Expansions of a chord diagram and alternating permutations

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Abstract

A chord diagram is a set of chords of a circle such that no pair of chords has a common endvertex. A chord diagram $E$ with $n$ chords is called an $n$-crossing if all chords of $E$ are mutually crossing. A chord diagram $E$ is called nonintersecting if $E$ contains no 2-crossing. For a chord diagram $E$ having a 2-crossing $S = \{x_1x_3, x_2x_4\}$, the expansion of $E$ with respect to $S$ is to replace $E$ with $E_1 = (E \setminus S) \cup \{x_2x_3, x_4x_1\}$ or $E_2 = (E \setminus S) \cup \{x_1x_2, x_3x_4\}$. It is shown that there is a one-to-one correspondence between the multiset of all nonintersecting chord diagrams generated from an $n$-crossing with a finite sequence of expansions and the set of alternating permutations of order $n + 1$.

Keywords: chord diagram; alternating permutation; Entringer number; Euler number; Ptolemy’s theorem

1 Introduction

Let us consider a set of chords of a circle. A set of chords is called a chord diagram, if they have no common endvertex. If a chord diagram consists of a set of $n$ mutually crossing chords, it is called an $n$-crossing. A 2-crossing is simply called a crossing as well. If a chord diagram contains no crossing, it is called nonintersecting.

Let $V$ be a set of $2n$ vertices on a circle, and let $E$ be a chord diagram of order $n$, where each chord has endvertices of $V$. We denote the family of all such chord diagrams by $\mathcal{CD}(V)$. Let $x_1, x_2, x_3, x_4 \in V$ be placed on a circle in clockwise order. Let $E \in \mathcal{CD}(V)$. For a crossing $S = \{x_1x_3, x_2x_4\} \subset E$, let $S_1 = \{x_2x_3, x_4x_1\}$, and $S_2 = \{x_1x_2, x_3x_4\}$. The expansion of $E$ with respect to $S$ is defined as a replacement of $E$ with $E_1 = (E \setminus S) \cup S_1$ or $E_2 = (E \setminus S) \cup S_2$ (see Figure 1). In this procedure, $E$ is called the predecessor of $E_1$.
and $E_2$, and $E_1$ and $E_2$ are called the successors of $E$. A chord of a chord diagram is called isolated, if it intersects no other chord.

For $E \in CD(V)$, let us denote the number of 2-crossings of $E$ by $c(E)$. Let $E'$ be a successor of $E$ such that $E' = (E \setminus S) \cup S'$, where $S$ is an original 2-crossing and $S'$ is a pair of additional chords.

We claim that $c(E') < c(E)$. Indeed, for $e \in E \cap E'$, let $t$ (resp. $t'$) be the number of chords of $S$ (resp. $S'$) intersecting $e$.

It is not difficult to see that if $t \leq 1$ then we have $t' = t$, and if $t = 2$ then we have $t' = 2$ or $t' = 0$. Hence, we have $t' \leq t$. Since $S$ is a crossing of $E$ which is removed in $E'$, we have $c(E') < c(E)$.

**Lemma 1.** Let $E \in CD(V)$ be a chord diagram. Then beginning from $E$, the resulting multiset of nonintersecting chord diagrams generated by a maximal set of expansions is uniquely determined.

**Proof.** We proceed by induction on the number of crossings $c$ of a chord diagram $E$.

If $c = 0$ or 1, there is nothing to prove. Let $c \geq 2$ and let $c(E) = c$. By inductive hypothesis, for a chord diagram $E'$ with $c(E') \leq c - 1$, we define $NCD(E')$ as the resulting multisets of nonintersecting chord diagrams generated by $E'$. Moreover, for a set of chord diagrams $\mathcal{E}$ such that $E' \in \mathcal{E}$ with $c(E') \leq c - 1$, let us denote $NCD(\mathcal{E}) = \cup_{E' \in \mathcal{E}} NCD(E')$.

Let $S_1$ and $S_2$ be two 2-crossings of $E$, and let $E_{i1}$ and $E_{i2}$ be two successors of $E$ by an expansion with respect to $S_i$ for $i = 1, 2$. Let $\mathcal{E}_i = \{E_{i1}, E_{i2}\}$ for $i = 1, 2$. What we want to show is that $NCD(\mathcal{E}_1) = NCD(\mathcal{E}_2)$.

**Case 1.** $S_1 \cap S_2 = \emptyset$.

For $E_{i1}$ and $E_{i2}$, by an expansion with respect to $S_2$, we have a set $\mathcal{E}'$ of four chord diagrams. Then we have $NCD(\mathcal{E}_1) = NCD(\mathcal{E}')$. In the same way, for $E_{21}$ and $E_{22}$, by an expansion with respect to $S_1$, we have $\mathcal{E}'$, and we have $NCD(\mathcal{E}_2) = NCD(\mathcal{E}')$. Hence, we have $NCD(\mathcal{E}_1) = NCD(\mathcal{E}_2)$.
Case 2. $S_1 \cap S_2 \neq \emptyset$.

We may assume $S_1 = \{e_0, e_1\}$ and $S_2 = \{e_0, e_2\}$, where $e_i = x_iy_i$ for $0 \leq i \leq 2$. Let $V_0 = \{x_0, x_1, x_2, y_0, y_1, y_2\}$ and let $E' = E \setminus \{e_0, e_1, e_2\}$. Beginning from $E_i$ with $i = 1, 2$, let us consider expansions with respect to a crossing induced by $V_0$.

Case 2.1. $e_1$ and $e_2$ are not crossing.

We may assume $x_0, x_1, x_2, y_0, y_1$ are placed on a circle in clockwise order. By iterating possible expansions, not depending on the order of the expansions, we always have a set of four chord diagrams $E' = \{E' \cup \{x_0x_1, x_2y_0, y_2y_1\}, E' \cup \{x_0x_1, x_2y_1, y_0y_2\}, E' \cup \{x_0y_1, x_1x_2, y_0y_2\} \}$. 

In any case, we have $NCD(E_i) = NCD(E')$ for $i = 1, 2$, as required. \hfill \Box

Let us denote the multiset of nonintersecting chord diagrams generated by $E \in CD(V)$ by $NCD(E)$. For $E \in CD(V)$, let us define $f(E)$ as the cardinality of $NCD(E)$ as a multiset.

Example 2. Let $C_n$ be an $n$-crossing. Then we have $f(C_2) = 2$, $f(C_3) = 5$ and $f(C_4) = 16$. (See Figure 2.)

A background of expansions of a chord diagram is Ptolemy’s theorem and its generalization. For two points $x, y$ on a circle, let $xy$ be the length of a chord $xy$. Ptolemy’s theorem states that if $E = \{x_1x_3, x_2x_4\}$ itself is a 2-crossing, then we have $x_1x_3 \cdot x_2x_4 = x_2x_3 \cdot x_4x_1 + x_1x_2 \cdot x_3x_4$. In other words, we have

$$\prod_{e \in E} e = \prod_{e \in E_1} e + \prod_{e \in E_2} e,$$

where $E_1$ and $E_2$ are two successors of $E$. In general, for a given $E \in CD(V)$, by iterating expansions with applications of Ptolemy’s theorem, we have

$$\prod_{e \in E} e = \sum_{E' \in NCD(E)} \prod_{e \in E'} e.$$

If $E$ is a 3-crossing, the equation (2) is known as Fuhrmann’s Theorem ([2]).

2 Main Results

For two nonnegative integers $k$ and $n$ with $k \leq n$, we define $A(n, k)$ as a chord diagram of order $n + 1$, in which there is an $n$-crossing $E_0$ with an extra chord $e$ such that $e$ crosses
Figure 2: Multisets of nonintersecting chord diagrams generated by a 2-crossing (upper), a 3-crossing (middle) and a 4-crossing (lower).
exactly \(k\) chords of \(E_0\). Note that \(A(n-1,n-1)\) is simply an \(n\)-crossing, and that \(A(n,0)\) is a union of an \(n\)-crossing and an isolated chord. Hence, we have \(f(A(n-1,n-1)) = f(A(n,0))\). The values of \(f(A(n,k))\) for small nonnegative integers \(n\) and \(k\) are shown in Table 1.

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A permutation \(\sigma\) of \([n] = \{1, 2, \ldots, n\}\) is called an alternating permutation if \((\sigma(i) - \sigma(i-1))(\sigma(i+1) - \sigma(i)) < 0\) for \(2 \leq i \leq n-1\) (see [9] for an excellent survey of alternating permutations). An alternating permutation \(\sigma\) is called an up-down permutation (resp. down-up permutation) if \(\sigma(1) < \sigma(2)\) (resp. \(\sigma(1) > \sigma(2)\)). Let \(\mathcal{UDP}(n,k)\) denote the set of up-down permutations of \([n]\) with the first term at most \(k\). Similarly, let \(\mathcal{DUP}(n,k)\) denote the set of down-up permutations of \([n]\) with the first term at least \(n-k+1\). Note that by definition, there is a natural bijection from \(\mathcal{UDP}(n,k)\) to \(\mathcal{DUP}(n,k)\).

The main result of the paper is the following theorem.

Theorem 3. For \(0 \leq k \leq n\), there is a bijection from \(\mathcal{NCD}(A(n,k))\) to \(\mathcal{UDP}(n+2,k+1)\).

For \(0 \leq k \leq n\), Entringer number \(E_{n,k}\) is defined as the number of down-up permutations of \([n]+1\) with the first term \(k+1\) [1], which equals the cardinality of \(\mathcal{UDP}(n,k)\). Since \(n \geq 1\), \(E_{n+1,1}\) equals Euler number \(E_n\), the number of down-up permutations of \([n]\), we have the following Corollary.

Corollary 4. For \(0 \leq k \leq n\), we have \(f(A(n,k)) = E_{n+2,k+1}\). In particular, we have \(f(A(n,0)) = E_{n+1,1}\).

Several combinatorial interpretations for Entringer numbers are known ([4, 5, 6, 7, 8]). The generating function for Entringer number is treated in [3] as an exercise, Exer. 6.75. According to [3], it follows that

\[
\sum_{n \geq 0} \sum_{k \geq 0} E_{n+k,k} \frac{x^n y^k}{n! k!} = \frac{\cos x + \sin y}{\cos(x+y)}.
\]

By Corollary 4, we have

\[
\sum_{n \geq 0} \sum_{k \geq 0} f(A(n+k,k)) \frac{x^n y^k}{n! k!} = \frac{\partial^2}{\partial x \partial y} \left( \sum_{n \geq 0} \sum_{k \geq 0} E_{n+k,k} \frac{x^n y^k}{n! k!} \right) \\
= \frac{\cos x + \sin y}{\cos(x+y)(1 - \sin(x+y))}.
\]
3 Proof of Theorem 3

For two chord diagrams $F_i \in CD(V_i)$ for $i = 1, 2$, let $F'_i \in CD(V'_i)$ be a chord diagram such that $F'_i$ consists of the set of all nonisolated chords of $F_i$. Suppose that $|F'_1| = |F'_2|$. Let $V'_i = \{v'_{i,0}, v'_{i,1}, \ldots, v'_{i,2w'-1}\}$, and the vertices are placed on a circle in clockwise order for each $i = 1, 2$. Suppose that $v'_{i,\alpha}v'_{i,\beta} \in F'_i$ holds if and only if $v'_{2,\alpha}v'_{2,\beta} \in F'_2$ holds. Then we say that $F'_1$ and $F'_2$ are isomorphic, and furthermore we say that $F_1$ and $F_2$ are isomorphic as well.

In order to prove Theorem 3, we will recursively construct a bijection from $NCD(A(n, k))$ to $UDP(n + 2, k + 1)$ for $0 \leq k \leq n$.

Firstly, we will show a recurrence for $NCD(A(n, k))$, which is a key ingredient for the proof of Theorem 3.

Lemma 5. For $1 \leq k \leq n$, we have a bijection between $NCD(A(n, k))$ and $NCD(A(n, k - 1)) \cup NCD(A(n - 1, n - k))$. In particular, we have $f(A(n, k)) = f(A(n, k - 1)) + f(A(n - 1, n - k))$.

Proof. Let $E$ be a chord diagram isomorphic to $A(n, k)$. We may assume $E$ contains an $n$-crossing $E_0$ and an extra edge $e = xz$ such that $e$ crosses exactly $k$ edges of $E_0$.

Let $f = yw$ be an edge of $E_0$ such that (1) $x, y, z, w$ are placed on a circle in clockwise order and (2) there is no endvertex of $E_0$ between $x$ and $y$. (See Figure 3.)

Put $S = \{xz, yw\}$. Let us expand $E$ with respect to $S$. We have two successors $E_1, E_2$ of $E$, where $E_1 = (E \setminus S) \cup \{yz, wx\}$ and $E_2 = (E \setminus S) \cup \{xy, zw\}$. Then $E_1$ is isomorphic to $A(n, k - 1)$ and $E_2$ is isomorphic to $A(n - 1, n - k)$. Hence, we have a bijection between $NCD(A(n, k))$ and $NCD(A(n, k - 1)) \cup NCD(A(n - 1, n - k))$. \qed
For the sake of completeness, we recall the well-known recurrence relation for $\mathcal{UDP}(n, k)$.

**Lemma 6.** For $1 \leq k \leq n$, we have a bijection between $\mathcal{UDP}(n + 2, k + 1)$ and $\mathcal{UDP}(n + 2, k) \cup \mathcal{UDP}(n + 1, n - k + 1)$.

**Proof.** By the definition, $\mathcal{UDP}(n + 2, k + 1)$ is a set of up-down permutations of $[n + 2]$ with the first term at most $k + 1$. $\mathcal{UDP}(n + 2, k + 1)$ is partitioned into $\mathcal{UDP}(n + 2, k)$ and $\mathcal{T} = \mathcal{UDP}(n + 2, k + 1) \setminus \mathcal{UDP}(n, k)$, where $\mathcal{T}$ is a set of up-down permutations of $[n + 2]$ with the first term $k + 1$.

For $\sigma \in \mathcal{T}$, let us remove the first term of $\sigma$. The resulting permutation $\sigma'$ is a down-up permutation of $[n + 2]\setminus\{k + 1\}$ with the first term at least $k + 2$. Hence, there is a natural bijection from $\mathcal{T}$ to $\mathcal{DUP}(n + 1, n - k + 1)$, which has a one-to-one correspondence to $\mathcal{UDP}(n + 1, n - k + 1)$.

Now, we return to the proof of Theorem 3.

For $n = 0$ and $k = 0$, a set of a single chord of $\mathcal{NCD}(A(0, 0))$ clearly corresponds to a single permutation 12 of $\mathcal{UDP}(2, 1)$.

Let $n \geq 1$ and $k \geq 0$. By the inductive hypothesis, we have a bijection from $\mathcal{NCD}(A(n', k'))$ to $\mathcal{UDP}(n' + 2, k' + 1)$ for $n' < n$ or $n' = n$ and $k' < k$.

For $k = 0$, $A(n, 0)$ is isomorphic to $A(n - 1, n - 1)$. Hence, there is a bijection from $\mathcal{NCD}(A(n, 0))$ to $\mathcal{NCD}(A(n - 1, n - 1))$. On the other hand, let $\sigma \in \mathcal{UDP}(n + 2, 1)$. By removing the first term of $\sigma$, we have a down-up permutation $\sigma'$ of $[n + 2]\setminus\{1\}$. Hence, there is a natural bijection from $\mathcal{UDP}(n + 2, 1)$ to $\mathcal{DUP}(n + 1, n)$, which has a one-to-one correspondence to $\mathcal{UDP}(n + 1, n)$.

Therefore, we have a bijection from $\mathcal{NCD}(A(n, 0))$ to $\mathcal{UDP}(n + 2, 1)$.

Let $k \geq 1$. In this case, by Lemma 5 and Lemma 6, we can recursively construct a bijection from $\mathcal{NCD}(A(n, n))$ to $\mathcal{UDP}(n + 2, k + 1)$.

This completes the proof.

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**References**


