mathcal{R}$ in a normal rational curve of order $c$.

The following remark illustrates that this necessary condition is not always satisfied.

**Remark 8.** For $n > 2$ by using the package FinInG [5] of GAP [6] examples can be given of $(n-1)$-subspaces disjoint from $S_{n-1,1,q}$ contained in $n$-subspaces intersecting the Segre variety in normal rational curves of distinct orders. We include one explicit example. Let $q = 4$, $\mathbb{F}_q = \mathbb{F}_2(\omega)$, with $\omega^2 + \omega + 1 = 0$. Let $\mathcal{R}$ be the regulus of 3-dimensional subspaces of $PG(7, 4)$ obtained from the standard subline $PG(1, q)$ in $PG(1, q^4)$, and put

$$S_3 := ((1, 0, 0, 0, \omega^2, 1, 0, 1), (0, 1, 0, 0, 1, \omega^2, 0, \omega^2), (0, 0, 1, 0, 0, \omega, 1, \omega), (0, 0, 0, 1, \omega^2, 0, \omega, 1)).$$

Then $S_3$ is a three-dimensional subspace disjoint from the regulus $\mathcal{R}$. Moreover, the 4-dimensional subspace $\langle S_3, (1, 0, 0, 0, 0, 0, 0, 0, 0) \rangle$ intersects the regulus $\mathcal{R}$ in a normal rational curve of order 4, while the 4-dimensional subspace $\langle S_3, (0, 1, 0, \omega^2, 0, 0, 0, 0) \rangle$ intersects $\mathcal{R}$ in a conic.
4 André-Bruck-Bose representation

The André-Bruck-Bose representation of a Desarguesian affine plane of order $q^n$ is related to the image of $\text{PG}(2, q^n)$, under the field reduction map $\mathcal{F}$, by means of the following straightforward result.

**Proposition 9.** Let $\mathcal{D}$ be the Desarguesian spread in $\text{PG}(3n-1, q)$ obtained after applying the field reduction map $\mathcal{F}$ to the set of points of $\text{PG}(2, q^n)$, $l_\infty$ a line in $\text{PG}(2, q^n)$, and $\mathcal{K}$ a $(2n)$-subspace of $\text{PG}(3n-1, q)$, containing the spread $\mathcal{F}(l_\infty)$. Take $\text{PG}(2, q^n) \setminus l_\infty$ and $\mathcal{K} \setminus \langle \mathcal{F}(l_\infty) \rangle_q$ as representatives of $\text{AG}(2, q^n)$ and $\text{AG}(2n, q)$, respectively. Then the map $\varphi : \text{AG}(2, q^n) \rightarrow \text{AG}(2n, q)$ defined by $\varphi(X) = \mathcal{F}(X) \cap \mathcal{K}$ for any $X \in \text{AG}(2, q^n)$ is a bijection, mapping lines of $\text{AG}(2, q^n)$ into $n$-subspaces of $\text{AG}(2n, q)$ whose $(n-1)$-subspaces at infinity belong to the spread $\mathcal{F}(l_\infty)$.

The notation in Proposition 9 is assumed to hold in the whole section. The following result improves [2, Theorems 3.3 and 3.5], by determining the order of the involved normal rational curves.

**Theorem 10.** Let $b$ be a $q$-subline of $\text{PG}(2, q^n)$, not contained in $l_\infty$. Set $\Theta = \langle b \rangle_q \cap l_\infty$. Then the André-Bruck-Bose representation $\varphi(b \setminus l_\infty)$ is the affine part of a normal rational curve whose order is $\delta = \min\{q, [\Theta : b]\}$. More precisely, if $\delta = 1$, then $\varphi(b \setminus l_\infty)$ is an affine line; if $\delta > 1$, then $b \cap l_\infty = \emptyset$, and $\varphi(b)$ is a normal rational curve with no points at infinity.

**Proof.** The intersection $\mathcal{H} = \langle \mathcal{F}(b) \rangle_q \cap \mathcal{K}$ is an $n$-space containing $\mathcal{F}(\Theta)$, and contained in the span of the Segre variety $S_{n-1, q} = \tilde{\mathcal{F}}(b)$, as defined at the start of Section 3. The result follows from Proposition 5 and Theorem 6.

The results in [2, Theorems 3.3 and 3.5] also characterize the normal rational curves arising from $q$-sublines in $\text{AG}(2, q^n)$.

In [3, 14, 16] for $n = 2$ and [2, Theorem 3.6 (a)(b)] for any $n$ the André-Bruck-Bose representation of a $q$-subplane tangent to a line at the infinity is described. Further properties are stated in the following theorem:

**Theorem 11.** Let $B$ be a $q$-subplane of $\text{PG}(2, q^n)$ that is tangent to $l_\infty$ at the point $T$. Let $b$ be a line of $B$ not through $T$, $\Theta = \langle b \rangle_q \cap l_\infty$, and $\delta = \min\{q, [\Theta : b]\}$. Then there are a normal rational curve $\mathcal{C}_0$ of order $\delta$ in the $n$-subspace $\varphi(\langle b \rangle_q)$, a normal rational curve $\mathcal{C}_1 \subset \mathcal{F}(T)$ of order $\delta'$, with

$$\delta' = \begin{cases} [\Theta : b] - 1 & \text{for } q > [\Theta : b] \\ \{q-1, q\} & \text{otherwise,} \end{cases}$$

and a projectivity $\kappa : \mathcal{C}_0 \rightarrow \mathcal{C}_1$ (in the sense of Remark 3), such that $\varphi(B \setminus l_\infty)$ is the ruled surface union of all lines $XX^\ast$ for $X \in \mathcal{C}_0$.  

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Proof. By Theorem 10, \( C_0 := \varphi(b) \) is a normal rational curve of order \( \delta \) in the \( n \)-subspace \( \varphi((b)_{q^n} \setminus l_{\infty}) \), and for any \( P = \varphi(X) \in C_0 \), the subline \( TX \) of \( B \) corresponds to an affine line \( PP^1 \) with \( P^1 \in \mathcal{F}(T) \) at infinity. Define \( C_1 = \{ P^x \mid P \in C_0 \} \).

By the field reduction map \( \mathcal{F} = \mathcal{F}_{3,n,q} \), the subplane \( B \) is mapped to \( \mathcal{F}(B) \) which is the set of all maximal subspaces of the first family in \( \mathcal{S}_{n-1,2,q} \subset \text{PG}(3n-1, q) \). Considering \( F_q^m \) as an \( F_q \)-vector space, the homomorphism

\[
F_q^m \times F_q^m \rightarrow F_q^{m} \otimes F_q^{m} : (\lambda, v) \mapsto \lambda \otimes v
\]
corresponds to a projective embedding \( g : \text{PG}(n-1, q) \times B \rightarrow \mathcal{S}_{n-1,2,q} \) whose image is \( \mathcal{S}_{n-1,2,q} \), and such that \( \mathcal{F}(X) = (\text{PG}(n-1, q) \times X)^{\varphi} \) for any point \( X \) in \( B \). It holds \( \varphi(B \setminus l_{\infty}) = \mathcal{S}_{n-1,2,q} \cap \mathcal{K} \setminus \mathcal{F}(T) \). For any point \( U \) in \( B \) define

\[
\kappa_U : (X,Y)^{\varphi} \in \mathcal{S}_{n-1,2,q} \mapsto (X,U)^{\varphi} \in \mathcal{F}(U).
\]

Note that for any \( Y \in B \), the restriction of \( \kappa_U \) to \( \mathcal{F}(Y) \) is a projectivity. For any \( U \in b \), using the notation from Lemma 1 it holds \( C_0^{TV} = S(C_0, \mathcal{F}(U)) \), and as a consequence, \( C_0^{TV} \) is a normal rational curve of order \( \delta' \) as in (3). Now, since for any \( P \in C_0 \), say \( P = (X_P, Y_P)^{\varphi} \), the points \( P, P^x \) and \( P^{x\tau} \) are on the plane \( (X_P \times B)^{\varphi} \in \mathcal{S}_{n-1,2,q} \) and \( P^x, P^{x\tau} \in \mathcal{F}(T) \), it follows that \( P^x = P^{x\tau} \). It also follows that \( \mathcal{C}_1 = \mathcal{C}_0^{TV\tau} = S(C_0, \mathcal{F}(U))^{x\tau} \), and hence \( \mathcal{C}_1 \) is a normal rational curve of order \( \delta' \) as in (3). Finally, \( \kappa_U : C_0 \rightarrow S(C_0, \mathcal{F}(U)) \) is a projectivity as defined in Remark 3, and hence so is \( \kappa \).

\[\square\]

5 On the classification of clubs

An \( F_q \)-club (or simply a club) in \( \text{PG}(1, q^n) \) is an \( F_q \)-linear set of rank three, having a point of weight two, called the head of the club. An \( F_q \)-club has \( q^2 + 1 \) points, and the non-head points have weight one. From now on it will be assumed that \( n > 2 \). The next proposition is a straightforward consequence of the representation of linear sets as projections of subgeometries [12, Theorem 2].

Proposition 12. Let \( L \) be an \( F_q \)-club in \( \text{PG}(1, q^n) \subset \text{PG}(2, q^n) \). Then there is a \( q \)-subplane \( \Sigma \) of \( \text{PG}(2, q^n) \), a \( q \)-subline \( b \) in \( \Sigma \), and a point \( \Theta \in (b)_{q^n} \setminus b \), such that \( L \) is the projection of \( \Sigma \) from the center \( \Theta \) onto the axis \( \text{PG}(1,q^n) \).

As before the notation \( \mathcal{F} \) and \( \tilde{\mathcal{F}} \) is used, where \( \mathcal{F} = \mathcal{F}_{2,n,q} \) denotes the field reduction map from \( \text{PG}(1,q^n) \) to \( \text{PG}(2n-1, q) \).

Proposition 13. Let \( L \) be an \( F_q \)-club of \( \text{PG}(1,q^n) \) with head \( \Upsilon \). Then \( \tilde{\mathcal{F}}(L) \) contains two collections of subspaces, say \( F_1 \) and \( F_2 \), satisfying the following properties.

(i) The subspaces in \( F_1 \) are \((n-1)\)-dimensional, are pairwise disjoint, and any subspace in \( F_1 \) is disjoint from \( \mathcal{F}(\Upsilon) \).

(ii) Any subspace in \( F_2 \) is a plane and intersects \( \mathcal{F}(\Upsilon) \) in precisely a line.

\[\text{THE ELECTRONIC JOURNAL OF COMBINATORICS 23(1) (2016), \#P1.37} \]
(iii) Any point of \( F(\Upsilon) \) belongs to exactly \( q + 1 \) planes in \( F_2 \).

(iv) If \( L \) is not isomorphic to \( \text{PG}(1, q^2) \), and \( l \) is any line of \( \text{PG}(2n-1, q) \) contained in \( \tilde{F}(L) \), then \( l \) is contained in \( F(\Upsilon) \) or in a subspace in \( F_1 \cup F_2 \).

Proof. The assumptions imply the existence of \( \Sigma \) and a \( q \)-subline \( b \) in \( \Sigma \) as in Proposition 12. The assertions are a consequence of the fact that \( \tilde{F}(\Sigma) \) is a Segre variety \( S_{n-1,2,q} \) in \( \text{PG}(3n-1, q) \). Let

\[ p_1 : \text{PG}(2, q^n) \setminus \Theta \rightarrow \text{PG}(1, q^n) \]

be the projection with center \( \Theta \), associated with

\[ p_2 : \text{PG}(3n-1, q) \setminus F(\Theta) \rightarrow \text{PG}(2n-1, q). \]

The collections \( F_1 \) and \( F_2 \) are defined as follows:

\[
F_1 = \{ F(p_1(X)) \mid X \in \Sigma \setminus b \} = F(L) \setminus F(\Upsilon), \quad F_2 = \{ p_2(V^{II}) \mid V^{II} \in \tilde{F}(\Sigma)^{II} \}.
\]

The assertion \( (i) \) is straightforward, as well as \( \dim(V) = 2 \) for any \( V \in F_2 \). For any \( V^{II} \in \tilde{F}(\Sigma)^{II} \), the intersection \( V^{II} \cap (\tilde{F}(b))_q \) is a line, and this with \( p_2^{-1}(F(\Upsilon)) = (\tilde{F}(b))_q \setminus F(\Theta) \) implies the second assertion in \( (ii) \). Next, let \( P \) be a point in \( F(\Upsilon) \). A plane \( V = p_2(V^{II}) \) contains \( P \) if, and only if, \( V^{II} \) intersects the \( n \)-subspace \( (F(\Theta), P)_q \), that is, \( V^{II} \) intersects the normal rational curve \( S_{n-1,2,q} \cap (F(\Theta), P)_q \); this implies \( (iii) \).

Assume that a line \( l \subset \tilde{F}(L) \) exists which is neither contained in \( F(\Upsilon) \), nor in a \( T \in F_1 \cup F_2 \). Let \( Q \) be a point in \( l \setminus F(\Upsilon) \), and let \( V \in F_2 \) such that \( Q \in V \). It holds \( L = B(V) \). Then \( B(l) \) is a \( q \)-subline of \( L \). Suppose that a line \( l' \in V \) exists such that \( B(l') = B(l) \). Since \( B(Q) \neq B(Q') \) for any \( Q' \in V, Q' \neq Q \), the line \( l' \) contains \( Q \). Then \( l, l' \) are two distinct transversal lines in \( B(l)^{II} \), a contradiction. Hence \( B(l') \neq B(l) \) for any line \( l' \in V \), that is, \( B(l) \) is a so-called irregular subline \([8]\). By \([8, \text{Corollary 13}]\), no irregular subline exists in \( L \), and this contradiction implies \( (iv) \).

Proposition 14. Let \( L \) be an \( \mathbb{F}_q \)-club with head \( \Upsilon \). Let \( \Theta \) be the point and \( b \) be the subline as defined in Proposition 12. Then for any point \( X \) in \( F(\Upsilon) \), the intersection lines of \( F(\Upsilon) \) with any \( q \) distinct planes in \( F_2 \) containing \( X \) span an \( s \)-dimensional subspace, where

\[
(i) \quad s = [\Theta : b] - 1 \text{ if } q > [\Theta : b];

(ii) \quad s \in \{ q - 1, q \} \text{ if } q \leq [\Theta : b].
\]

Proof. Let \( p_2 \) be the projection map as defined in the proof of Proposition 13, \( X = p_2(P) \), and \( \mathcal{H} = (F(\Theta), P)_q \). For any plane \( V = p_2(V^{II}) \), it holds \( X \in V \) if, and only if \( V^{II} \cap \mathcal{H} \neq \emptyset \). The intersection \( \mathcal{H} \cap \tilde{F}(b) \) is a normal rational curve of order \( \min\{ q, [\Theta : b] \} \) (cf. Theorem 6). Let \( V_0 = p_2(V_0^{II}) \) be the unique plane of \( F_2 \) through \( X \) distinct from the \( q \) planes chosen in the assumptions (cf. Proposition 13). Let \( Q = \tilde{F}(b) \cap V_0^{II}; B(Q) \) is an \((n-1)\)-subspace of \( \tilde{F}(b)^{I} \). Such \( B(Q) \) is mapped onto \( B(X) = F(\Upsilon) \) by \( p_2 \). Assume \( V_i = p_2(V_i^{II}), i = 1,2,\ldots,q \), are the \( q \) planes chosen in the assumptions. Any \( V_i^{II} \),
\[ \mathcal{H}, \text{ hence } V_i^{II} \cap B(Q) \text{ is the intersection of } B(Q) \text{ with a transversal line of } \mathcal{F}(b) \text{ intersecting the normal rational curve } \mathcal{H} \cap \mathcal{F}(b). \]

By Lemma 1 (ii), the set

\[ S = \{ V_i^{II} \cap B(Q) \mid i = 1, 2, \ldots, q \} \cup \{ Q \} \]

is a normal rational curve of order \( s \) where \( s \) takes the values as stated in (i) and (ii). Since \( V_i \cap \mathcal{F}(\Upsilon) \) is the line through \( X \) and a point of \( p_2(S) \), distinct from \( X \), the span of the intersection lines is the same as the span of \( p_2(S) \).

\[ \square \]

**Theorem 15.** Let \( \mathcal{I}_{n,q} \) be the set of integers \( h \) dividing \( n \) and such that \( 1 < h < q \). For any \( h \in \mathcal{I}_{n,q} \), let \( L_h \) be the linear set obtained by projecting a \( q \)-subplane \( \Sigma \) of \( \text{PG}(2,q^n) \) from a point \( \Theta_h \) collinear with a \( q \)-subline \( b \) in \( \Sigma \) and such that \([\Theta_h : b] = h\). Then the set \( \Lambda = \{ L_h \mid h \in \mathcal{I}_{n,q} \} \) contains \( \mathbb{F}_q \)-clubs in \( \text{PG}(1,q^n) \) all belonging to distinct orbits under \( \text{PGL}(2,q^n) \).

**Proof.** If \( n \) is odd, then no club is isomorphic to \( \text{PG}(1,q^2) \). So, by Proposition 13 (iv), the families \( F_1 \) and \( F_2 \) are uniquely determined. The thesis is a consequence of Proposition 14, taking into account that if \( L \) and \( L' \) are projectively equivalent, then \( \mathcal{F}(L) \) and \( \mathcal{F}(L') \) are projectively equivalent in \( \text{PG}(2n-1,q^n) \).

In order to deal with the case \( n \) even, it is enough to show that in \( \Lambda \) at most one club is isomorphic to \( \text{PG}(1,q^2) \). So assume \( L_h \cong \text{PG}(1,q^2) \). Then \( \mathcal{F}(L_h) \) has a partition \( \mathcal{P}_1 \) in \((n-1)\)-subspaces, and a partition \( \mathcal{P}_2 \) in 3-subspaces. From [8, Lemma 11] it can be deduced that any line contained in \( \mathcal{F}(L_h) \) is contained in an element of \( \mathcal{P}_1 \) or \( \mathcal{P}_2 \). The intersections of a subspace \( U \) of a family \( \mathcal{P}_i \) with the elements of the other family form a line spread of \( U \). Hence all planes in \( F_2 \) are contained in 3-subspaces of \( \mathcal{P}_2 \), and all planes of \( F_2 \) through a point \( X \) in \( \mathcal{F}(\Upsilon) \) meet \( \mathcal{F}(\Upsilon) \) in the same line. By Proposition 14 this implies \( h = 2 \).

\[ \square \]

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**References**


A Appendix: On a result in [4]

In [4, p.175] the following result (Korollar) is stated for $F = \mathbb{C}$.

**Corollary 16.** Let $F$ be an algebraically closed field. If an $s$-subspace $S_s$ of $\text{PG}(2s - 1, F)$ meets all $S^l \in S^l_{s-1,1,F}$ only in points, then such points span $S_s$.

In [4] the previous result is seemingly proved using methods valid in any field with enough elements. However such a generalisation would contradict Theorem 3.3. In the opinion of the authors the proof in [4] is obtained using an erroneous argument. As a matter of fact, it is claimed in the proof at page 174 that the assumption $\langle \Phi \rangle = S_s$ is not used. However the contradiction $S_s \subset \langle S_{s-2,1,F} \rangle$ is inferred from $\Phi \subset S_{s-2,1,F}$.

A further counterexample, which exists whenever a hyperbolic quadric $Q^+(3, F)$ in a three-dimensional projective space admits an external line (a condition which is not met
when the field $F$ is algebraically closed) is the following. If $\ell$ is the line corresponding to the two-dimensional vector space $\langle e_1 \rangle \otimes \langle e'_1, e'_2 \rangle$ and $m$ is a line external to the hyperbolic quadric obtained by the intersection of the Segre variety $S_{2,1,F}$ with the 3-space corresponding to the vector space $\langle e_2, e_3 \rangle \otimes \langle e'_1, e'_2 \rangle$, then the 3-dimensional subspace $\langle \ell, m \rangle$ intersects $S_{2,1,F}$ in the line $\ell$ belonging to $S_{2,1,F}^\perp$.

For the sake of completeness, a proof for corollary 16 is given.

**Proof of corollary 16.** Define

$$S_t = \langle S_s \cap S_{s-1,1,F} \rangle, \; t = \dim S_t$$

and suppose $t < s$. It is proved in [4, p.173 (6)] that $S_t \subset \langle S_{t-1,1,F} \rangle$ for some $S_{t-1,1,F} \subset S_{s-1,1,F}$.

Note that $S_s \cap \langle S_{t-1,1,F} \rangle = S_t$; otherwise, comparing dimensions, $S_s$ would intersect each $S^I \subset S_{t-1,1,F}$ in more than one point. Now choose

- a subspace $S_{s-t-1} \subset S_s$ such that $S_{s-t-1} \cap \langle S_{t-1,1,F} \rangle = \emptyset$;
- a Segre variety $S_{s-t-1,1,F} \subset S_{s-1,1,F}$, such that $\langle S_{s-t-1,1,F} \rangle \cap \langle S_{t-1,1,F} \rangle = \emptyset$;
- two distinct $A^I, B^I \in S_{s-t-1,1,F}^I$.

Since $\langle S_{s-t-1,1,F} \rangle$ and $\langle S_{t-1,1,F} \rangle$ are complementary subspaces of $\langle S_{s-1,1,F} \rangle$, a projection map

$$\pi : \langle S_{s-1,1,F} \rangle \setminus \langle S_{t-1,1,F} \rangle \to \langle S_{s-t-1,1,F} \rangle$$

is defined by $\pi(P) = \langle P \cup S_{t-1,1,F} \rangle \cap \langle S_{s-t-1,1,F} \rangle$.

Now suppose $\pi(S_{s-t-1}) \cap S_{s-t-1,1,F} = \emptyset$. In $\langle S_{s-t-1,1,F} \rangle$ consider

- the regulus $R$ corresponding to $S^I_{s-t-1,1,F}$, and the projectivity $\kappa : A^I \to B^I$ such that, for any $P \in A^I$, the line $\langle P, \kappa(P) \rangle$ belongs to $S_{s-t-1,1,F}^\perp$;
- the regulus $R'$ containing $A^I, B^I$ and $\pi(S_{s-t-1})$, and the projectivity $\kappa' : A^I \to B^I$ such that, for any $P \in A^I$, the line $\langle P, \kappa'(P) \rangle$ is a transversal line of $R'$.

Since $F$ is an algebraically closed field, $\kappa^{-1} \circ \kappa$ has a fixed point $P$. Therefore $\kappa(P) = \kappa'(P)$, so $R$ and $R'$ have a common transversal. This contradicts $\pi(S_{s-t-1}) \cap S_{s-t-1,1,F} = \emptyset$. So, a point $P \in S_{s-t-1}$ exists such that $\pi(P) \in S_{s-t-1,1,F}$.

Next, let $C^I \in S^I_{s-1,1,F}$ be such that $\pi(P) \subset C^I$, and $Q$ the point in $\langle S_{t-1,1,F} \rangle$ such that $Q, P$, and $\pi(P)$ are collinear. If $Q \notin S_t$, then $\pi(P) \subset S_s$, a contradiction; also $Q \in C^I$ leads to a contradiction (since it implies $P \in C^I$). So $Q \notin S_t \cup C^I$ and by a dimension argument two points $Q_1 \in C^I \setminus S_t$ and $Q_2 \in S_t \setminus C^I$ exist such that $Q, Q_1$ and $Q_2$ are collinear: they are on the unique line through $Q$ meeting both $C^I \cap \langle S_{t-1,1,F} \rangle$ and a $(t-1)$-subspace of $S_t$ disjoint from $C^I$.

The plane $\langle P, Q_1, Q_2 \rangle$ contains the lines $PQ_2 \subset S_s$ and $\pi(P)Q_1 \subset S_{s-1,1,F}$ which meet outside $\langle S_{t-1,1,F} \rangle$. This is again a contradiction. 

\[\square\]